Pettis operators and their integrals

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$L_{w^*,w}(X^*, Y) = L_{w^*,w}(Y^*, X)$, using

$$\langle x^*, F^*(y^*)\rangle = \langle y^*, F(x^*)\rangle \qquad \text{for all } x^* \in X^*,\ y^* \in Y^*,$$

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In particular we can associate a vector-valued measure

$$m_f(A) = (B) - \int_A f d\mu \in X$$

for any $A \in \Sigma$. It is well known that $|m_f| = \int_\Omega \|f\| d\mu$.

Bochner integral and operators

Also we can associate the operators $T_f : L^\infty(\mu) \to X$ and $S_f : X^* \to L_1(\mu)$ defined by

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- $t \to \delta_t$ does not belong to $L_1(m, M(\mathbb{T}))$.

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Now, by definition, the (indefinite) *Pettis (μ-) integral of f* is the (countably additive) vector measure $m_f : \Sigma \to X$ given by the formula

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$t \to (r_n(t))_{n \in \mathbb{N}}$, where r_n stand for the Rademacher functions, does not belong to $P_1(m, \ell^\infty)$.

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- Not every Pettis integrable function admits a conditional expectation with respect to a sub-σ-algebra;
- the Radon-Nikodym theorem does not hold even if one restricts to vector measures of bounded variation.
- the Fubini theorem fails to hold in a dramatic way.
- the Fatou theorem for harmonic functions fails. In particular, whenever X is infinite dimensional, there exists a Pettis-integrable function $f : \mathbb{T} \to X$ such that $\lim_{r\to\infty} \|P_r * F(t)\| = \infty$ uniformly in $t \in \mathbb{T}$.

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More generally, for any $\psi \in L_\infty(\mu)$ we set $F^*(\psi) = (P) - \int_\psi F\, d\mu$, and is a unique element of X such that

$$\left\langle x^*, (P) - \int_\psi F\, d\mu \right\rangle = \int_S \psi \cdot F(x^*)\, d\mu \qquad \text{for all } x^* \in X^*.$$

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- Let $X = L_p([0,1])$ for $1 < p < \infty$ and let $F : L_{p'}([0,1]) \to L_1([0,1])$ be the inclusion map. Then $F \in \mathbb{P}(L_{p'}([0,1]), L_1([0,1]) \setminus P_1(m, L_p([0,1]))$.

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- Let $X = \ell_\infty$ and let $F : (\ell_\infty)^* \to L_1([0,1])$ the operator defined by
$$x^* \to t \to \langle x^*, (r_n(t)) \rangle.$$
Then $F \in \mathbb{P}((\ell_\infty)^*, L_1([0,1])) \setminus P_1(\eta, \ell_\infty)$.

On the equality $P_1(\mu, X) = \mathbb{P}(X^*, L_1(\mu))$

If $S = \mathbb{N}$ and $\mu = \eta$ the counting measure $P_1(\eta, X)$ can be identified with $\mathbf{x} = (x_n) \in X$ such that the series $\sum_n x_n$ is unconditionally (or subseries) convergent in X.

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Clearly, the Pettis operator $P_{\mathbf{x}} : X^* \to \ell_1$ associated with $\mathbf{x}$ given by $P_{\mathbf{x}}(x^*) = \big(x^*(x_n)\big)_{n\in\mathbb{N}}$ is weak*-weakly continuous, because $P_{\mathbf{x}}^* : \ell_\infty \to X$.

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Proof: Let $F \in \mathbb{P}(X^*, \ell_1)$ and let (e_n) be the sequence of unit vectors in ℓ_∞. Then for each $n \in \mathbb{N}$,
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$x_n =: (P) - \int_{e_n} F\, d\eta = F^*(e_n) \in X$.By the Orlicz-Pettis theorem, the series $\sum_n x_n$ is subseries convergent. Hence
$\mathbf{x} \in P_1(\eta, X)$

Pettis operators versus vector-valued measures

Let us denote by $ca_\mu(\Sigma, X)$ the Banach space of all μ-continuous countably additive vector measures $m : \Sigma \to X$, endowed with the norm given by the semivariation $\|m\| = \sup_{\|x^*\| \leq 1} |x^* m|(S)$.

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Let $F : X^* \to L_1(\mu)$ be a Pettis operator. Then its indefinite integral $m_F : \Sigma \to X$ is a vector measure of finite or σ-finite variation if and only if the set $F(B(X^*))$ is order bounded in $L^1(\mu)$ or $L_0(\mu)$, respectively.

Vector-valued harmonic functions

Here $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $\mu_{\mathbb{T}}$ the Lebesgue measure on $\mathbb{T}$.

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For $z \in \mathbb{D}$, we denote by $P : \mathbb{D} \to \mathbb{R}^+$ the harmonic function $P(z) = \Re \frac{1+z}{1-z}$ and by P_z the *Poisson kernel* on $\mathbb{T}$ as

$$P_z(\xi) = P(z\bar{\xi}) = \frac{1-|z|^2}{|1-z\bar{\xi}|^2} \qquad (\xi \in \mathbb{T}).$$

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For a bounded linear operator $T : C(\mathbb{T}) \to X$ we define the *Poisson integral of T* by the formula

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Then $P(T)$ is a vector-valued harmonic function.

On weak Hardy spaces of harmonic functions

Let $h^1(\mathbb{D})$ and $H^1_{max}(\mathbb{D})$ stand for the spaces of harmonic functions ϕ in the unit disc such that $\|\phi\|_{h^1} = \sup_{0<r<1} \|\phi_r\|_1 < \infty$ and that the Poisson maximal function $P^*\phi(\xi) = \sup_{0<r<1} |\phi_r(\xi)| \in L_1(\mathbb{T})$ respectively, with $\|\phi\|_{H^1_{max}} = \|P^*\phi\|_1$, where $\phi_r(z) = \phi(rz)$.

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A well known fact is that $\phi \in H^1_{max}(\mathbb{D})$ then there exists $\phi^\circ(\xi) = \lim_{r\to 1} \phi_r(\xi)$ a.e and $\phi^\circ \in L^1(\mathbb{T})$.

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The "weak" vector-valued versions $wh^1(\mathbb{D}, X)$ and $wH^1_{max}(\mathbb{D}, X)$ consist of those functions $f : \mathbb{D} \to X$ such that $x^*f \in h^1(\mathbb{D})$ and $x^*f \in H^1_{max}(\mathbb{D})$ for all $x^* \in X^*$, respectively, where $x^*f(z) = \langle x^*, f(z)\rangle$ for each $z \in \mathbb{D}$.

Operators and weak Hardy spaces of harmonic functions

Proposition

Let $F \in \mathbb{P}(X^*, L_1(\mathbb{T}))$. *Then*
(a) $f = P(F^*) \in wh^1(\mathbb{D}, X)$.

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(a) $f = P(F^*) \in wh^1(\mathbb{D}, X)$*.*
(b) $\lim_{r\to 1} P_{f_r} = F$ *in the strong operator topology, i.e.*
$\lim_{r\to 1} P_{f_r}(x^*) = F(x^*)$ *for all* $x^* \in X^*$*.*

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(c) If F *is compact, then* $\lim_{r\to 1} P_{f_r} = F$ *in the uniform norm of operators.*

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What can be said for functions in $wH^1_{max}(\mathbb{D}, X)$ **?**

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Proof: Let us define $F : X^* \to L_1(\mathbb{T})$ by $F(x^*) = (x^* f)^\circ$ where we have used the inclusion $H^1_{max}(\mathbb{D}) \subseteq L_1(\mathbb{T})$ given $\phi \to \phi^\circ$. Hence for $0 < r < 1$ and $\xi \in \mathbb{T}$,

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Observe that

$$\langle x^*, f(z)\rangle = (x^* f)(z) = P((x^* f)^\circ)(z) = \langle P_z, F(x^*)\rangle = \langle F^*(P_z), x^*\rangle.$$

Hence for $x^* \in X^*$ and $z \in \mathbb{D}$ one has $F^*(P_z) \in X$ and $F^*(P_z) = f(z)$. We shall show that $F \in \mathbb{P}(X^*, L_1(\mathbb{T}))$.

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Observe that

$$\langle x^*, f(z)\rangle = (x^* f)(z) = P((x^* f)^\circ)(z) = \langle P_z, F(x^*)\rangle = \langle F^*(P_z), x^*\rangle.$$

Hence for $x^* \in X^*$ and $z \in \mathbb{D}$ one has $F^*(P_z) \in X$ and $F^*(P_z) = f(z)$. We shall show that $F \in \mathbb{P}(X^*, L_1(\mathbb{T}))$.

Let (s_n) be a sequence converging to 1 and consider $F_n = P_{s_n} * F$. Notice that $F_n^* = F^* * P_{s_n} : L_\infty(\mathbb{T}) \to X^{**}$ and one can show that

$$\langle F_n^*(\psi), x^* \rangle = \langle x^*, \int \psi(\xi) f(s_n \xi) d\mu_{\mathbb{T}}(\xi) \rangle.$$

Hence $F_n^*(\psi) = \int \psi(\xi) f(s_n \xi) d\mu_{\mathbb{T}}(\xi) \in X$. This gives that F_n are Pettis operators.

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Hence $F_n^*(\psi) = \int \psi(\xi) f(s_n \xi) d\mu_{\mathbb{T}}(\xi) \in X$. This gives that F_n are Pettis operators. On the other hand, since $f \in H^1_{max}(\mathbb{D}, X)$, for each $x^* \in X^*$

$$\lim_n \langle x^*, f(s_n \xi) \rangle = (x^* f)^\circ(\xi), \quad a.e$$

and for each $\psi \in L_\infty(\mathbb{T})$ one has

$$\sup_n |\psi(\xi) \langle x^*, f(s_n \xi) \rangle| \in L_1(\mathbb{T}).$$

This allows us to conclude that $F_n^*(\psi)$ is weakly convergent to $F^*(\psi)$ for any $\psi \in L_\infty(\mathbb{T})$. Hence F is a Pettis operator.

- O. Blasco, *Boundary values of vector-valued harmonic functions considered as operators*, Studia Math. **86** (1987), 19–33.
- O. Blasco, L. Drewnowsky *Extension of Pettis integration: Pettis operators and their integrals* Submitted

THANKS!

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