Functional Inequalities and convergence of diffusion processes. From the classical heat equation to nonlinear and fractional equations

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Outline

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Energy estimates

- We are going to use energy functions of different types to study the evolution of diffusion equations. This will show a fruitful application of Functional Analysis in the theory of Partial Differential Equations that has been happening for a century and is very active now in new directions.
- The basic equation is the classical heat equation, but the scope is quite general. Our aim is not to establish the convergence of general solutions to the fundamental solution (which in the heat equation can be done by other methods), and a bit more: to find the speed of convergence. This is what the functional analysis does well.
- After change of variables (renormalization) this speed reads as the rate of convergence to equilibrium, and relies on important functional inequalities for typical variable-coefficient equations, like the Ornstein-Uhlenbeck equation.
- The methods will apply to more general linear parabolic equations that generate (linear or nonlinear) semigroups, $S_t : X \to X$, where X is the base space (a space of functions or measures), and S_t is the evolution mapping, $t > 0$.

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Other Equations. Nonlinear, nonlocal, geometric

- Since 2000 we have been studying these functional methods for nonlinear diffusion equations.
- Nonlinear models: porous medium equation, fast diffusion equation, p-Laplacian evolution equation, chemotaxis system, thin films, ...
  plus
- Since 2007: fractional heat equation and fractional porous medium equations, ...
- The method works for equations evolving on manifolds. This is a challenging connection with differential geometry.
- It has been an intense effort. The work related to our research is reported in the survey paper

  ♡ *The mathematical theories of diffusion. Nonlinear and fractional diffusion*, by J. L. Vázquez. CIME Summer Course 2016. Springer Lecture Notes in Mathematics, To appear.
- Many important problems have been solved for the main models, many important problems are still open. We will mention some open problems having a functional flavor.

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Energy estimates for the Heat Equation

recordando la charla del grupo T4 en la Escuela Taller

- Take the classical Heat Equation posed in the whole space $\mathbb{R}^N$ for $\tau > 0$:

$$u_\tau = \frac{1}{2}\Delta_y u$$

with notation $u = u(y,\tau)$ that is useful as we will see. We know the (self-similar) fundamental solution, that is an attractor of its basin

$$U(y,\tau) = C\,\tau^{-N/2} e^{-y^2/2\tau}.$$

- First step: the logarithmic time-space rescaling

$$u(y,\tau) = v(x,t)\,(1+\tau)^{-N/2}, \quad y = x(1+\tau)^{1/2}, \quad t = \log(1+\tau),$$

that leads to the well-known Fokker-Plank equation for $v(x,t)$:

$$v_t = \frac{1}{2}\Delta_x v + \frac{1}{2}\nabla\cdot(x\,v)$$

- If we now pass to the quotient $w = v/G$, where $G = c\,e^{-x^2/2}$ is the stationary state (Gaussiann kernel), to get the Ornstein-Uhlenbeck version

$$w_t = \frac{1}{2}G^{-1}\,\nabla\cdot\left(\,G\,\nabla w\,\right) = \frac{1}{2}\Delta w - \frac{1}{2}x\cdot\nabla w\,,$$

a symmetrically weighted heat equation. The equivalence of these three equations is a main tool in Linear Diffusion and Semigroup Theory.

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Energy estimates for the Heat Equation II

- We may assume without lack of generality that $\int w\,d\mu = \int v\,dx = \int u\,dy = 1$. We now make a crucial estimate on the time decay of the energy for the OUE:

$$\mathcal{F}(w(t)) = \int_{\mathbb{R}^N} |w-1|^2\, G\,dx, \quad \frac{d\mathcal{F}(w(t))}{dt} = -\int_{\mathbb{R}^N} |\nabla w|^2\, G\,dx = -\mathcal{D}(w(t)).$$

- We can now use a result from abstract functional analysis: the Gaussian Poincaré inequality with measure $d\mu = G(x)\,dx$:

$$\int_{\mathbb{R}^N} w^2\,d\mu - \left(\int_{\mathbb{R}^N} w\,d\mu)^2\right) \le C_g \int_{\mathbb{R}^N} |\nabla w|^2\,d\mu, \quad C_g = 1.$$

- Then, the left-hand side is just $\mathcal{F}$ and the inequality implies after the ODE integration $-d\mathcal{F}/dt \ge \mathcal{F}$, that:

$$\int_{\mathbb{R}^N} |w-1|^2\,\mathrm{d}\mu \le \mathrm{e}^{-t} \int_{\mathbb{R}^d} |w_0-1|^2\,\mathrm{d}\mu \quad \forall\, t \ge 0$$

In other words, $\quad \|w(t)-1\|_{L^2(Gdx)} \le \|w_0)-1\|_{L^2(Gdx)} e^{-t/2}$

These are the convergence estimates of solutions to the HE. The rate of convergence is given by the constant C_g of the GPI. Here $C_g/2 = 1/2$.

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Entropy estimates for the Heat Equation III

- There is another approach that starts the analysis from Boltzmann's ideas on entropy dissipation. We start from the Fokker-Planck equation $v_t = \Delta v + \nabla \cdot (xv)$ and consider the functional called **entropy**

$$\mathcal{E}(v) = \int_{\mathbb{R}^N} v \, \log(v/G) \, dx = \int_{\mathbb{R}^N} v \, \log(v) dx + \frac{1}{2} \int_{\mathbb{R}^N} x^2 v \, dx + C \, .$$

- Differentiating along the flow (i.e., for a solution) leads to

$$\frac{d\mathcal{E}(v)}{dt} = -\mathcal{I}(v), \quad \mathcal{I}(v) = \int_{\mathbb{R}^N} v \left| \frac{\nabla v}{v} + x \right|^2 dx = \int_{\mathbb{R}^N} v \, |\nabla \log(v/G)|^2 \, dx \, .$$

- Put now $v = Gf^2$ to find that

$$\mathcal{E}(v) = 2 \int_{\mathbb{R}^N} f^2 \, \log(f) \, d\mu, \quad \mathcal{I}(v) = 4 \int_{\mathbb{R}^N} |\nabla f|^2 \, d\mu.$$

- The famous logarithmic Sobolev inequality [Gross 75] says than that (for all functions, not only solutions)

$$\mathcal{E} \le \frac{1}{2} \mathcal{I}$$

and we obtain the decay $\mathcal{E}(t) \le \mathcal{E}(0) \, e^{-2t}$. Translate that into a good norm.

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About entropy in physics

- Physics books say that entropy was introduced as a state function in Thermodynamics by R. Clausius in 1865, in the framework of the second law of thermodynamics, in order to interpret the results of S. Carnot.
- A statistical physics approach: Boltzmann's formula (1877) defines the entropy of a system in terms of a counting of the micro-states of a physical system.

  The Boltzmann's equation is $\partial_t f + v \cdot \nabla_x f = Q(f, f)$. It describes the evolution of a gas of particles having binary collisions at the kinetic level; $f(t, x, v)$ is a time dependent distribution function (probability density) defined on the phase space $\mathbb{R}^N \times \mathbb{R}^N$.
- The Boltzmann entropy: $H[f] := \iint f \log(f) dxdv$ measures irreversibility: The famous H-Theorem (1872) says that

$$\frac{d}{dt} H[f] = \iint Q(f, f) log(f) dxdv \leq 0 .$$

- Other notions of entropy: The Shannon entropy in information theory, entropy in probability theory (with reference to an arbitrary measure).
  Other approaches: Carathéodory (1908), Lieb-Yngvason (1997).

Porous Medium / Fast Diffusion Equations

- The simplest model of nonlinear diffusion equation is maybe

$$\boxed{u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)}$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1}[= m|u|^{m-1}]$$

- If $m > 1$ it degenerates at $u = 0$, $\Longrightarrow$ slow diffusion. The equation is called Porous Medium Equation, PME.
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \Longrightarrow$ Fast Diffusion, FDE.

- A more general model of nonlinear diffusion takes the divergence form

$$\partial_t H(u) = \nabla \cdot \vec{\mathcal{A}}(x, u, Du) + \mathcal{B}(x, t, u, Du)$$

with monotonicity conditions on H and $\nabla_p \vec{\mathcal{A}}(x, t, u, p)$ and structural conditions on $\vec{\mathcal{A}}$ and $\mathcal{B}$. This generality includes Stefan Problems, p-Laplacian flows (including $p = \infty$ and total variation flow $p = 1$) and many others.

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$$\partial_t H(u) = \nabla \cdot \vec{\mathcal{A}}(x, u, Du) + \mathcal{B}(x, t, u, Du)$$

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Porous Medium / Fast Diffusion Equations

- The simplest model of nonlinear diffusion equation is maybe

$$\boxed{u_t = \Delta u^m = \nabla \cdot (c(u)\nabla u)}$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = m u^{m-1} [= m|u|^{m-1}]$$

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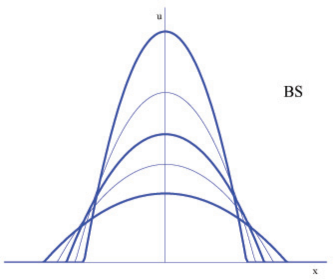
Barenblatt profiles and Asymptotics

- These profiles are the alternative to the Gaussian profiles that have such a star role for the HE. The Barenblatt profiles are the model behaviour for the PME.

  They are source solutions. *Source means that $u(x,t) \to M\,\delta(x)$ as $t \to 0$.*
- They have explicit formulas (1950, 52), they are self-similar:

$$\mathbf{B}(x,t;M) = t^{-\alpha}\mathbf{F}(x/t^{\beta}), \quad \mathbf{F}(\xi) = \left(C - k\xi^2\right)_+^{1/(m-1)}$$



$\alpha = \frac{n}{2+n(m-1)}$

$\beta = \frac{1}{2+n(m-1)} < 1/2$

Height $u = Ct^{-\alpha}$

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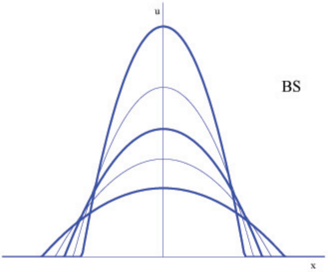

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Asymptotic behaviour I

Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^N$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^N)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x)\,dx + \iint f\,dxdt$.

Asymptotic Theorem [Friedman-Kamin, 1980; V. 2001] *Let $B(x,t;M)$ be the Barenblatt with the asymptotic mass M; u converges to B after renormalization*

$$\boxed{t^{\alpha}|u(x,t) - B(x,t)| \to 0}$$

Let $f = 0$ (or small at infinity in L^p). For every $p \ge 1$ we have

$$\boxed{\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p-1).}$$

Note: α *and* $\beta = \alpha/n = 1/(2 + n(m-1))$ *are the zooming exponents as in* $B(x,t)$.

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Calculations of the entropy rates

- This is next step of information after proving plain convergence. We go back to the ideas of the second proof of convergence for the heat equation, and use rescaling and entropies.
- We rescale the function as $u(x,t) = r(t)^n\, v(y\, r(t), s)$ where $r(t)$ is the Barenblatt radius at $t+1$, and "new time" is $s = \log(1+t)$. The equation becomes
$$v_s = \mathrm{div}\,(v(\nabla v^{m-1} + \frac{c}{2}\nabla y^2)).$$
- Then define a new entropy (not Boltzmann entropy, but a new type called Rényi entropy)
$$\mathcal{E}(u)(t) = \int (\frac{1}{m} v^m + \frac{c}{2} v y^2)\, dy$$
The minimum of entropy is precisely the Barenblatt profile.
- Calculate
$$\frac{d\mathcal{E}}{ds} = -\int v|\nabla v^{m-1} + cy|^2\, dy = -\mathcal{D}$$
Moreover, a difficult calculation known as Bakry-Emery method gives
$$\frac{d\mathcal{D}}{ds} = -R, \quad \mathcal{R} \sim \lambda \mathcal{D}.$$
- *We conclude exponential decay of $\mathcal{D}$ in new time s, i.e., a power rate in real time t. It follows that $\mathcal{E}$ decays to a minimum $\mathcal{E}_\infty > 0$ and we then prove that this is the level of the Barenblatt solution, which attains the functional minimum.*

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Rates through entropies for Fast Diffusion

Large effort has been invested in making the entropy machinery work for fast diffusion, $-\infty < m < 1$.

The nice properties of the entropies from the view of transport theory (cf. Villani's book) are lost soon, when $m = (N-1)/N$.

Finite entropy is lost when the second moment is infinite, i.e. for $m = (N-1)/(N+1)$.

Finite-mass, stable states (Barenblatt solutions) are lost for $m = (N-2)/N$.

Functional inequalities play a crucial role in the asymptotic analysis, there are so to say "equivalent".

There is work by many authors: Blanchet, Bonforte, Carrillo, Dolbeault, Del Pino, Denzler, Grillo, McCann, Markowich, Otto, Slepcev, Vazquez, ...

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Fractional diffusion

- Replacing Laplacians by fractional Laplacians is motivated by the need to represent anomalous diffusion. In probabilistic terms it replaces next-neighbour interaction and Brownian motion by long-distance interaction and what they call Lévy processes. The solutions do not have exponential decay in space like the Gaussian, but larger, power-like tails. The main mathematical models are the Fractional Laplacians that have special symmetry and invariance properties that makes analysis easier. In practice, other nonlocal integral operators are also used, but I will not mention them below.

- Basic evolution equation

$$\boxed{\partial_t u + (-\Delta)^s u = 0}$$

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The fractional Laplacian operator

- **Different formulas for fractional Laplacian operator**.
  We assume that the space variable $x \in \mathbb{R}^N$, and the fractional exponent is $0 < s < 1$. First, pseudo differential operator given by the Fourier transform:

$$\widehat{(-\Delta)^s} u(\xi) = |\xi|^{2s} \widehat{u}(\xi)$$

- Singular integral operator:

$$(-\Delta)^s u(x) = C_{n,s} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy$$

  With this definition, it is the inverse of the Riesz integral operator $I_{2s} = (-\Delta)^{-s} u$. This one has kernel $C_1 |x-y|^{n-2s}$, which is not integrable, this time at infinity.

- Take the random walk for Lévy processes:

$$u_j^{n+1} = \sum_k P_{jk} u_k^n$$

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The fractional Laplacian operator II

- The α-harmonic extension: Find first the solution of the $(n+1)$ problem

$$\nabla \cdot (y^{1-\alpha}\nabla v) = 0 \quad (x,y) \in \mathbb{R}^N \times \mathbb{R}_+; \quad v(x,0) = u(x), \quad x \in \mathbb{R}^N.$$

Then, putting $\alpha = 2s$ we have

$$(-\Delta)^s u(x) = -C_\alpha \lim_{y\to 0} y^{1-\alpha} \frac{\partial v}{\partial y}$$

When $s = 1/2$ i.e. $\alpha = 1$, the extended function v is harmonic (in $n+1$ variables) and the operator is the Dirichlet-to-Neumann map on the base space $x \in \mathbb{R}^N$. It was proposed in PDEs by Caffarelli and Silvestre.

Remark. In $\mathbb{R}^N$ all these versions are equivalent. In a bounded domain we have to re-examine all of them. Three main alternatives are studied in probability and PDEs, corresponding to different options about what happens to particles at the boundary or what is the domain of the functionals.

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Nonlocal diffusion model. The problem

- The nonlinear diffusion model with nonlocal effects proposed in 2007 with Luis Caffarelli uses the derivation of the PME but with a closure relation between pressure and density of the form $p = \mathcal{K}(u)$, where $\mathcal{K}$ is a linear integral operator, which we assume in practice to be the inverse of a fractional Laplacian. Hence, p es related to u through a fractional potential operator, $\mathcal{K} = (-\Delta)^{-s}$, $0 < s < 1$, with kernel $k(x, y) = c|x - y|^{-(n-2s)}$, (i.e., a Riesz operator). We have $(-\Delta)^s p = u$.
- The diffusion model with nonlocal effects is thus given by the system

$$(1) \qquad \boxed{u_t = \nabla \cdot (u \nabla p), \quad p = \mathcal{K}(u).}$$

where u is a function of the variables (x, t) to be thought of as a density or concentration, and therefore nonnegative, while p is the pressure, which is related to u via a linear operator $\mathcal{K}$: $u_t = \nabla \cdot (u \nabla (-\Delta)^{-s} u)$

- The problem is posed for $x \in \mathbb{R}^N$, $n \geq 1$, and $t > 0$, and we give initial conditions

$$(2) \qquad u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N,$$

where u_0 is a nonnegative, bounded and integrable function in $\mathbb{R}^N$.

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$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[\sigma_s(\rho) \nabla \frac{\delta F(\rho)}{\delta \rho} \right]$$

  See also (GL2) and the review paper (GLP). The model is used to study phase segregation in (GLM, 2000).
- More generally, it could be assumed that $\mathcal{K}$ is an operator of integral type defined by convolution on all of $\mathbb{R}^n$, with the assumptions that is positive and symmetric. The fact the $\mathcal{K}$ is a homogeneous operator of degree $2s$, $0 < s < 1$, will be important in the proofs. An interesting variant would be the Bessel kernel $\mathcal{K} = (-\Delta + cI)^{-s}$. We are not exploring such extensions.

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- Existence of weak energy solutions and property of finite propagation

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- Existence of self-similar profiles, renormalized Fokker-Planck equation and entropy-based proof of stabilization

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- Regularity in three levels: $L^1 \to L^2$, $L^2 \to L^\infty$, and bounded implies C^α

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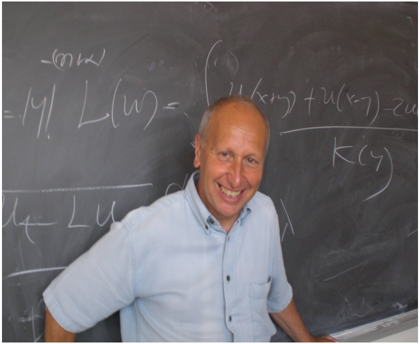
Asymptotic behavior for the nonlocal PME



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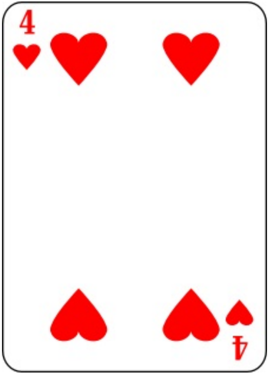
La respuesta es inesperada



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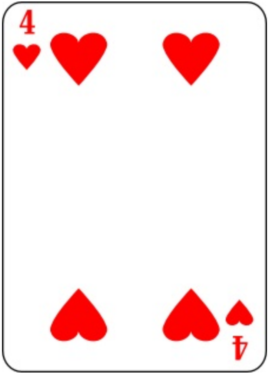

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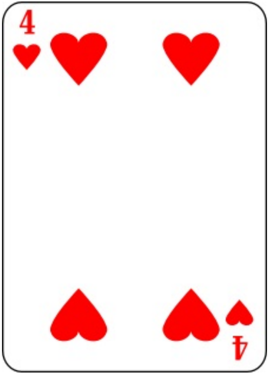

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Rescaling for the NL-PME

We now begin the study of the large time behavior.

- Inspired by the asymptotics of the standard porous medium equation, we define the renormalized flow through the transformation

(3) $$u(x,t) = t^{-\alpha} v(x/t^{\beta}, \tau)$$

with new time $\tau = \log(1+t)$. We also put $y = x/t^{\beta}$ as rescaled space variable. In order to cancel the factors including t explicitly, we get the condition on the exponents

(4) $$\alpha + (2-2s)\beta = 1$$

- We use the homogeneity of $K = (-\Delta)^{-s}$ in the form

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- If we also want conservation of (finite) mass, then we must put $\alpha = n\beta$, and we arrive at the the precise value of the exponents:

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Renormalized flow

- We thus arrive at the *nonlinear, nonlocal Fokker-Plank equation*

(6) $$v_\tau = \nabla_y \cdot (v\,(\nabla_y K(v) + \beta y))$$

- **Stationary renormalized solutions**. They are the solutions $U(y)$ of

(7) $$\nabla_y \cdot (U\,\nabla_y(P + a|y|^2)) = 0, \quad P = K(U).$$

where $a = \beta/2$, and β defined just above. Since we are looking for asymptotic profiles of the standard solutions of the NL-PME we also want $U \geq 0$ and integrable. The simplest possibility is integrating once and getting the radial version

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Obstacle problem

- Indeed, if we solve the obstacle problem with fractional Laplacian we will obtain a unique solution $P(y)$ of the problem:

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(9) \qquad \begin{array}{l} P \geq \Phi, \quad U = (-\Delta)^s P \geq 0; \\ \text{either } \; P = \Phi \text{ or } \; U = 0. \end{array}
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  with $0 < s < 1$. Here we have to choose as obstacle

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\Phi = C - a\,|y|^2,
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  where C is any positive constant and $a = \beta/2$. For uniqueness we also need the condition $P \to 0$ as $|y| \to \infty$.

- The obstacle problem theory by Caffarelli and collaborators says that the solution is unique and belongs to the space H^{-s} with pressure in H^s. The solutions have $P \in C^{1,s}$ and $U \in C^{1-s}$.

- Note that for $C \leq 0$ the solution is trivial, $P = 0$, $U = 0$, hence we choose $C > 0$. We also note the pressure is defined but for a constant, so that we may take without loss of generality $C = 0$ and take as pressure $\widehat{P} = P - C$ instead of P. But then $P \to 0$ implies that $\widehat{P} \to -C$ as $|y| \to \infty$, so we get a one parameter family of stationary profiles that we denote $U_C(y)$.

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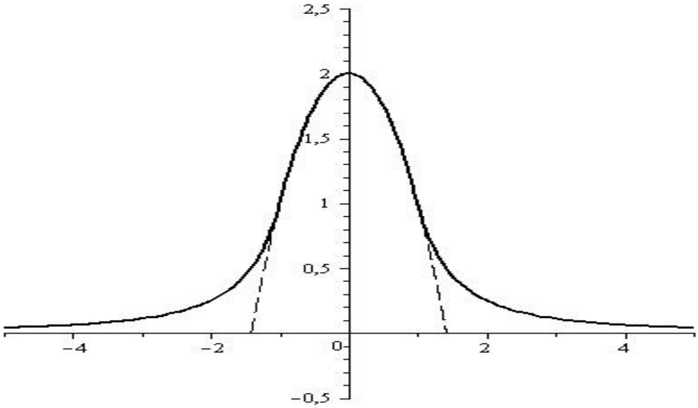

The solution of the obstacle problem with parabolic obstacle

The variable is the pressure P and $U = (-\Delta)^s U$ has compact support

Estimates for the renormalized problem. Entropy dissipation.

- Our main problem is now to prove that these profiles are attractors for the renormalized flow.
- We review the estimates of Main Estimates of Section above in order to adapt them to the renormalized problem.
- There is no problem is reproving mass conservation or positivity.
- First energy estimate becomes (recall that $H = K^{1/2}$)

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&\frac{d}{d\tau}\int v(y,\tau)\log v(y,\tau)\,dy \\
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$$\mathcal{E}(v(\tau)) := \frac{1}{2}\int_{\mathbb{R}^n} (v\,K(v) + \beta y^2 v)\,dy \tag{11}$$

The entropy contains two terms. The first is

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This means that whenever the initial entropy is finite, then $\mathcal{E}(v(\tau))$ is uniformly bounded for all $\tau > 0$, $\mathcal{I}(v)$ is integrable in $(0,\infty)$ and

$$\mathcal{E}(v(\tau)) + \iint \left|\nabla(Kv + \frac{\beta}{2}y^2)\right|^2 vdy\,dt \le \mathcal{E}(v_0).$$

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This means that whenever the initial entropy is finite, then $\mathcal{E}(v(\tau))$ is uniformly bounded for all $\tau > 0$, $\mathcal{I}(v)$ is integrable in $(0,\infty)$ and

$$\mathcal{E}(v(\tau)) + \iint \left|\nabla(Kv + \frac{\beta}{2}y^2)\right|^2 vdy\, dt \le \mathcal{E}(v_0).$$

Estimates for the renormalized problem. Entropy dissipation.

- However, the second energy estimate has an essential change. We need to define the entropy of the renormalized flow as

$$\mathcal{E}(v(\tau)) := \frac{1}{2}\int_{\mathbb{R}^n} (v\,K(v) + \beta y^2 v)\,dy \tag{11}$$

The entropy contains two terms. The first is

$$E_1(v(\tau)) := \int_{\mathbb{R}^n} v\,K(v)\,dy = \int_{\mathbb{R}^n} |Hv|^2\,dy, \quad H = K^{1/2}$$

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Convergence.

- The standard idea is to let $t \to \infty$ in the renormalized flow. Since the entropy goes down there is a limit
$$E_* = \lim_{t\to\infty} \mathcal{E}(t) \geq 0.$$
Since u is bounded in L^1_x unif. in t, and also ux^2 is bounded in L^1_x unif. in t, and moreover $|\nabla H(u)| \in L^2_x$ unif in t, we have that $u(t)$ is a compact family that there is a subsequence $t_j \to \infty$ that converges in L^1_x and almost everywhere to a limit $u_* \geq 0$. The mass of u_* is the same mass of u. One consequence is that the lim inf of the component $E_2(u(t_j))$ is equal or larger that $M_2(u_*)$.
- The rest of the convergence depends on Fractional Sobolev spaces and compact embedding theorems.
Thus, we also have $H(u) \in L^2_x$ uniformly in t. The boundedness of $\nabla H(u)$ in L^2_x implies the compactness of $H(u)$ in space, so that it converges along a subsequence to v_* . This allows to pass to the limit in $E_1(u(t_j))$ and obtain a correct limit. We have $v_* = H(u_*)$.
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- Now we get the consequence that for every $h > 0$ fixed

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  This implies that if $w(x,t) = Ku + \frac{\beta}{2}x^2$ and $w_h(x,t) = w(x,t+h)$, then $u_h|\nabla w_h|^2$ converges to zero as $h \to \infty$ in $L^1(\mathbb{R}^n \times (0,T))$. Then w_h converges to a constant in space wherever u is not zero, and that constant must be $Ku_* + \frac{\beta}{2}x^2$ along the said subsequence, hence constant also in time

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$$u_t = \nabla \cdot (u\Lambda^{\alpha-1}u^m), \quad 0 < \alpha < 2$$

  and obtain our type of Barenblatt solutions for every $m > 1$ with a very nice added information, they happen to be explicit of the form

$$u(x,t) = Ct^{-\mu}(R^2 - x^2 t^{-2\nu})_+^{\alpha/2(m-1)}$$

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- Uniqueness and comparison. These questions of are solved in dimension $N = 1$ thanks to the trick of integration in space used previously by Biler, Karch, and Monneau (2008). New tools are needed to make progress in several dimensions.

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Functional analysis and convergence rates

- Proving that the self-similar solutions (Caffareli-Vazquez, Biler-Imbert-Karch-Monneau) are attractors with a calculated rate is done in 1D in

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  It uses entropy analysis and Bakry Emery method, and new functional inequalities.

- Details of the proof are as follows: the nonlinear nonlocal equation, is written after renormalization as

$$\rho_t = \nabla \cdot \left(\rho(\nabla(-\Delta)^{-s}\rho + \lambda x)\right), \qquad \lambda > 0,\ x \in \mathbb{R}^d, \tag{13}$$

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- the free energy $\mathcal{E}(\rho)$ defined as

$$\text{(14)} \qquad \mathcal{E}(\rho) = \frac{1}{2}\int_{\mathbb{R}^d} \{(-\Delta)^{-s}\rho(x) + \lambda|x|^2\}\rho(x)\,dx$$
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  is a Lyapunov functional for $0 < s < \min(1, d/2)$. *One can similarly define the Lyapunov functional for $1/2 \le s < 1$ in one dimension, assuming that ρ satisfies a growth condition at infinity, namely $\rho \log|x| \in L^1(\mathbb{R})$ if $s = 1/2$ and $\rho|x|^{2s-1} \in L^1(\mathbb{R})$ if $1/2 < s < 1$.*

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Functional analysis and convergence rates

- Now, we can consider the difference $\mathcal{E}(\rho|\rho_\infty) := \mathcal{E}(\rho) - \mathcal{E}(\rho_\infty)$ as a measure of convergence towards equilibrium. We first rewrite the equation (13) as

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  After many calculations we get $d\mathcal{I}(\rho)/dt = -2\lambda\mathcal{I}(\rho) - 2\mathcal{R}(\rho)$. By good fortune in trying to put $\mathcal{R}(\rho)$ in good shape we get the signed version

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  $$\mathcal{R}(\rho) = \frac{c^+_{d,s}}{2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(x)\rho(y)\big\langle \nabla\xi(x) - \nabla\xi(y), \mathbf{K}(x-y)\big(\nabla\xi(x) - \nabla\xi(y)\big)\big\rangle dydx,$$

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Work to Do

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- Study the equation and regularity of the free boundary
- Study fine asymptotic behavior in other classes of data
- Study these nonlocal problems in bounded domains
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- Decide conditions of comparison
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- Consider different nonlocal nonlinear diffusion problems
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