Lineability in sequence spaces

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Dpto. Análisis Matemático

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Previous Concepts

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- *A is maximal-(dense)-lineable if $dim(M) = dim(X)$.*

Everywhere Surjective Functions

Recall that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is an everywhere surjective function if $f(I) = \mathbb{R}$ for all interval $I \subset \mathbb{R}$.

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- Take any bijection $\Phi_n : C_n \longrightarrow \mathbb{R}$.
- Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) := \begin{cases} \Phi_n(x) & \text{if } x \in C_n, \\ 0 & \text{in other case.} \end{cases}$$

Everywhere Surjective Functions

Theorem (Araújo, Bernal, Muñoz, Prado and Seoane, 2017)

The set of measureable everywhere surjective functions $\mathcal{MES}$ is $\mathfrak{c}$-lineable.

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Theorem (A, B, M, P and S, 2017)

The family of sequences $(f_n)_{n\in\mathbb{N}}$ of Lebesgue measurable functions such that $f_n \longrightarrow 0$ pointwise and $f_n \in \mathcal{MES}$ is $\mathfrak{c}$-lineable.

Measure versus Almost Convergence

Recall that $f_n \longrightarrow f$ in measure if $\forall\, \varepsilon > 0$ we have

$$\mu\left(\{x \in X \ :\ |f_n(x) - f(x)| \geq \varepsilon\}\right) \longrightarrow 0, \quad (n \to \infty).$$

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- Let $f_n(x) = \chi_{\left[\frac{1}{n+1}, \frac{1}{n}\right]}(x)$.

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- Consider now

$$f_{n,t}(x) = \chi_{[\frac{1}{n+1}, \frac{1}{n}]}\left(\frac{1}{2}(x-t)\right) = \chi_{[\frac{2}{n+1}+t, \frac{2}{n}+t]}(x),\ t \in (-1, 0).$$

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- Let $M := \text{span}\{(f_{n,t}) \ : \ t \in (-1,0)\}$. Then $\dim(M) = \mathfrak{c}$, so A is maximal-lineable.
- Take $X = L_0^{\mathbb{N}}$, $B = \widetilde{L} := \{\Phi = (f_n) \in L_0^{\mathbb{N}} \ : \ \exists N = N(\Phi) \in \mathbb{N} \mid f_n = 0\ \forall n \geq N\}$

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- Thus, A is maximal-dense-lineable.

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Thank you very much for your attention