Greedy algorithm and embbeddings

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N-term approximation and greedy algorithm

$\mathbb{X}$ = Banach space, $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^\infty$ = seminormalized, complete, biorthogonal system, $x = \sum_{n=1}^\infty e_n^*(x) e_n$

- **N-term approximation:** Given $x \in \mathbb{X} \longrightarrow$ want good approximant within
$$\Sigma_N = \Big\{ \sum_{\lambda \in \Lambda} c_\lambda \mathbf{e}_\lambda \; : \; |\Lambda| \leq N \Big\}$$

- A natural choice are **greedy operators**:
$$x \in \mathbb{X} \longmapsto \mathbb{G}_N x = \sum_{j \in \Lambda(x)} \mathbf{e}_j^*(x) \mathbf{e}_j \in \Sigma_N$$
where $\min_{j \in \Lambda(x)} |\mathbf{e}_j^*(x)| \geq \max_{n \notin \Lambda(x)} |\mathbf{e}_n^*(x)|$ and $|\Lambda(x)| = N$.

N-term approximation and greedy algorithm

- **Q:** How good is $\|x - \mathbb{G}_N x\|$ vs $\sigma_N(x) = \inf_{x_N \in \Sigma_N} \|x - x_N\|$?

- **Goal:** find smallest $\mathbf{L}_N = \mathbf{L}_N(\mathbb{X}, \{\mathbf{e}_n\})$ s.t.

$$\|x - \mathbb{G}_N x\| \leq \mathbf{L}_N \, \sigma_N(x), \quad \forall \, x \in \mathbb{X}$$

- Examples
  - $\{\mathbf{e}_n\} = \text{ONB} \implies \mathbf{L}_N = 1$
  - $\{e^{2\pi ikx}\}_{k=-\infty}^{\infty} = \mathcal{T} \implies \mathbf{L}_N(L^p, \mathcal{T}) \approx N^{|\frac{1}{p} - \frac{1}{2}|}$ [Tem'98]
  - $\{h_{j,k}\} = \mathcal{H} \implies \mathbf{L}_N(L^p, \mathcal{H}) = O(1)$, $1 < p < \infty$, [Tem'98]

Greedy bases

Given $\mathbb{X}$ and $\mathcal{B} = \{\mathbf{e}_n\}_{n=1}^{\infty}$ (basis), when is $\mathbb{G}_N x$ "essentially" optimal?

- **Theorem** [Konyagin-Temlyakov'99]:
  $\mathbf{L}_N = O(1)$ **iff** $\mathcal{B}$ is unconditional and democratic.
  Moreover, $\mathbf{L}_N \leq K + 4K^3\Delta$, provided

$$\|P_A x\| \leq K\|x\|, \qquad \|\mathbf{1}_A\| \leq \Delta \, \|\mathbf{1}_B\|, \ \forall \, |A| = |B| < \infty$$

  where $\mathbf{1}_A = \sum_{n\in A} e_n$.

- Examples:
  - $\{\mathbf{e}_n\}$ in ℓ^p, $\mathcal{W}$ in $F^s_{p,q}$,... are greedy bases
  - ... but $\{\mathbf{e}_n\}$ in $\ell^p \oplus \ell^q$, or $\mathcal{W}$ in $B^s_{p,q}$ are not democratic if $p \neq q$...
  - $\mathcal{H}$ is **not** unconditional in L^1, BV...
- To handle such examples need more general bounds for $\mathbf{L}_N$...

- **Definition:** $\{\mathbf{e}_n, \mathbf{e}_n^*\}$ is a quasi-greedy system in $\mathbb{X}$ if $\|G_N x\| \leq \mathbf{q}\|x\|$, $\forall\, N, x$ (equivalently, $\mathbb{G}_N x \to x$, $\forall\, x \in \mathbb{X}$ [Wo'00])

- Examples:
  - Every greedy basis is quasi-greedy.
  - $\mathcal{H}$ q-greedy and democratic in $BV([0,1]^d)$, $d \geq 2$, [Cohen'99],[Wo'03]
  - Lindenstrauss basis: $\mathcal{L} = \{\mathbf{e}_n - \frac{1}{2}(\mathbf{e}_{2n} + \mathbf{e}_{2n+1})\}$ in ℓ^1 is q-greedy and democratic [Dilworth-Mitra'01]

- Consider the parameters

$$K_N = \sup_{|A| \le N} \|P_A\|, \qquad \Delta_N = \sup_{|A|=|B|\le N} \frac{\|\mathbf{1}_A\|}{\|\mathbf{1}_B\|}$$

- A nice remark [DKK'03]: $\mathcal{B}$ = quasi-greedy $\Longrightarrow K_N \lesssim \log N$!!

Thm [GHO'13]:

If $\mathcal{B}$ is quasi-greedy then

$$\mathbf{L}_N \approx \max\left\{ K_N,\ \Delta_N \right\}$$

Moreover, $\mathbf{L}_N \le K_{2N} + 8\mathbf{q}^4 \Delta_N$ $\longrightarrow$ actually $\mathbf{q}^3$... [DKO'15]

- **Q:** Bound $\mathbf{L}_N$ for non q-greedy bases?

Let $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^{\infty}$ complete, biorthogonal in $\mathbb{X} \times \mathbb{X}^*$ (seminormalized)

- Notation: $\mathbf{1}_{\varepsilon A} = \sum_{n\in A} \varepsilon_n \mathbf{e}_n\,, \quad \mathbf{1}^*_{\varepsilon A} = \sum_{n\in A} \varepsilon_n \mathbf{e}^*_n\,, \quad A \subset \mathbb{N},\ |\varepsilon| = 1.$

- Suppose one knows upper bounds for:

$$\left\|\mathbf{1}_{\varepsilon A}\right\|_{\mathbb{X}} \le \eta_1(N), \quad \left\|\mathbf{1}^*_{\varepsilon A}\right\|_{\mathbb{X}^*} \le \eta_2(N), \quad \forall\, |A| = N,\ |\varepsilon| = 1,$$

  with η_1, η_2 increasing concave sequences, that is
  $\Delta^2\eta(n) = \Delta\eta(n) - \Delta\eta(n+1) \ge 0$ and $\Delta\eta(n) = \eta(n) - \eta(n-1)$

- Define $S_N(\eta_1, \eta_2) = \sum_{n=1}^{N} \Delta\eta_1(n)\,\Delta\eta_2(n)$

Main result

Theorem 1: [BBGHO'17]

- $K_N \leq S_N(\eta_1, \eta_2)$ and $\mathbf{L}_N \leq 1 + 3S_N(\eta_1, \eta_2)$.
  Also, $K_N^* \leq S_N(\eta_1, \eta_2)$ and $\mathbf{L}_N^* \leq 1 + 3S_N(\eta_1, \eta_2)$ (symmetry)
- These estimates are best possible, i. e. there exists $\mathbb{X}$ and $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^{\infty}$ for which all the equalities hold.

1. The best η_1, η_2 we can take in Theorem 1 are

$$D(N) = \sup_{|A|=N, |\varepsilon|=1} \|\mathbf{1}_{\varepsilon A}\|, \quad \text{and} \quad D^*(N) = \sup_{|A|=N, |\varepsilon|=1} \|\mathbf{1}^*_{\varepsilon A}\|_* .$$

...if they are concave...

- A sequence space $\mathbb{S}$ *embeds into* $\mathbb{X}$ *via* $\mathcal{B}$ (with norm c), denoted $\mathbb{S} \overset{\mathcal{B},\, c}{\hookrightarrow} \mathbb{X}$, if for every $\mathbf{s} = \{s_n\}_{n=1}^{\infty} \in \mathbb{S}$, there exists a **unique** $x \in \mathbb{X}$ such that $\mathbf{e}_n^*(x) = s_n$ and it holds:

$$\|x\| \leq c\,\|\mathbf{s}\|_{\mathbb{S}} = c\,\|\{\mathbf{e}_j^*(x)\}_{j=1}^{\infty}\|_{\mathbb{S}}\,.$$

- Discrete weighted Lorentz spaces: $\eta \in \mathbb{W}$,

$$\ell_\eta^1 = \Big\{ \mathbf{s} \in c_0 \;:\; \|\mathbf{s}\|_{\ell_\eta^1} := \sum_{j=1}^{\infty} s_j^* \frac{\eta(j)}{j} < \infty \Big\}.$$

Theorem 2:

The following are equivalent:

i) $\|\mathbf{1}_{\varepsilon A}\| \leq \eta(|A|)$ for all finite $A \subset \mathbb{N}$ and all $|\varepsilon| = 1$, .

ii) $\|\sum a_n e_n\|_{\mathbb{X}} \leq \|\mathbf{a}\|_{\ell^1_{\hat{\eta}}}$, for all $\mathbf{a} = \{a_n\} \in c_{00}$.

If $\mathcal{B}^*$ is total, then each of the above is equivalent to iii) $\ell^1_{\hat{\eta}} \overset{\mathcal{B},\, 1}{\hookrightarrow} \mathbb{X}$.

Sharp embeddings of $\mathbb{X}$ [BBGHO'17]

- The space $\mathbb{X}$ *embeds into* $\mathbb{S}$ *via* $\mathcal{B}$ (with norm c), denoted $\mathbb{X} \overset{\mathcal{B},\, c}{\hookrightarrow} \mathbb{S}$, if for every $x \in \mathbb{X}$ it holds: $\|\{\mathbf{e}_j^*(x)\}_{j=1}^{\infty}\|_{\mathbb{S}} \leq c\|x\|$.

- Discrete weighted Marcinkiewicz spaces: $\eta \geq 0$,

$$m(\eta) = \left\{ \mathbf{s} \in c_0 \;\; : \;\; \|\mathbf{s}\|_{m(\eta)} := \sup_{k\in\mathbb{N}} \frac{\eta(k)}{k} \sum_{j=1}^{k} s_j^* < \infty \right\}.$$

- **Remark:** When $\eta' = \{j/\eta(j)\}_{j=1}^{\infty}$, is the "dual" weight, then $(\ell_\eta^1)^* = m(\eta')$ (... if $\eta \in \mathbb{W}_d$ and $\inf_n \frac{\eta(n)}{n} = 0$.)

Theorem 3:

The following are equivalent:

i) $\|\mathbf{1}_{\varepsilon A}^*\|_* \leq \eta(|A|)$ for all finite $A \subset \mathbb{N}$ and all $|\varepsilon| = 1$.

ii) $\mathbb{X} \overset{\mathcal{B},\, 1}{\hookrightarrow} m(\eta')$, with $\eta' = \{j/\eta(j)\}_{j=1}^{\infty}$.

Sketch of proof: Thm 1 ($K_N \leq S_N(\eta_1, \eta_2)$)

We follow the strategy developed in [DKO'15]

- Thm 2 gives
$$\|P_A x\| \leq \sum_{j=1}^{N} a_j^*(P_A x)\,\Delta\eta_1(j) \leq \sum_{j=1}^{N} a_j^*(x)\,\Delta\eta_1(j) := A_N(x)$$
- Let $S_J(x) := \sum_{j=1}^{J} a_j^*(x)$. By Abel summation
$$A_N(x) = \sum_{j=1}^{N}[S_j(x) - S_{j-1}(x)]\,\Delta\eta_1(j) = \sum_{j=1}^{N-1} \Delta^2\eta_1(j)\,S_j(x) + \Delta\eta_1(N)\,S_N(x)\,.$$
- From Thm 3, $\frac{1}{\eta_2(j)} S_j(x) = \frac{\eta_2'(j)}{j} \sum_{n=1}^{j} a_n^*(x) \leq \|x\|$.
- By Abel summation again
$$\|P_A x\| \leq \Big[\sum_{j=1}^{N-1} \Delta^2\eta_1(j)\,\eta_2(j) + \Delta\eta_1(N)\,\eta_2(N)\Big]\|x\| = S_N(\eta_1, \eta_2)\|x\|.$$
- Thus $K_N \leq S_N(\eta_1, \eta_2)$

Let $\{\mathbf{e}_n\}_{n=1}^{\infty}$ be the canonical basis in ℓ^1.

- **Difference basis**: $\mathbf{x}_1 = \mathbf{e}_1\,, \mathbf{x}_n = \mathbf{e}_n - \mathbf{e}_{n-1}\,, n \geq 2$.
  $\mathbb{X} = \overline{\text{span}}^{\ell^1}\{\mathbf{x}_n\}_{n=1}^{\infty}$.

- **Dual system**: vectors in ℓ^∞ of the form $\mathbf{x}_n^* = \sum_{m=n}^{\infty} e_m^*$ and

$$\Big\| \sum_{n=1}^{\infty} c_n \mathbf{x}_n^* \Big\|_* = \sup_{n \geq 1} \Big| \sum_{j=1}^{n} c_j \Big|, \qquad \{c_n\} \in c_{00}\,.$$

(**Summing basis**, [LT'1977])

- **Lemma 4:** $\{\mathbf{x}_n, \mathbf{x}_n^*\}_{n=1}^{\infty}$ as above: $D(N) = 2N$ and $D^*(N) = N$.
- Since $S_N(D, D^*) = \sum_{j=1}^{N} 2 \times 1 = 2N$, Thm 1 gives $K_N, K_N^* \leq 2N$ and $L_N, L_N^* \leq 1 + 6N$
- Equality can be proved by testing with particular elements .... this gives the announced sharpness of Thm 1.... .... (the values of K_N^* and L_N^* were known, [BBG'17].)

Let $\{\mathbf{e}_n\}_{n=1}^{\infty}$ be the canonical basis in ℓ^1.

- (Lindenstrauss, 1964):
  $\mathcal{L}: \ \mathbf{x}_n = \mathbf{e}_n - \frac{1}{2}\,\mathbf{e}_{2n+1} - \frac{1}{2}\,\mathbf{e}_{2n+2}\,, \ n = 1, 2, 3, \dots$
  $\mathcal{L}$ is a basis for $\mathbb{D} = \overline{\text{span}}\,\{\mathcal{L}\}$ in ℓ^1.
- Dual system (Holub-Retherford, 1970):
  $\mathcal{Y}: \ \mathbf{y}_n := \sum_{j=0}^{n} 2^{-j}\mathbf{e}_{\gamma_j(n)} \in c_0, \ n = 1, 2, 3\dots$ where $\gamma_0(n) = n$ and $\gamma_{j+1}(n) = \lfloor \frac{\gamma_j(n)-1}{2} \rfloor \ (j \geq 0)$, with the convention $\mathbf{e}_\gamma = \mathbf{0}$ if $\gamma \leq 0$.
  ($\mathcal{Y}$ is a Shauder basis for c_0)
- **Lemma 5:** $D(N) = 2N$ and $D^*(N) \approx \ln(N+1) := \eta_2(N)$.
- Since $S_N(D, \eta_2) = \sum_{j=1}^{N} 2\,\Delta\eta_2(j) = 2\ln(N+1)$, Thm 1 gives $K_N, K_N^* \lesssim \ln(N+1)$ and $L_N, L_N^* \lesssim \ln(N+1)$
- Equivalence can be proved by testing with particular elements .... (the values of K_N and L_N were known.)

Example 3: The trigonometric system

$\mathcal{T} = \{e^{2\pi ikx}\}_{k\in\mathbb{Z}^d}$ in $L^p(\mathbb{T}^d)$, $1 \leq p \leq \infty$, with $L^\infty(\mathbb{T}^d) = C(\mathbb{T}^d)$.

- For $1 \leq p \leq 2$, $D(N) \leq N^{1/2} := \eta_1(N)$ and $D^*(N) \leq N^{1/p} := \eta_2(N)$.

$$S_N(\eta_1, \eta_2) \leq \sum_{j=1}^{N} j^{-\frac{1}{2}} j^{\frac{1}{p}-1} \leq c_p N^{\frac{1}{p}-\frac{1}{2}}, \ p \neq 2$$

- For $2 \leq p \leq \infty$, $D(N) \leq N^{1/p'} := \eta_1(N)$ and $D^*(N) \leq N^{1/2} := \eta_2(N)$.

$$S_N(\eta_1, \eta_2) \leq \sum_{j=1}^{N} j^{\frac{1}{p'}-1} j^{-\frac{1}{2}} \leq c_p N^{\frac{1}{p'}-\frac{1}{2}} = c_p N^{\frac{1}{2}-\frac{1}{p}}, \ p \neq 2$$

- By Theorem 1, $K_N(\mathcal{T}, L^p), L_N(\mathcal{T}, L^p) \leq c_p N^{|\frac{1}{p}-\frac{1}{2}|}$, $p \neq 2$. ([T'98])
The lower bound also holds: Remark 2 in [T'98]

- Drawback: we cannot recover the trivial case $p = 2$.

More examples

- For the Haar system $\mathcal{H} = \{h_{j,k}\}$ in L^1,

$$D(N) = D^*(N) = N.$$

Theorem 1 gives $L_N \leq 1 + 3N$ ([Oswald'2001] with equality)

- For the Haar system $\mathcal{H} = \{h_{j,k}\}$ in BMO_d,

$$D(N) \approx \sqrt{\ln(N+1)}, \quad D^*(N) = N.$$

Theorem 1 gives $L_N \lesssim \sqrt{\ln(N+1)}$ ([Oswald'2001] with equality)

- Drawback: For greedy bases, $S_N \lesssim \ln(N+1)$, and Theorem 1 gives $L_N \lesssim \ln(N+1)$, while $L_N \approx 1$.

THANKS FOR YOUR ATTENTION!!