A NEW APPROACH FOR THE CONVEX FEASIBILITY PROBLEM VIA MONOTROPIC PROGRAMMING

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Convex Feasibility Problem

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- Numerous problems in mathematics and physical sciences can be recast as Convex Feasibility Problem:

  $\{C_i\}_{i=1}^m$ nonempty closed convex subsets of H Hilbert

$$\boxed{\text{find } x \in \bigcap_{i=1}^m C_i} \qquad \text{(CFP)}$$

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- broad applicability in different disciplines:
  - *image and signal reconstruction* (computerized tomography) Combettes (1996)

Combettes, The Convex Feasibility Problem in Image Recovery, Advances in Imaging and Electron Physics 95, Academic Press, New York, 1996

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- widely studied from diverse frameworks:
  - iterative projection methods provided the intersection is nonempty, Bauschke-Borwein (1996)

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$$\boxed{\text{What if } \bigcap_{i=1}^m C_i = \emptyset?}$$

Translation to the product space

$\mathbf{H} = \underbrace{H \times \cdots \times H}_{m \text{ times}}$ Hilbert space endowed with the scalar product

$$\langle\langle \mathbf{x}, \mathbf{y} \rangle\rangle = \sum_{i=1}^{m} w_i \langle x_i, y_i \rangle, \quad \mathbf{x}, \mathbf{y} \in \mathbf{H}$$

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Let $\mathbf{C} = C_1 \times \cdots \times C_m = \{\mathbf{x} \in \mathbf{H} : x_i \in C_i\}$, cartesian product of the sets and $\mathbf{D} = \{(x, \cdots, x) \in \mathbf{H} : x \in H\}$, closed diagonal subspace of $\mathbf{H}$

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Thus $\mathbf{C} \cap \mathbf{D} = \{(x, \cdots, x) \in \mathbf{H} : x \in \bigcap_{i=1}^{m} C_i\}$

and in the product space the CFP can be reformulated as finding

$$\boxed{\text{find } \mathbf{x} \in \mathbf{C} \cap \mathbf{D}} \qquad \text{(CFP)}$$

Monotropic Programming Problem

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Extended monotropic programming problem:

$$\boxed{\inf_{(x_1,\cdots,x_m)\in S} f_1(x_1)+f_2(x_2)+\cdots+f_m(x_m)} \qquad \text{(P)}$$

where $f_i : X_i \to \mathbb{R}$ **proper convex function**, $i = 1, \cdots, m$
X_i **separately locally convex spaces**, $i = 1, \cdots, m$
$S \subseteq \prod_{i=1}^m X_i$ **linear closed subspace** such that $S \cap \prod_{i=1}^m \mathrm{dom} f_i \neq \emptyset$

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The dual problem is

$$\sup_{(x_1^*,\cdots,x_m^*)\in S^\perp} -f_1^*(x_1^*)-f_2^*(x_2^*)-\cdots-f_m^*(x_m^*) \qquad \text{(D)}$$

where $f^*(x^*) := \sup_{x\in X}\{\langle x^*,x\rangle - f(x)\}$ is the **Fenchel conjugate function** of f
$S^\perp = \{x^* : \langle x^*,x\rangle = 0, \forall x \in S\}$ is the **orthogonal subspace** of S.

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$$\inf_{(x_1,\cdots,x_m)\in S} f_1(x_1)+f_2(x_2)+\cdots+f_m(x_m)=v(P) \qquad \text{(P)}$$

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the situation $v(P) = v(D)$ is called *zero duality gap*

Important results in Monotropic Programming Problems

Rockafellar was the first to prove a zero duality gap result for the original class of monotropic programs when each space X_i is $\mathbb{R}$.

Rockafellar, Network Flows and Monotropic Optimization. Wiley, New York, **1984**

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Bertsekas generalized Rockafellar's result to *extended monotropic programs* in which the X_i's are finite-dimensional spaces, assuming:

- f_i **lower semicontinuous** in $\mathrm{dom} f_i$, for all $i = 1, \cdots, m$,
- $S^{\perp} + \prod_{i=1}^{m} \partial_{\varepsilon} f_i(x_i)$ **closed**, $\forall \varepsilon > 0$, $\forall (x_1, \cdots, x_m) \in \prod_{i=1}^{m} \mathrm{dom} f_i \bigcap S$

where $\partial_{\varepsilon} f_i$ denotes the ε-subdifferential of f_i:

$$\partial_{\varepsilon} f(x) := \begin{cases} \{v \in H \mid \langle v, y - x\rangle - \varepsilon \leq f(y) - f(x), \text{ for all } y \in H\} & \text{if } f(x) \in \mathbb{R} \\ \emptyset & \text{otherwise} \end{cases}$$

Bertsekas, Extended monotropic programming and duality. JOTA. 139, 209–225, **2008**

Important results in Monotropic Programming Problems

Boţ and Csetnek extended Bertsekas' result to the general case:

Zero Duality Gap Theorem (Bot et al., 2010)

X_i **separately locally convex spaces**, $i = 1, \cdots, m$

$f_i : X_i \to \overline{\mathbb{R}} := (-\infty, +\infty]$ **proper convex functions**, $i = 1, \cdots, m$

$S \subseteq \prod_{i=1}^m X_i$ **linear closed subspace** such that $\prod_{i=1}^m \operatorname{dom} f_i \cap S \neq \emptyset$

$g : \prod_{i=1}^m X_i \to \mathbb{R}$ defined by $g(x_1, \cdots, x_m) = \sum_{i=1}^m f_i(x_i)$,

$\operatorname{cl} f_i$ proper functions and $g(x) = \operatorname{cl} g(x)$ for all $x \in \operatorname{dom} \operatorname{cl} g \cap S$

$S^\perp + \prod_{i=1}^m \partial_\varepsilon f_i(x_i)$ **closed**, $\forall \varepsilon > 0$, $\forall (x_1, \cdots, x_m) \in \prod_{i=1}^m \operatorname{dom} f_i \bigcap S$

then $v(P) = v(D)$.

Boţ et al., On a zero duality gap result in extended monotropic programming. JOTA 147, 473–482, **2010**

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C_1, C_2 closed convex sets in $X = H$, Hilbert space, $C_1 \cap C_2$ possibly empty

$$\boxed{\text{find } x \in C_1 \cap C_2} \quad ? \tag{CFP}$$

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equivalent to the monotropic optimization problem

$$\boxed{\inf_{(x,y) \in S} d_{C_1}(x) + d_{C_2}(y)} \qquad \text{(P)}$$

where $S = \{(x,y) \in H^2 : x = y\}$

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$f_1 := d_{C_1}(\cdot)$ and $f_2 := d_{C_2}(\cdot)$ are **convex and continuous everywhere** monotropic optimization problem

$$\inf_{(x,y)\in S} d_{C_1}(x) + d_{C_2}(y) \qquad \text{(P)}$$

Then the dual problem of (P) is

$$\sup_{(v,w)\in S^\perp} -d^*_{C_1}(v) - d^*_{C_2}(w) \qquad \text{(D)}$$

where $S^\perp = \{(v,w) \in H^2 : v + w = 0\}$ and $d^*_C(v) = \sigma_C(v) + \iota_B(v)$

$$\sigma_C(v) := \sup_{y\in C}\langle v, y\rangle, \quad \text{support function of } C$$

$$\iota_B(x) := \begin{cases} 0 & \text{if } x \in B \\ +\infty & \text{if } x \notin B \end{cases} \quad \text{indicator function of the unit ball in } H$$

Zero duality gap

(P) and (D) satisfy the **zero duality gap** property

Consequence of Boţ and Csetnek's result:

- C_i closed and convex $\Rightarrow f_i = d_{C_i}$ convex and continuous.

  $\Rightarrow$ Thus functions f_i **satisfy the assumptions of Theorem**

- f_i real-valued $\Rightarrow \partial_\varepsilon f_i(x_i)$ $(x_i \in H)$ nonempty and weakly compact

  $\Rightarrow S^\perp + \prod_{i=1}^m \partial_\varepsilon f_i(x_i)$ is **weakly closed**

  (every weakly closed convex set is closed for the strong topology)

Strong duality and optimality conditions

To derive **strong duality** (existence of a dual solution)

and **first order optimality conditions** for primal-dual problems (P) and (D)

we use these classical primal-dual problems in Fenchel duality

$$\boxed{\inf_{(x,y)\in H^2} f(x,y) + g(x,y)} \qquad (P_0)$$

$$\boxed{\sup_{(v,w)\in H^2} -f^*(v,w) - g^*(-v,-w)} \qquad (D_0)$$

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Proposition (Strong duality)

$f,g : H^2 \to \overline{\mathbb{R}}$ **proper and lsc functions** such that $\boxed{0 \in \mathrm{core}(\mathrm{dom}\, g - \mathrm{dom} f)}$

Then $$\inf_{x\in H^2} f(x)+g(x) = -\min_{v\in H^2} f^*(v)+g^*(-v)$$

$$\mathrm{core}(C) := \{x \in C \,:\, cone(C-x) = H\}$$

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Theorem (First order optimality conditions)

$f, g : H^2 \to \overline{\mathbb{R}}$ **proper and lsc functions** such that $\operatorname{dom} g \cap \operatorname{dom} f \neq \emptyset$

Then the following are equivalent:

(i) (x_1, x_2) solves (P_0), and (v_1, v_2) solves (D_0)

(ii) $(v_1, v_2) \in \partial f(x_1, x_2)$ and $-(v_1, v_2) \in \partial g(x_1, x_2)$

Strong duality and optimality conditions

$$\inf_{(x,y)\in S} d_{C_1}(x) + d_{C_2}(y) \qquad \text{(P)}$$

$$\sup_{(v,w)\in S^\perp} -d^*_{C_1}(v) - d^*_{C_2}(w) \qquad \text{(D)}$$

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Lemma

Defining $f(x,y) := d_{C_1}(x) + \iota_S(x,y)$ and $g(x,y) := d_{C_2}(y)$:

(a) $\inf_{(x,y)\in S} d_{C_1}(x) + d_{C_2}(y) = \inf_{(x,y)\in H^2} f(x,y) + g(x,y)$

(b) $(z_1, z_2) \in S$ solves (P) if and only if (z_1, z_2) solves (P_0)

(c) $\sup_{(v,w)\in S^\perp} -d^*_{C_1}(v) - d^*_{C_2}(w) = \sup_{(v,w)\in H^2} -f^*(v,w) - g^*(-v,-w)$

(d) $(u,-u) \in H^2$ solves (D) if and only if $(0,u)$ solves (D_0).

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Proposition

Problems (P) and (D) satisfy **strong duality**:

the dual problem always has a solution.

$$\begin{cases} (x,y) \text{ primal solution to } (P) \\ (u,v) \text{ dual solution to } (D) \end{cases} \iff \begin{cases} (x,y) \in S \\ (u,v) \in S^\perp \\ u \in \partial d_{C_1}(x), v \in \partial d_{C_2}(y) \end{cases}$$

Separation of sets

C_1, C_2 are *separated* if there exist $v \in H$, $\|v\| = 1$, and $\delta \in \mathbb{R}$ such that

$$C_1 \subseteq H^{\leq}_{v,\delta} := \{x \in H : \langle v, x \rangle \leq \delta\}$$

$$C_2 \subseteq H^{\geq}_{v\delta} := \{y \in H : \langle v, y \rangle \geq \delta\}.$$

The separating hyperplane is $H = H_{v,\delta} := \{x \in H : \langle v, x \rangle = \delta\}$.

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The separating hyperplane is $H = H_{v,\delta} := \{x \in H : \langle v, x \rangle = \delta\}$. This separation is said to be:

- *proper* if C_1 and C_2 are not contained in H;
- *nice* if the hyperplane H is disjoint from C_1 or C_2;
- *strict* if the hyperplane H is disjoint from both C_1 and C_2;
- *strong* if there exist $\varepsilon > 0$ such that $C_1 + \varepsilon B$ is contained in one of the open half-spaces bounded by H and $C_2 + \varepsilon B$ is contained in the other;

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Standard Separation Theorem

C_1 and C_2 nonempty convex sets such that $\operatorname{int} C_1 \neq \emptyset$

$$\text{properly separated} \quad \Leftrightarrow \quad (\operatorname{int} C_1) \cap C_2 = \emptyset$$

Consistency of CFP and the optimal dual values

The dual problem (D):

$$
\begin{aligned}
\sup_{v\in H} -d^*_{C_1}(v) - d^*_{C_2}(-v) &= \sup_{v\in H} -[\sigma_{C_1}(v)+\iota_B(v)] - [\sigma_{C_2}(-v)+\iota_B(-v)] \\
&= \sup_{\|v\|\le 1} -\sigma_{C_1}(v) - \sigma_{C_2}(-v) \\
&= \max_{t\in[0,1]} \sup_{\|v\|=t} -\sigma_{C_1}(v) - \sigma_{C_2}(-v) \\
&= -\min_{t\in[0,1]} \underbrace{\inf_{\|v\|\le t} \sigma_{C_1}(v) + \sigma_{C_2}(-v)}_{\Phi(t)} \\
&= -\min_{t\in[0,1]} \{t\Phi(1)\} \\
&= \begin{cases} -\Phi(1)(>0) & \text{if } \Phi(1)<0, \\ 0 & \text{if } \Phi(1)=0. \end{cases}
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By studying the values of $\Phi(1)$, we can obtain information about $C_1 \cap C_2$.

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The key point is the relation between Φ and the **infimal convolution of the support functions**

Consistency of CFP and the optimal dual values

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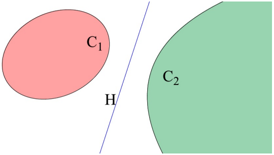
\end{aligned}
$$

Theorem

1. If $\Phi(1) < 0$, then $\boxed{C_1 \cap C_2 = \emptyset}$.
   C_1 and C_2 are strongly separated
2. If $\Phi(1) = 0$:
   - 2.1 If $(\sigma_{C_1}\Box\sigma_{C_2})(\cdot)$ is not lower-semicontinuous at 0 then $\boxed{C_1 \cap C_2 = \emptyset}$.
     C_1 and C_2 are separated but not strongly
   - 2.2 If $(\sigma_{C_1}\Box\sigma_{C_2})(\cdot)$ is lower-semicontinuous at 0 then $\boxed{C_1 \cap C_2 \neq \emptyset}$.

Separation of sets

- $\Phi(1) < 0$:



Strong separation

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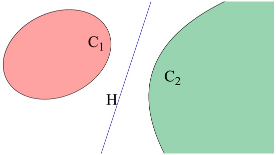
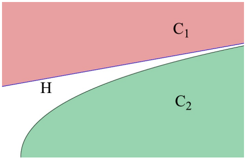
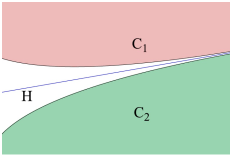


Strong separation

- $\Phi(1) = 0$:
    - $(\sigma_{C_1} \Box \sigma_{C_2})(0)$ not lsc



Nice separation



Strict separation

Separation of sets

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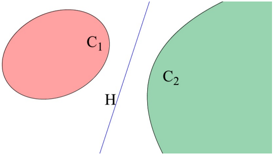
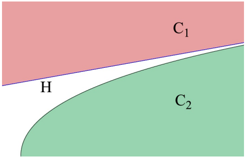
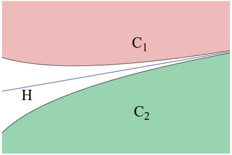


Strong separation

- $\Phi(1) = 0$:
  - $(\sigma_{C_1} \square \sigma_{C_2})(0)$ not lsc

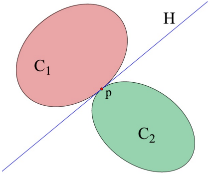


Nice separation

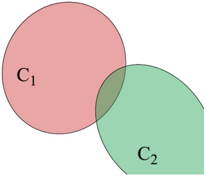


Strict separation

  - $(\sigma_{C_1} \square \sigma_{C_2})(0)$ lsc



No nice separation



No separation

Characterization of $C_2 - C_1$

The value of $\Phi(1)$ in the dual problem (D) characterizes also the Minkowski difference set

$$C_2 - C_1 := \{x - y \in H : x \in C_2, y \in C_1\}.$$

Theorem

(i) $\Phi(1) < 0$ if and only if $0 \notin \overline{C_2 - C_1}$ the closure of $C_2 - C_1$.

(ii) $\Phi(1) = 0$ and $(\sigma_{C_1} \Box \sigma_{C_2})(0)$ not lsc if and only if $0 \in \text{Bd}\,(C_2 - C_1)$, the boundary of $C_2 - C_1$.

(iii) $\Phi(1) = 0$ and $(\sigma_{C_1} \Box \sigma_{C_2})(0)$ lsc if and only if $0 \in \text{int}\,(C_2 - C_1)$, the interior of $C_2 - C_1$.

Lower semicontinuity of infimal convolution

Lower semicontinuity of infimal convolution

Geometric Condition for the lower semicontinuity of the infimal convolution:

Corollary

If $(\sigma_{C_1}\Box\sigma_{C_2})(0) > -\infty$ then $(\sigma_{C_1}\Box\sigma_{C_2})$ is proper, and the following statements are equivalent and satisfied:

(i) $C_1 \cap C_2 \neq \emptyset$,

(ii) $(\sigma_{C_1}\Box\sigma_{C_2})$ is lsc at 0,

(iii) $\{0\} \times \mathbb{R} \cap \operatorname{epi}(\sigma_{C_1}\Box\sigma_{C_2}) = \{0\} \times \mathbb{R}_+$

Consequently, if $\operatorname{epi}\sigma_{C_1} + \operatorname{epi}\sigma_{C_2}$ is closed, then $C_1 \cap C_2 \neq \emptyset$.

Consistency of CFP and dual solutions

We have seen that

- (D) always has a solution.
- If $v(D)$ is positive then CFP has no solution (strong separation).
- If $v(D) = 0$, CFP may or may not have solution.

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The following result gives us information about consistency of the CFP when $v(D) = 0$.

Corollary

Assume that $v(D) = 0$

(a) $v = 0$ unique solution to the dual problem $(D) \Leftrightarrow C_1 \cap C_2 \neq \emptyset$.

(b) The dual problem (D) has multiple solutions $\Leftrightarrow C_1 \cap C_2 = \emptyset$.
In this situation, every nonzero dual solution induces a separation of the sets.

Consistency of CFP and primal solutions

Assume that problem (P) has a solution

we study the set $C_1 \cap C_2$ in terms of the location of the solutions.

For that the subdifferential of the distance function d_C is given by

$$\partial d_C(x) = \begin{cases} 0 & \text{if } x \in \operatorname{int} C, \\ N_C(x) \cap B & \text{if } x \in \operatorname{Bd} C, \\ \dfrac{x - P_C(x)}{\|x - P_C(x)\|} & \text{if } x \notin C, \end{cases}$$

where $P_C(x)$ is the metric projection of x onto C.

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Theorem

(a) $\inf_{x \in H} \|P_{C_1}(x) - P_{C_2}(x)\| = d(C_1, C_2)$.

(b) The set of solutions of (P) is the set

$$\operatorname{sol}(P) = \{x \in H : d(C_1, C_2) = \|P_{C_1}(x) - P_{C_2}(x)\|\}.$$

Corollary

Assume that $d(C_1, C_2) = 0$. In this situation, the following statements are equivalent.

(i) (P) has no solutions.

(ii) $0 \in \text{cl}(C_1 - C_2) \setminus (C_1 - C_2)$.

(iii) $\sigma_{C_1} \Box \sigma_{C_1}$ is not lsc at 0.

(iv) $C_1 \cap C_2 = \emptyset$

(v) $\{0\} \times \mathbb{R}_{--} \cap \text{ epi } (\sigma_{C_1} \Box \sigma_{C_2}) \neq \emptyset$.

THANK YOU! - ¡MUCHAS GRACIAS!