A class of matrices with operator entries

XIV Encuentro de la Red de Análisis Funcional y Aplicaciones

Ismael García Bayona

an ongoing work with
Óscar Blasco de la Cruz

Universitat de València

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1. Introduction
2. The operator setting
3. Some results

Introduction - Definitions

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Note the following important fact:

Let $f \sim \sum a_k e^{ikt}, g \sim \sum b_k e^{ikt}$, and define:

$$A_f = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n & \cdots \\ a_{-1} & a_0 & a_1 & \cdots & a_{n-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ a_{-n} & a_{-n+1} & a_{-n+2} & \cdots & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}, A_g = \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_n & \cdots \\ b_{-1} & b_0 & b_1 & \cdots & b_{n-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ b_{-n} & b_{-n+1} & b_{-n+2} & \cdots & b_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

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Introduction - Previous results

- (Toeplitz) If A is a Toeplitz matrix given by the sequence $(a_k)_{k\in\mathbb{Z}}$, then

$$A \in \mathcal{B}(\ell^2) \Leftrightarrow f_A \in L^\infty(\mathbb{T}) \qquad \text{where } \hat{f}_A(k) = a_k,\ \forall k \in \mathbb{Z}$$

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The space $B(\ell^2(H))$

We will work with the following space:

Given a matrix $A = (T_{kj})$ with entries $T_{kj} \in \mathcal{B}(H)$ and $x \in c_{00}(H)$ we write $A(x)$ for the sequence $(\sum_{j=1}^{\infty} T_{k,j}(x_j))_k$.

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We say that $A \in B(\ell^2(H))$ if the map $x \to A(x)$ extends to a bounded linear operator in $\ell^2(H)$, that is

$$\Big(\sum_{k=1}^{\infty} \|\sum_{j=1}^{\infty} T_{kj}(x_j)\|^2\Big)^{1/2} \leq C\Big(\sum_{j=1}^{\infty} \|x_j\|^2\Big)^{1/2}.$$

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- Recall that $\ell^2(H) = \{\mathbf{x} = (x_1, x_2, \cdots) \,/\, (\sum_{i=1}^{\infty} \|x_i\|^2)^{\frac{1}{2}} < \infty\}$

The space $B(\ell^2(H))$

Example

Given $\mathbf{x}^* = (x_j^*) \in \ell^2(H)$ and $\mathbf{y} = (y_j) \in \ell^2(H)$, we define $(\mathbf{x}^* \otimes \mathbf{y})$ as

$$(\mathbf{x}^* \otimes \mathbf{y})(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x}^* \rangle \mathbf{y}, \quad \mathbf{y} \in \ell^2(H)$$

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Proposition

Let $A = (a_{k,j}) \in \mathcal{B}(\ell^2)$ and $T \in \mathcal{B}(H)$. Then

$$\mathbf{A} = (a_{k,j}T) \in B(\ell^2(H)) \quad \text{and } \|\mathbf{A}\|_{B(\ell^2(H))} = \|A\|_{\mathcal{B}(\ell^2)}\|T\|_{\mathcal{B}(H)}$$

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Recall that $\|D_l\|_{B(\ell^2(H))} = \|D_l\|_{\mathcal{M}(\ell^2(H))} = \sup_{k-j=l} \|T_{k,j}\|$.

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We define $L^1(\ell^2(H))$ as the closure of $\mathcal{P}(\ell^2(H))$ in $\mathcal{M}(\ell^2(H))$.

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Schur product for matrices with operator entries

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$$A * B := (T_{k,j} \circ S_{k,j})_{k,j}.$$

Schur multipliers

Multipliers

Given a matrix $A = (T_{k,j})$. We say that A is a left Schur multiplier, to be denoted by $A \in \mathcal{M}_l(\ell^2(H))$ whenever $A * B \in B(\ell^2(H))$ for any $B \in B(\ell^2(H))$. We shall write

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Definition

Given $A = (T_{kj})_{k,j}$,

$$f_A(t) := M_t * A, \quad t \in [-\pi, \pi)$$

If $A \in \mathcal{P}(\ell^2(H))$ one has that $f_A(t) \in \mathcal{P}(\mathbb{T}, B(\ell^2(H))$.

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- Let $A \in \mathcal{M}_l(\ell^2(H))$ (respect. $A \in \mathcal{M}_r(\ell^2(H))$) and $\mathbf{x}^* \in \ell^2(H)$ and $\mathbf{y} \in \ell^2(H)$. Then the map

$$f_A(t) * (\mathbf{x}^* \otimes \mathbf{y}) = \left(e^{i(j-k)t} x_j^* \otimes T_{k,j}(y_k) \right)_{k,j} \in C(\mathbb{T}, B(\ell^2(H))$$

(respect. $(\mathbf{x}^* \otimes \mathbf{y}) * f_A(t) = \left(e^{i(j-k)t} (x_j^* \circ T_{k,j}) \otimes y_k \right) \quad)$

The function f_A

Example: A matrix A such that $t \to f_A(t)$ is not continuous

Consider a matrix A such that $A(k, 2k+1) = Id$ and $A(k,s) = 0\ \forall s \neq 2k+1$. It is clear that $A \in B(\ell^2(H))$. However, taking $x = (x_i)_i \in \ell^2(H)$, we observe that

$$(f_A(t) - f_A(0))(x) = \left((e^{i(k-1)t} - 1) \cdot x_{2k-1}\right)_k$$

And taking supremums, we have

$$\begin{aligned}\|(f_A(t) - f_A(0))\| &= \sup_{\sum_i \|x_i\|^2 \leq 1} \left(\sum_k |e^{i(k-1)t} - 1|^2 \|x_{2k-1}\|^2\right)^{1/2} \\ &= \sup_{k \in \mathbb{N}} |e^{ikt} - 1| \geq \sqrt{2}.\end{aligned}$$

Therefore, $t \to f_A(t)$ is not a continuous function.

The function f_A

Proof of $\|f_A(t)\|_{B(\ell^2(H))} = \|A\|_{B(\ell^2(H))}, \quad \forall t \in [-\pi, \pi)$.

$$\|f_A(t)\|_{B(\ell^2(H))} = \|A * (e^{i(j-k)t} \cdot Id)_{k,j}\|_{B(\ell^2(H))} \leq$$

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□

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3) If $\{f_n\} \subseteq L^1(\mathbb{T})$ is a summability kernel, and we define the matrix $\mathbf{M}_n := (\hat{f}_n(j-k) \cdot Id)_{k,j}$, then $\|M_n * A - A\|_{B(\ell^2(H))} \xrightarrow[n\to\infty]{} 0$

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Remark

Matrices of the type $(\mathbf{x}^* \otimes \mathbf{y})$ are in $C(\ell^2(H))$. $\mathbf{x}^* \in \ell^2(H), y \in \ell^2(H)$

The space $C(\ell^2(H))$

Proof: $C(\ell^2(H)) \Rightarrow \|M_n * A - A\|_{B(\ell^2(H))} \to 0.$

Let $E > 0$, and $\varepsilon > 0$ such that $\varepsilon < \frac{3E}{2+M}$, where $\|f_n\|_{L^1} < M \ \ \forall n \in \mathbb{N}$.

Select $P = (S_{k,j})_{k,j} = \sum_{l=-N}^{N} D_l \in \mathcal{P}(\ell^2(H))$ such that $\|A - P\|_{B(\ell^2(X))} < \varepsilon/3$. Then,

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$$\|M_n * P - P\|_{B(\ell^2(H))} = \| \sum_{l=-N}^{N} (\hat{f}_n(l) - 1) D_l \|_{B(\ell^2(H))}$$

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Proof: $C(\ell^2(H)) \Rightarrow \|M_n * A - A\|_{B(\ell^2(H))} \to 0.$

Let $E > 0$, and $\varepsilon > 0$ such that $\varepsilon < \frac{3E}{2+M}$, where $\|f_n\|_{L^1} < M \ \ \forall n \in \mathbb{N}$.

Select $P = (S_{k,j})_{k,j} = \sum_{l=-N}^{N} D_l \in \mathcal{P}(\ell^2(H))$ such that $\|A - P\|_{B(\ell^2(X))} < \varepsilon/3$. Then,

$$\|M_n * P - P\|_{B(\ell^2(H))} = \|\sum_{l=-N}^{N} (\hat{f}_n(l) - 1)D_l\|_{B(\ell^2(H))} \leq \sup_{k,j} \|S_{k,j}\| \cdot (2N+1) \cdot \max_{|l| \leq N} |\hat{f}_n(l) - 1|$$

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Since $\{f_n\}$ is a summability Kernel, we know that $\lim_n f_n * g = g \ \forall g \in L^1(\mathbb{T})$, therefore $\lim_n \hat{f}_n(l) = 1 \ \forall l \in \mathbb{Z}$.

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So, we can choose $n_0 \in \mathbb{N}$ such that $|\hat{f}_n(l) - 1| < \frac{\varepsilon}{3(2N+1)\sup_{k,j} \|S_{k,j}\|} \ \forall n \geq n_0$ and $\forall |l| \leq N$. Hence,

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$$\|M_n * A - A\|_{B(\ell^2(H))} \leq$$

$$\leq \|M_n * (A - P)\|_{B(\ell^2(H))} + \|M_n * P - P\|_{B(\ell^2(H))} + \|P - A\|_{B(\ell^2(H))} \leq$$

$$\leq \|M_n\|_{\mathcal{M}(\ell^2(H))} \cdot \|A - P\|_{B(\ell^2(X))} + \varepsilon/3 + \varepsilon/3 =$$

$$\leq \|f_n\|_{L^1} \cdot \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon((2+M)/3) < E$$

The space $C(\ell^2(H))$

Proof: $\|\sigma_n(A) - A\| \to 0 \quad \Leftrightarrow \quad f_A(t) \in C(\mathbb{T}, B(\ell^2(H)))$.

Let us assume 2).

The space $C(\ell^2(H))$

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□

The space $C(\ell^2(H))$

$$\mathcal{M}(\ell^2(H)) = (B(\ell^2(H)), B(\ell^2(H)))$$

Proposition

$$A \in \mathcal{M}(\ell^2(H)) \Leftrightarrow A \in (C(\ell^2(H)), C(\ell^2(H))$$

Proof.

$\Rightarrow$) Let us assume that $A \in \mathcal{M}(\ell^2(H))$. If $B \in \mathcal{P}(\ell^2(H))$, it is obvious that $A * B$ is also in $\mathcal{P}(\ell^2(H))$, which in particular implies $A * B \in C(\ell^2(H))$. That is, $A \in (\mathcal{P}(\ell^2(H)), C(\ell^2(H))$. For the general case, use approximation.

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$\Leftarrow$) Suppose now that $A \in (C(\ell^2(H)), C(\ell^2(H))$, and let $B \in B(\ell^2(H))$. Note that $\sigma_n(B) \in \mathcal{P}(\ell^2(H)) \subset C(\ell^2(H))$. Therefore, by hypothesis, we have that for all $n \in \mathbb{N}$,

□

The space $C(\ell^2(H))$

Proof.

$$\|\sigma_n(A * B)\|_{B(\ell^2(H))} = \|A * \sigma_n(B)\|_{B(\ell^2(H))} \leq \|A\|_{(C,C)} \cdot \|\sigma_n(B)\|_{B(\ell^2(H))} =$$

$$= \|A\|_{(C,C)} \cdot \|(\hat{K}_n(j-k) \cdot Id)_{k,j} * B\|_{B(\ell^2(H))} \leq \|A\|_{(C,C)}\|B\|_{B(\ell^2(H))}$$

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This, in particular, means that for all $x \in \ell^2(H)$ and $x^* \in \ell^2(H)$ unitary,

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and taking supremums, we get

$$\|A * B\|_{B(\ell^2(H))} \leq \|A\|_{(C,C)} \cdot \|B\|_{B(\ell^2(H))},$$

so $A \in \mathcal{M}_l(\ell^2(H))$. □

The space $C(\ell^2(H))$

Lemma

Let $f \in C(\mathbb{T}, \mathcal{B}(H))$, and consider $A_f = (T_{k,j})_{k,j}$ with $T_{k,j} := \hat{f}(j-k)$. Then, $A_f \in B(\ell^2(H))$, with

$$\|A_f\|_{B(\ell^2(H))} = \|f\|_\infty$$

Theorem

Let A be an infinite Toeplitz matrix whose entries are in $\mathcal{B}(H)$, $A = (T_{j-k})_{k,j}$. Then,

$$A \in C(\ell^2(H)) \Leftrightarrow \exists f_A \in C(\mathbb{T}, \mathcal{B}(H)) \text{ such that } \hat{f}(l) = T_l.$$

Furthermore, $\|f_A\|_\infty = \|A\|_{B(\ell^2(H))}$.

Adjoint measure

- Given $\mu \in \mathcal{M}(\mathbb{T}, \mathcal{L}(E, F)) = \mathcal{L}(C(\mathbb{T}), \mathcal{L}(E, F))$, mapping $\varphi \in C(\mathbb{T})$ to $T_\mu(\varphi)$, we can define its **adjoint measure**, $\mu^* : \mathcal{M}(\mathbb{T}, \mathcal{L}(F^*, E^*)) = \mathcal{L}(C(\mathbb{T}), \mathcal{L}(F^*, E^*))$, as the measure mapping $\varphi \in C(\mathbb{T})$ to $(T_\mu(\varphi))^*$.

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Lemma

The fact that $\Psi : C(\mathbb{T}, X) \to Y$ is continuous is equivalent to the associate measure $T_{\mu_\Psi} : C(\mathbb{T}) \to \mathcal{L}(X, Y)$ having its adjoint measure in $M_{SOT}(\mathbb{T}, \mathcal{L}(Y^*, X^*))$, that is $\sup_{\|y^*\|=1} |(T^*_{\mu_\Psi})_{y^*}| < \infty$. Also,

$$\|\Psi\| = \sup_{\|y^*\|=1|} |(T^*_{\mu_\Psi})_{y^*}| < \infty.$$

A description of $\mathcal{M}_\tau(\ell^2(H))$

$$\mathcal{M}_\tau(\ell^2(H)) := \big(B(\ell^2(H)) \cap \tau, B(\ell^2(H)) \cap \tau\big)$$

Theorem

Let A be an infinite Toeplitz matrix whose entries are elements in $\mathcal{B}(H)$, and we denote by $f_A \sim \sum_{k\in\mathbb{Z}} T_k e^{ikt}$ the distribution associated to A. Then:

$$A \in \mathcal{M}_\tau(\ell^2(H)) \quad \Leftrightarrow \quad (f_A)^* \in M_{SOT}\,(\mathbb{T}, \mathcal{B}(H))$$

A description of $\mathcal{M}_\tau(\ell^2(H))$

Proof. $(f_A)^* \in M_{SOT}(\mathbb{T}, H) \quad \Rightarrow \quad A \in \mathcal{M}_\tau(\ell^2(H))$

$$\left.\begin{aligned} (C(\ell^2(H), C(\ell^2(H)) = M(\ell^2(H)) = (B(\ell^2(H)), B(\ell^2(H))) \\ C(\ell^2(H)) \cap \mathcal{T} = C(\mathbb{T}, \mathcal{B}(H)) \end{aligned}\right\} \Rightarrow \mathcal{M}_\tau(\ell^2(H)) = (C(\mathbb{T}, \mathcal{B}(H)), C(\mathbb{T}, \mathcal{B}(H)))$$

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Let $T : C(\mathbb{T}) \to \mathcal{B}(H)$ be the operator whose associated adjoint measure is f_A^*. Since $f_A^* \in M_{SOT}$, previous lemma gives that the associated operator $\Psi : C(\mathbb{T}, H) \to H$ (defined as follows) is continuous.

$$\Psi\left(\sum_k x_k e^{ikt}\right) = \sum_k T(e^{ikt})(x_k), \quad \text{where } T(e^{ikt}) = \widehat{f_A}(k),$$

where $\|\Psi\| = \|(f_A)^*\|_{M_{SOT}(\mathbb{T},\mathcal{B}(H))}$. Taking $f(t) = \sum_k \hat{f}(k)e^{ikt} \in C(\mathbb{T}, \mathcal{B}(H))$ we have,

A description of $\mathcal{M}_\tau(\ell^2(H))$

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$$\sup_t \|\sum_k \widehat{f_A}(k) \circ \hat{f}(k)e^{ikt}\|_{\mathcal{B}(H)} = \sup_{\substack{t \\ \|x\|=1}} \|\sum_k \widehat{f_A}(k) \circ \hat{f}(k)(x)e^{ikt}\|_H = \sup_{\substack{t \\ \|x\|=1}} \|\Psi(\sum_k \hat{f}(k)xe^{ikt}\varphi_k)\| \leq$$

$$\leq \sup_{\substack{t \\ \|x\|=1}} \|\Psi\| \sup_s \|\sum_k \hat{f}(k)xe^{ik(t+s)}\| \leq$$

$$\leq \sup_{\substack{t \\ \|x\|=1}} \|\Psi\| \cdot \|x\| \cdot \sup_s \|\sum_k \hat{f}(k)e^{ik(t+s)}\| =$$

$$= \|\Psi\| \cdot \|f\|_\infty = \|(f_A)^*\|_{M_{SOT}(\mathbb{T}, \mathcal{B}(X))} \cdot \|f\|_\infty$$

A description of $\mathcal{M}_\tau(\ell^2(H))$

Proof. $A \in \mathcal{M}_\tau(\ell^2(H)) \cap \mathcal{T} \quad \Rightarrow \quad (f_A)^* \in M_{SOT}(\mathbb{T}, H)$

Let $B \in B(\ell^2(H)) \cap \mathcal{T}$. We know that $A * B \in B(\ell^2(H))$. Which is equivalent to:

$$\sup_{\substack{\|y\|=1 \\ \|x\|=1 \\ N,M}} \left| \int P^{[N,M]} f_A(s) \left(\int P^{[N,M]} f_B(t-s)(x(-t)) \otimes y(t)\, dt \right) ds \right| \leq C. \tag{1}$$

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We select $x(t) = xF_n(t)$, $y(t) = yF_n(t)$, where $F_n(t) = \frac{1}{\sqrt{2n+1}} D_n(t) e^{2\pi i(n+1)t}$, and $D_n(t)$ is the Dirichlet kernel.

Recall that

- $D_n(t) = \sum_{j=-n}^{n} e^{2\pi ijt} = \frac{\sin \pi(2n+1)t}{\sin \pi t}$
- $K_n(t) = \sum_{j=-n}^{n} (1 - \frac{|j|}{n+1}) e^{2\pi ijt} = \frac{1}{n+1} \left(\frac{\sin(n+1)\pi t}{\sin \pi t} \right)^2$.

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And we are able to rewrite (1) as:

$$\left| \int P^{[N,M]} f_A(s) \left(\int K_l(t) P^{[N,M]} f_B(t-s)(x) \otimes y\, dt \right) ds \right| = \cdots =$$

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And we are able to rewrite (1) as:

$$\left| \int P^{[N,M]} f_A(s) \left(\int K_l(t) P^{[N,M]} f_B(t-s)(x) \otimes y\, dt \right) ds \right| = \cdots =$$

$$= \left| \int P^{[N,M]} f_B(t) \left[x \otimes \left(K_l * (P^{[N,M]} f_A)_y^* \right)(-t) \right] dt \right| = \left| \left\langle f_B, x \otimes \left(K_l * (P^{[N,M]} f_A)_y^* \right)(-t) \right\rangle \right|.$$

A description of $\mathcal{M}_{\mathcal{T}}(\ell^2(H))$

Proof. $A \in \mathcal{M}(\ell^2(H)) \cap \mathcal{T} \quad \Rightarrow \quad (f_A)^* \in M_{SOT}(\mathbb{T}, H)$

So we have:

$$\sup_{\substack{\|y\|=1 \\ \|x\|=1 \\ N,M}} \left| \left\langle f_B, x \otimes \left(K_l * (P^{[N,M]} f_A)_y^* \right)(-t) \right\rangle \right| \leq C \qquad \forall f_B \in V^{\infty}(\mathbb{T}, \mathcal{B}(H)) \tag{2}$$

A description of $\mathcal{M}_\tau(\ell^2(H))$

Proof. $A \in \mathcal{M}(\ell^2(H)) \cap \mathcal{T} \quad \Rightarrow \quad (f_A)^* \in M_{SOT}(\mathbb{T}, H)$

So we have:

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$$\Rightarrow \sup_{\substack{\|y\|=1\\ \|x\|=1\\ N,M}} \|x \otimes \left(K_l * (P^{[N,M]} f_A)_y^*\right)(-t)\|_{L^1(\mathbb{T}, H \hat{\otimes} H)} \leq C \quad \Rightarrow \sup_{\substack{\|y\|=1\\ N,M}} \|\sigma_l\left(P^{[N,M]} f_A\right)_y^*)(-t)\|_{L^1(\mathbb{T}, H)} \leq C.$$

A description of $\mathcal{M}_\tau(\ell^2(H))$

Proof. $A \in \mathcal{M}(\ell^2(H)) \cap \mathcal{T} \quad \Rightarrow \quad (f_A)^* \in M_{SOT}(\mathbb{T}, H)$

So we have:

$$\sup_{\substack{\|y\|=1\\ \|x\|=1\\ N,M}} \left| \left\langle f_B, x \otimes \left(K_I * (P^{[N,M]} f_A)^*_y\right)(-t) \right\rangle \right| \leq C \qquad \forall f_B \in V^\infty(\mathbb{T}, \mathcal{B}(H)) \tag{2}$$

$$\Rightarrow \sup_{\substack{\|y\|=1\\ \|x\|=1\\ N,M}} \|x \otimes \left(K_I * (P^{[N,M]} f_A)^*_y\right)(-t)\|_{L^1(\mathbb{T}, H\hat{\otimes}H)} \leq C \quad \Rightarrow \sup_{\substack{\|y\|=1\\ N,M}} \|\sigma_I \left(P^{[N,M]} f_A)^*_y\right)(-t)\|_{L^1(\mathbb{T},H)} \leq C.$$

$$\Rightarrow \sup_{\substack{\|y\|=1\\ N,M}} \| \left(P^{[N,M]} f_A)^*_y\right)(-t)\|_{L^1(\mathbb{T},H)} \leq C \quad \Rightarrow \sup_{\substack{\|y\|=1\\ N,M}} \| \left(P^{[N,M]} f_A)^*_y\right)(-t)\|_{(C(\mathbb{T},H))^*} \leq C.$$

A description of $\mathcal{M}_\tau(\ell^2(H))$

Proof. $A \in \mathcal{M}(\ell^2(H)) \cap \mathcal{T} \quad \Rightarrow \quad (f_A)^* \in M_{SOT}(\mathbb{T}, H)$

So we have:

$$\sup_{\substack{\|y\|=1\\ \|x\|=1\\ N,M}} \left| \left\langle f_B, x \otimes \left(K_I * (P^{[N,M]} f_A)_y^* \right)(-t) \right\rangle \right| \leq C \qquad \forall f_B \in V^\infty(\mathbb{T}, \mathcal{B}(H)) \tag{2}$$

$$\Rightarrow \sup_{\substack{\|y\|=1\\ \|x\|=1\\ N,M}} \|x \otimes \left(K_I * (P^{[N,M]} f_A)_y^* \right)(-t)\|_{L^1(\mathbb{T}, H \hat{\otimes} H)} \leq C \quad \Rightarrow \sup_{\substack{\|y\|=1\\ N,M}} \|\sigma_I \left(P^{[N,M]} f_A)_y^* \right)(-t)\|_{L^1(\mathbb{T},H)} \leq C.$$

$$\Rightarrow \sup_{\substack{\|y\|=1\\ N,M}} \| \left(P^{[N,M]} f_A)_y^* \right)(-t)\|_{L^1(\mathbb{T},H)} \leq C \quad \Rightarrow \sup_{\substack{\|y\|=1\\ N,M}} \| \left(P^{[N,M]} f_A)_y^* \right)(-t)\|_{(C(\mathbb{T},H))^*} \leq C.$$

By Banach-Alaoglu $(P^{[N,M]} f_A)^*{}_y \xrightarrow{w^*} \mu_{A_y}$ for certain $\mu_{A_y} \in M(\mathbb{T}, H)$. Therefore

$(f_A)_y^* \in M(\mathbb{T}, H) \quad \forall y \in H.$ □

Thanks for your attention!