Approximation of Sobolev-type embeddings
Recent results and open problems

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Subject of the talk

Approximation of functions on the d-dimensional torus $\mathbb{T}^d$

- Functions: from quite general spaces, including e.g. classical Sobolev spaces (isotropic, dominating mixed smoothness,...)
- Quality of approximation: expressed via approximation numbers
- Error: with respect to the L_2-norm or the sup-norm

Special emphasis on

- Optimal asymptotic rates and sharp constants
- Preasymptotic estimates

This subject is related to

- Functional Analysis, Approximation Theory, Numerical Analysis,...

The talk is based on the following recent papers

- T. Kühn, W. Sickel and T. Ullrich, *Approximation numbers of Sobolev embeddings – Sharp constants and tractability*, J. Complexity 30 (2014), 95–116.
- T. Kühn, W. Sickel and T. Ullrich, *Approximation of mixed order Sobolev functions on the d-torus – Asymptotics, preasymptotics and d-dependence*, Constr. Approx. 42 (2015), 353–398.
- F. Cobos, T. Kühn and W. Sickel, *Optimal approximation of multivariate periodic Sobolev functions in the sup-norm*, J. Funct. Anal. 270 (2016), 4196–4212.
- T. Kühn, S. Mayer and T. Ullrich, *Counting via entropy: New preasymptotics for the approximation numbers of Sobolev embeddings*, SIAM J. Numer. Anal. 54 (2016), 3625–3647.
- T. Kühn and M. Petersen, *Approximation in periodic Gevrey spaces*, in progress

Approximation numbers

- For (bounded linear) operators $T : X \to Y$ between two Banach spaces the approximation numbers are defined as

$$a_n(T : X \to Y) := \inf\{\|T - A\| : \operatorname{rank} A < n\}$$

- $\lim_{n\to\infty} a_n(T) = 0 \quad \Longrightarrow \quad T$ compact

  $\Longleftarrow \quad$ fails by Enflo's counter-example

  Rate of decay of $a_n(T)$ describes the 'degree' of compactness of T

- For compact operators between Hilbert spaces

$$a_n(T) = s_n(T) = \sqrt{\lambda_n(T^*T)} = n\text{-th singular number}$$

Interpretation in Numerical Analysis

- Every operator $A : X \to Y$ of finite rank n can be written as

$$Ax = \sum_{j=1}^{n} L_j(x)\, y_j \qquad \text{for all } x \in X$$

with linear functionals $L_j \in X'$ and vectors $y_j \in Y$.
$\curvearrowright$ A is a linear algorithm using n arbitrary linear informations

- worst-case error of the algorithm A

$$err^{wor}(A) := \sup_{\|x\| \le 1} \|Tx - Ax\| = \|T - A\|$$

- n-th minimal worst-case error of the problem T
(with respect to linear algorithms and arbitrary linear information)

$$err_n^{wor}(T) := \inf_{\operatorname{rank} A \le n} err^{wor}(A) = a_{n+1}(T)$$

Sobolev embeddings

- **Well-known**

  – For isotropic Sobolev spaces on the d-dimensional torus $\mathbb{T}^d$

$$c_{s,d} \cdot n^{-s/d} \le a_n(I_d : H^s(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le C_{s,d} \cdot n^{-s/d}$$

  – For Sobolev spaces of dominating mixed smoothness

$$c_{s,d} \cdot \left[\frac{(\log n)^{d-1}}{n}\right]^s \le a_n(I_d : H^s_{mix}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le C_{s,d} \cdot \left[\frac{(\log n)^{d-1}}{n}\right]^s$$

- **Almost nothing known**

  How do the constants $c_{s,d}$ and $C_{s,d}$ depend on s and d ???

  This is essential for high-dimensional numerical problems, and also for tractability questions in information-based complexity.

Some remarks

- Of course, the constants heavily depend on the chosen norms.
  $\curvearrowright$ First we have to fix (somehow natural) norms.
  For all our norms, we will have norm one embeddings into $L_2(\mathbb{T}^d)$.
- For example, for smoothness $s = 1$, the asymptotic rates are

$$\alpha_n := n^{-1/d} \quad \text{and} \quad \beta_n := \frac{(\log n)^{d-1}}{n} .$$

  In high dimensions, one has to wait exponentially long until these rates become visible, as one can see from the following examples.
- Isotropic case.
  $n = 10^d$ (very large) $\curvearrowright$ $\alpha_n = \frac{1}{10}$ (poor error estimate)
- Mixed case. (Dimension $d + 1$)
  Even worse, $n = d^d$ $\curvearrowright$ $\beta_n = (\log d)^d \gg 1$ (trivial estimate)
- $\curvearrowright$ We need precise information on the constants and preasymptotic estimates (for small n, say $n \leq 2^d$)

Periodic function spaces

- The Fourier coefficients of a function $f \in L_2(\mathbb{T}^d)$ on the d-dimensional torus $\mathbb{T}^d = [0, 2\pi]^d$ are

$$c_k(f) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-ikx} dx \quad , \quad k \in \mathbb{Z}^d$$

- Given any weights $w(k) > 0$, we define $F_d(w)$ as the space of all $f \in L_2(\mathbb{T}^d)$ such that

$$\|f|F_d(w)\| := \Big(\sum_{k \in \mathbb{Z}^d} w(k)^2 |c_k(f)|^2 \Big)^{1/2} < \infty \, .$$

- There are compact embeddings

$$F_d(w) \hookrightarrow L_2(\mathbb{T}^d) \iff \lim_{|k| \to \infty} 1/w(k) = 0$$

$$F_d(w) \hookrightarrow L_\infty(\mathbb{T}^d) \iff \sum_{k \in \mathbb{Z}^d} 1/w(k)^2 < \infty \, .$$

Isotropic periodic Sobolev spaces, integer smoothness

- The Sobolev space $H^m(\mathbb{T}^d)$, $m \in \mathbb{N}$, consists of all $f \in L_2(\mathbb{T}^d)$ such that the following (equivalent!) norms are finite.
- Natural norm (all partial derivatives)

$$\| f \,|H^m(\mathbb{T}^d)\| := \Big(\sum_{|\alpha|\le m} \| D^\alpha f \,|L_2(\mathbb{T}^d)\|^2 \Big)^{1/2}$$

- Modified natural norm (only highest derivatives in each coordinate)

$$\| f \,|H^m(\mathbb{T}^d)\|^* := \Big(\| f \,|L_2(\mathbb{T}^d)\|^2 + \sum_{j=1}^{d} \Big\| \frac{\partial^m f}{\partial x_j^m} \Big| L_2(\mathbb{T}^d) \Big\|^2 \Big)^{1/2}$$

Norms via Fourier coefficients

- These norms can be rewritten in terms of Fourier coefficients, using Parseval's identity and $c_k(D^\alpha f) = (ik)^\alpha c_k(f)$.
- For the natural norm one has

$$\| f \,|H^m(\mathbb{T}^d)\| \sim \left(\sum_{k\in\mathbb{Z}^d} \Big(1 + \sum_{j=1}^d |k_j|^2\Big)^m |c_k(f)|^2 \right)^{1/2}$$

with equivalence constants independent on d.

- For the modified natural norm one has even equality

$$\| f \,|H^m(\mathbb{T}^d)\|^* = \left(\sum_{k\in\mathbb{Z}^d} \Big(1 + \sum_{j=1}^d |k_j|^{2m}\Big) |c_k(f)|^2 \right)^{1/2} .$$

Fractional smoothness $s > 0$

- Let $s > 0$, $d \in \mathbb{N}$ and $0 < p \le \infty$.
  $H^{s,p}(\mathbb{T}^d)$ consists of all $f \in L_2(\mathbb{T}^d)$ such that

$$\|f|H^{s,p}(\mathbb{T}^d)\| := \Big(\sum_{k\in\mathbb{Z}^d} w_{s,p}(k)^2 |c_k(f)|^2 \Big)^{1/2} < \infty \,,$$

  where the weights are $\quad w_{s,p}(k) := \Big(1 + \sum_{j=1}^{d} |k_j|^p\Big)^{s/p}.$

- For fixed $s > 0$ and $d \in \mathbb{N}$, all these norms are equivalent.
  Clearly, the equivalence constants depend on d.
  But all spaces $H^{s,p}(\mathbb{T}^d)$, $0 < p \le \infty$, coincide as vector spaces.

- These spaces are of the general form $F_d(w)$.

Relation to the classical norms

- For the natural norm we have **equivalence**

$$\| f \,|H^m(\mathbb{T}^d)\| \sim \| f \,|H^{m,2}(\mathbb{T}^d)\|$$

with equivalence constants independent on d.

- For the modified natural norm one has even **equality**

$$\| f \,|H^m(\mathbb{T}^d)\|^* = \| f \,|H^{m,2m}(\mathbb{T}^d)\|$$

Sobolev spaces of dominating mixed smoothness

- Let $s > 0$, $d \in \mathbb{N}$ and $0 < p \leq \infty$.

  $H^{s,p}_{mix}(\mathbb{T}^d)$ consists of all $f \in L_2(\mathbb{T}^d)$ such that

$$\|f|H^{s,p}(\mathbb{T}^d)\| := \Big(\sum_{k\in\mathbb{Z}^d} w^{mix}_{s,p}(k)^2 |c_k(f)|^2 \Big)^{1/2} < \infty ,$$

  where the weights are now $\quad w^{mix}_{s,p}(k) := \prod_{j=1}^{d} (1 + |k_j|^p)^{s/p} .$

- For integer smoothness $s \in \mathbb{N}$, all $H^{s,p}_{mix}(\mathbb{T}^d)$, $0 < p \leq \infty$, coincide with the classical Sobolev space of dominating mixed smoothness

$$H^s_{mix}(\mathbb{T}^d) = \{ f \in L_2(\mathbb{T}^d) : D^\alpha f \in L_2(\mathbb{T}^d) \ \forall \alpha \in \{0, 1, ..., s\}^d \}$$

  The parameter p indicates which of the equivalent norms we are using.

Reduction to sequence spaces

$$\begin{array}{ccc} F_d(w) & \xrightarrow{\quad I_d \quad} & L_2(\mathbb{T}^d) \\ \Big\downarrow A & & \Big\uparrow B \\ \ell_2(\mathbb{Z}^d) & \xrightarrow{\quad D \quad} & \ell_2(\mathbb{Z}^d) \end{array}$$

$$Af := (w(k)\, c_k(f))_{k\in\mathbb{Z}^d} \quad , \quad B\xi := \sum_{k\in\mathbb{Z}^d} \xi_k\, e^{ikx} \quad , \quad D(\xi_k) := (\xi_k / w(k))$$

Let $(\sigma_n)_{n\in\mathbb{N}}$ is the non-increasing rearrangement of $(1/w(k))_{k\in\mathbb{Z}^d}$

A and B are unitary operators $\curvearrowright$ $\boxed{a_n(I_d) = a_n(D) = s_n(D) = \sigma_n}$

Isotropic Sobolev spaces

- $H^{s,p}(\mathbb{T}^d) = F_d(w)$ with $w(k) = \left(1 + \sum_{j=1}^{d} |k_j|^p\right)^{s/p}$.

  $\curvearrowright$ $(\sigma_n)_n$ attains the values $(1 + r^p)^{-s/p}$, $r \in \mathbb{N}$, but each of them at least $2d$ times.

- Define $N(r,d) := card\{k \in \mathbb{Z}^d : \sum_{j=1}^d |k_j|^p \leq r^p\}$.

Lemma

If $N(r-1,d) < n \leq N(r,d)$, then

$$a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = (1 + r^p)^{-s/p}.$$

- In principle, this gives $a_n(I_d)$ for all n, but the exact computation of the cardinalities $N(r,d)$ is impossible. The hard work is to find good estimates, using combinatorial and volume arguments.

Asymptotic constants, $n \to \infty$

- Let B_p^d denote the unit ball in $(\mathbb{R}^d, \|.\|_p)$. Using volume estimates, we can show the existence of asymptotically optimal constants.

Theorem (KSU 2014)

Let $0 < s, p < \infty$ *and* $d \in \mathbb{N}$. *Then*

$$\lim_{n\to\infty} n^{s/d} a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = vol(B_p^d)^{s/d} \sim d^{-s/p}$$

- The asymptotic constant is of order
  $d^{-s/2}$ for the natural norm $(p = 2)$,
  $d^{-1/2}$ for the modified natural norm $(p = 2s)$.
- We get the correct order $n^{-s/d}$ of the a_n in n and the
  exact decay rate $d^{-s/p}$ of the constants in d.
- Polynomial decay in d of the constants helps in error estimates!

Estimates for large n

Theorem (KSU 2014, case $p = 1$)

Let $s > 0$ and $n \geq 6^d/3$. Then

$$d^{-s} n^{-s/d} \leq a_n(I_d : H^{s,1}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq (4e)^s d^{-s} n^{-s/d} .$$

- We have similar estimates for all other $0 < p < \infty$, but for $p = 1$ the constants are nicer.
- Note the correct d-dependence d^{-s} of the constants!
- Proof: via combinatorial estimates of the cardinalities $N(r, d)$

Preasymptotic estimates – small n

Theorem (KSU 2014)

Let $p = 1$ and $2 \le n \le 2^d$. Then

$$\left(\frac{1}{2+\log_2 n}\right)^s \le a_n(I_d : H^{s,1}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le \left(\frac{\log_2(2d+1)}{\log_2 n}\right)^s.$$

- Using a relation to entropy numbers,
  – the gap between lower and upper bounds was closed
  – arbitrary p's could be treated, shows the influence of the norm

Theorem (KMU 2016)

Let $s > 0$, $0 < p < \infty$ and $2 \le n \le 2^d$. Then

$$a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \sim \left(\frac{\log_2(1 + d/\log_2 n)}{\log_2 n}\right)^{s/p}.$$

(We have explicit expressions for the hidden constants.)

Sobolev spaces of dominating mixed smoothness

- Same strategy as for isotropic spaces, but the combinatorial estimates are more complicated.

Theorem (KSU 2015 - optimal asymptotic constants)

Let $s > 0$ and $d \in \mathbb{N}$. Then, for all $0 < p < \infty$, it holds

$$\lim_{n\to\infty} \frac{n^s a_n(I_d : H^{s,p}_{mix}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))}{(\log n)^{s(d-1)}} = \left[\frac{2^d}{(d-1)!}\right]^s$$

- Interesting fact: For all $0 < p < \infty$ the limit is the same.
- The asymptotic constant decays super-exponentially in d.

Preasymptotic estimates, small n

Theorem (KSU 2015)

Let $s > 0$ and $d \in \mathbb{N}$, $d \geq 2$. Then, for $9 \leq n \leq d\,2^{2d-1}$, it holds

$$a_n(I_d : H^{s,1}_{mix}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq \left(\frac{e^2}{n}\right)^{\frac{s}{2+\log_2 d}}$$

- The bound is non-trivial in the given range, since $n \geq 9 > e^2$.
- We have also similar (non-matching) lower estimates.
  But they show, that one has to wait exponentially long until one can "see" the correct asymptotic rate n^{-s}, ignoring the log-terms.

Approximation in the sup-norm

- It is well-known that

$$H^s(\mathbb{T}^d) \hookrightarrow L_\infty(\mathbb{T}^d) \quad \Longleftrightarrow \quad s > \frac{d}{2}$$

$$H^s_{mix}(\mathbb{T}^d) \hookrightarrow L_\infty(\mathbb{T}^d) \quad \Longleftrightarrow \quad s > \frac{1}{2}$$

- The asymptotic behaviour of the approximation numbers is also well-known, up to multiplicative constants,

$$a_n(I_d : H^s(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)) \sim n^{d/2-s}$$

$$a_n(I_d : H^s_{mix}(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)) \sim n^{1/2-s}(\log n)^{s(d-1)}$$

- **Problem.** Find estimates for the hidden constants and the families of norms, with parameters $0 < p < \infty$.

From L_2-approximation to L_∞-approximation

- More general: $F_d(w) \hookrightarrow L_\infty(\mathbb{T}^d) \iff \sum_{k\in\mathbb{Z}^d} 1/w(k)^2 < \infty.$

  In this case, the embedding is even compact.

Theorem (CKS 2016)

Let $F_d(w) \hookrightarrow L_\infty(\mathbb{T}^d)$. Then

$$a_n(I_d : F_d(w) \to L_\infty(\mathbb{T}^d)) = \Big(\sum_{j=n}^{\infty} a_j(I_d : F_d(w) \to L_2(\mathbb{T}^d))^2\Big)^{1/2}$$

- – Upper estimate by factorization of $I_d : F_d \to L_\infty(\mathbb{T}^d)$ through a diagonal operator $D : \ell_2 \to \ell_1$, and known results for $a_n(D)$

  – Lower estimate via absolutely 2-summing operators

- L_2-approximation can be "translated" into L_∞-approximation!

Application to isotropic Sobolev spaces

- The relation

$$\lim_{n\to\infty} n^{s/d} a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \mathrm{vol}(B_p^d)^{s/d}$$

implies

Theorem (CKS 2016, asymptotic constants - isotropic spaces)

Let $d \in \mathbb{N}$, $s > d/2$ and $0 < p < \infty$. Then

$$\lim_{n\to\infty} n^{s/d-1/2} a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)) = \sqrt{\frac{d}{2s-d}} \cdot vol(B_p^d)^{s/d}$$

- Shift in the exponent of n by $\frac{1}{2}$, additional correction factor $\sqrt{\frac{d}{2s-d}}$.
- The same holds for the target space $C(\mathbb{T}^d)$, and also for the Wiener algebra $A(\mathbb{T}^d)$.
- Similarly one can translate estimates of a_n for large n / small n.

Application to mixed Sobolev spaces

- The relation

$$\lim_{n\to\infty} \frac{n^s a_n(I_d : H^{s,p}_{mix}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))}{(\log n)^{s(d-1)}} = \left[\frac{2^d}{(d-1)!}\right]^s$$

implies the following

Theorem (CKS 2014, asymptotic constants - mixed spaces)

Let $d \in \mathbb{N}$, $s > 1/2$ *and* $0 < p < \infty$. *Then*

$$\lim_{n\to\infty} \frac{n^{s-1/2} a_n(I_d : H^{s,p}_{mix}(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d))}{(\log n)^{s(d-1)}} = \frac{1}{\sqrt{2s-1}} \left[\frac{2^d}{(d-1)!}\right]^s$$

- Again: shift in the exponent by $\frac{1}{2}$ and additional correction factor.

Open problems

Open problems for 'our' Hilbert spaces $F_d(w)$

- Approximation with respect to L_p-norms, $1 \leq p \neq 2 < \infty$?
- Preasymptotic estimates for L_∞-approximation ?

Open problems for other spaces

- Sharp constants for approximation numbers and preasymptotic estimates
  - non-periodic Sobolev spaces $H^s([0,1]^d)$
  - Sobolev spaces $W_p^s(\Omega)$ with $p \neq 2$ (non-Hilbert case)
  - Besov spaces $B_{p,q}^s(\Omega)$

Thank you for your attention!