Some open problems in Banach Space Theory

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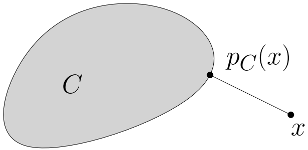

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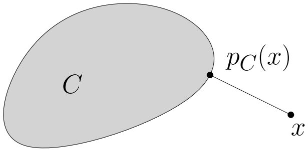

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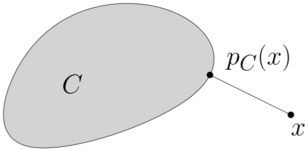

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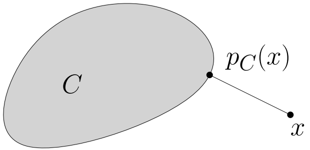

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[Bunt'1934, Motzkin'1935, et alt.] X Euclidean plane, then C Chebyshev $\Leftrightarrow$ closed convex (and P_C is continuous).
Easy: X (R) and reflexive $\Leftrightarrow$ every closed convex set $C \subset X$ is Chebyshev.

Convexity of Chebyshev sets

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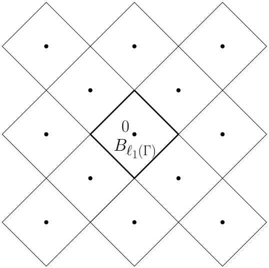

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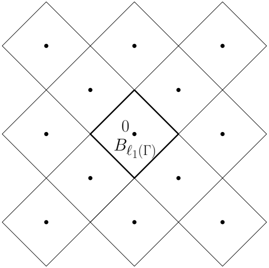

Remark The centers form a (nonconvex) Chebyshev set.

Convexity of Chebyshev sets

Problem

C Chebyshev in $\ell_2 \Rightarrow C$ convex?

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[V. Klee'1961] $C \subset \ell_2$ w-closed Chebyshev, then C convex (true for X uniformly convex or uniformly smooth).

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$\exists S$ not singleton $S \subset \ell_2$ st every $x \in \ell_2$ has farthest point in S?

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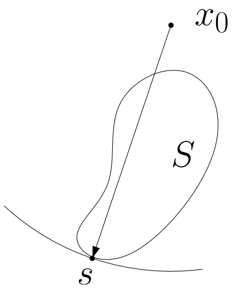

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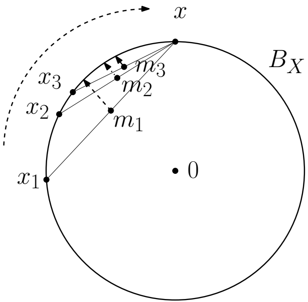

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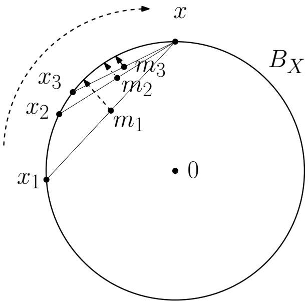

Loc. unif. rotunf (LUR)

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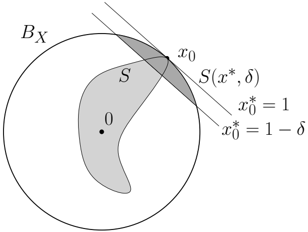

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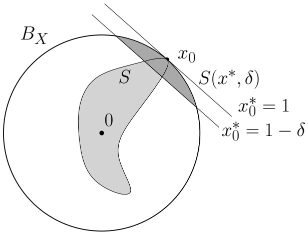

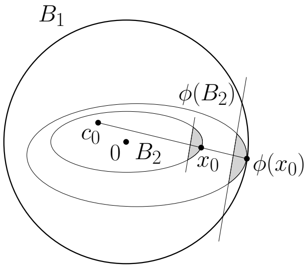

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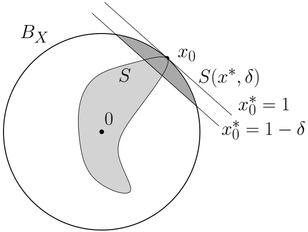

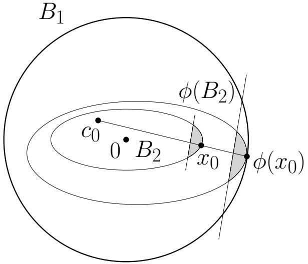

We gave (with P. and V. Zizler) an alternative, much easier, proof in 2011.

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Theorem (Vlasov'1970)

X such that X^* rotund. C Chebyshev, p_C continuous. Then C convex.

Tilings

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[Fonf, Lindenstrauss] $\exists$ reflexive X tiled by shifts of a single closed convex S with nonempty interior?

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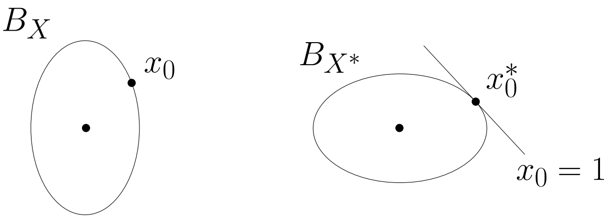

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Theorem (Guirao–M–Zizler'2012)

X nonreflexive, $X \subset$WCG, then $\exists\ |||\cdot|||$ LUR, Gâteaux, $|||\cdot|||^*$ not rotund. If moreover, X Asplund, then $|||\cdot|||$ even Fréchet, and $w = w^*$ on dual sphere.

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Problem

[Troyanski] X (uncountable) unconditional basis and Gâteaux norm. Has X^* dual rotund renorming?

M-bases

X Banach. $\{x_\gamma, x^*_\gamma\}_{\gamma\in\Gamma}$ biorthogonal, $\{x_\gamma\}$ linearly dense, $\{x^*_\gamma\}$ w^*-linearly dense is called Markushevich basis (M-basis).

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Problem

Can the constant be diminished to $2+\varepsilon$, for all $\varepsilon > 0$?

Auerbach bases

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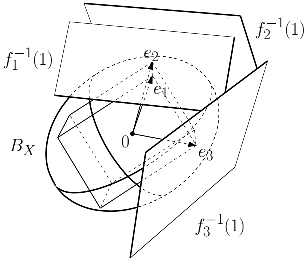

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[Pełczyński] X separable. Does X has an Auerbach basis?

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Theorem (Day)

Every infinite-dimensional Banach has an infinite-dimensional subspace with Auerbach basis.

Norming subspaces

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2. If $x^{**} \in X^{**} \setminus X$ then $\ker x^{**} \subset X^*$ is norming.
3. If $\{e_n; e_n^*\}$ is a Schauder basis, then $\overline{\mathrm{span}}\{e_n^*\}$ is norming.

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A space T is angelic (Fremlin) if all RNK$\subset T$ are RK and $\overline{RNK} =$ sequential closure (RNK).

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[Davis–Lindenstrauss'72] If X^{**}/X infinite-dimensional, then $\exists$ w^*-dense non-norming subspace.

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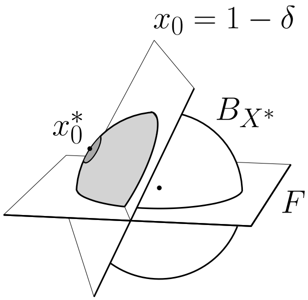

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Problem [Godefroy–Kalton]

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Every non-reflexive space has a proper closed norming subspace (the kernel of $x^{**} \in (X^{**} \setminus X)$).

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Every separable X has a norming M-basis.

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Problem [Godefroy]

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Problem [Fabian]

Characterize K compact st $C(K)$ hereditary WCG.

Fréchet norm then X Asplund.

Asplund spaces

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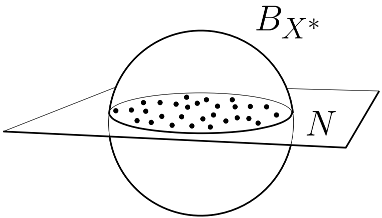

Asplund spaces

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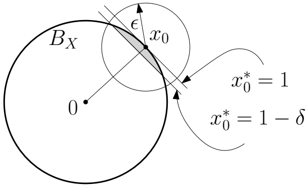

LUR renorming

Problem

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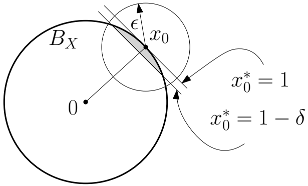

Problem [Hájek–Talponen' 2013]

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SSD norms

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Problem [Godefroy]

In ZFC, $\exists$ Asplund with no SSD norm?

SSD norms

Theorem (Godefroy–M–Zizler'94)

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SSD norms

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Theorem (James'1957)

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Theorem (Acosta–M.'2007)

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[Bandyopadhyay–Godefroy'2006] $\exists$ nonreflexive X st $\mathrm{NA}(X^*)$ is vector subspace of X^{**}?

Theorem (Rmoutil'2015, question of Godefroy)

$\exists$ X Banach, $NA(X)$ does not contain any 2-dimensional subspace.

Norm attaining operators

Theorem (Lindenstrauss'1963)

$\{T : X \to Y : \; T^{**}$ *attains the norm*$\}$ *dense in* $L(X, Y)$.

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Problem

[Ostrovski] Does there exists X infinite-dimensional separable such that every $T : X \to X$ bounded attains its norm?

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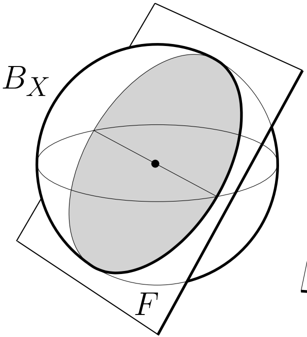
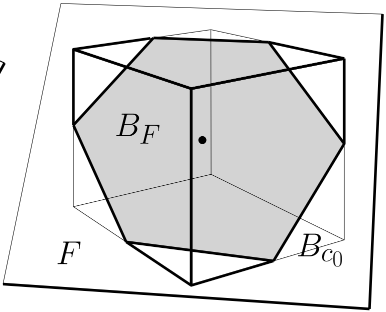
Problem

What if $n > 2$?

Polyhedral spaces





non-polyhedral Polyhedral

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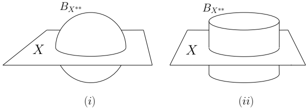

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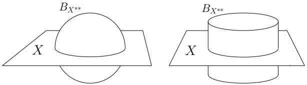

(i) (ii)

Theorem (Morris'83)

X separable $c_0 \subset X$, then $\exists$ (R) $|||\cdot|||$ st all $x \in S_X$ are unpreserved.

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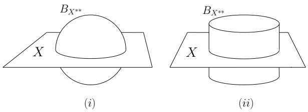

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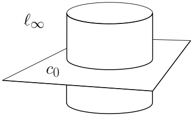

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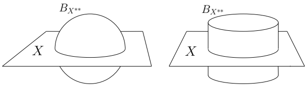

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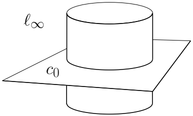

Theorem (Guirao–M–Zizler'2013)

X separable polyhedral, then $\exists$ C^∞-smooth (R) norm $|||\cdot|||$ all $x \in S_X$ unpreserved.

Polyhedral spaces

Theorem (Fonf'1980-81, Hájek)

X separable polyhedral $\Leftrightarrow \exists \, |||\cdot|||$ depending locally of finitely many coordinates.

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Problem

X nonseparable. X polyhedral $\Leftrightarrow \exists \; |||\cdot|||$ depending locally on finitely many coordinates?

Problem

X separable with a bump that depends locally on finitely many coordinates. Is X polyhedral?

Support sets

$C \subset X$ convex, closed, is a support set whenever $\forall x_0 \in C$, x_0 is proper support point, i.e., $\exists f \in X^*$
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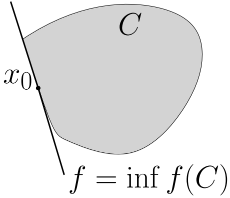

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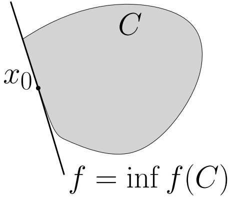

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If X separable, then there are no (bounded) support sets.

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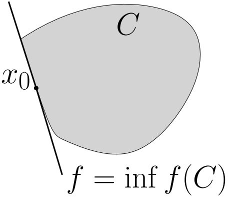

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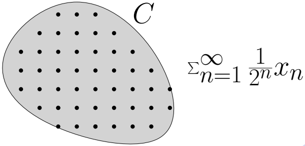

Support sets

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Problem

[Rolewicz] X nonseparable Banach. Do there exist support sets?

Support sets

Theorem (M.'1985)

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Theorem (Granero, Jiménez, Moreno' 98)

K compact, $\exists$ regular measure nonseparable. Then $C(K)$ has a support set.

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Problem

[Todorcevic] X with density $> \aleph_1$ has a support set?