A Dirichlet problem involving the 1–Laplacian operator

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We study a nonlinear elliptic equation with Dirichlet boundary condition:

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Goal

Our purpose is to study the role of the function g and how it affect our solutions.

L. Boccardo, F. Murat and J.–P. Puel, 1982

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Special features of the 1–Laplacian operator

Natural energy space

A function $u : \Omega \to \mathbb{R}$ is a function of **bounded variation** $BV(\Omega)$ if $u \in L^1(\Omega)$ and its derivative in the sense of distributions Du is a vector valued Radon measure with finite total variation.

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Definition

Let $\varphi \in C_0^\infty(\Omega)$. We define the functional

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- $u \in BV(\Omega)$ and $\mathbf{z} \in \mathcal{DM}^\infty(\Omega)$ such that $\operatorname{div} \mathbf{z} = \mu + f$ for a Radon measure $\mu \geq 0$ or $\mu \leq 0$ and $f \in L^{N,\infty}(\Omega)$.

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- Green's formula:

$$\int_\Omega u \operatorname{div} \mathbf{z} + \int_\Omega (\mathbf{z}, Du) = \int_{\partial\Omega} u\, [\mathbf{z}, \nu]\; d\mathcal{H}^{N-1}\,.$$

- We say that $x \in \Omega$ is an **approximate jump point** of $u \in BV(\Omega)$ if there exist two real numbers $u_+(x) > u_-(x)$ and $\nu_u(x) \in S^{N-1}$ such that

$$\lim_{\rho\downarrow 0} \frac{1}{|B^+_\rho(x,\nu_u(x))|}\int_{B^+_\rho(x,\nu_u(x))} |u(y)-u_+(x)|\,dy = 0\,,$$

$$\lim_{\rho\downarrow 0} \frac{1}{|B^-_\rho(x,\nu_u(x))|}\int_{B^-_\rho(x,\nu_u(x))} |u(y)-u_-(x)|\,dy = 0\,,$$

where

$$B^+_\rho(x,\nu_u(x)) = \{y \in B_\rho(x) : \langle y-x, \nu_u(x)\rangle > 0\}\,,$$
$$B^-_\rho(x,\nu_u(x)) = \{y \in B_\rho(x) : \langle y-x, \nu_u(x)\rangle < 0\}\,.$$

- We denote by J_u the set all approximate jump points of u.
- $D^j u = (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner J_u$.

- Let $1 < q < \infty$. The **Marcinkiewicz space** $L^{q,\infty}(\Omega)$ is the space of all Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ such that

$$[u]_q = \sup_{t>0} t \, |\{|u| > t\}|^{1/q} < +\infty \, .$$

The relationship with Lebesgue spaces is given by the following inclusions

$$L^q(\Omega) \hookrightarrow L^{q,\infty}(\Omega) \hookrightarrow L^{q-\epsilon}(\Omega) \, .$$

$g \equiv 1$ and $L^{N,\infty}$–data

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Theorem

Let $f \in L^{N,\infty}(\Omega)$ with $f \geq 0$. There is a $u \in BV(\Omega)$ solution to problem

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Comparison principle

Let f_1 and $f_2 \in L^{N,\infty}(\Omega)$ with $0 \leq f_1 \leq f_2$. If u_1 and u_2 are the solution to problem with data f_1 and f_2, respectively, then, $u_1 \leq u_2$.

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Moreover, $u \in L^q(\Omega)$ for all $1 \leq q < \infty$.

Example

Let $R > 0$ and $\Omega = B_R(0)$. We consider

$$\begin{cases} -\mathrm{div}\left(\dfrac{Du}{|Du|}\right) + |Du| = \dfrac{\lambda}{|x|} & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

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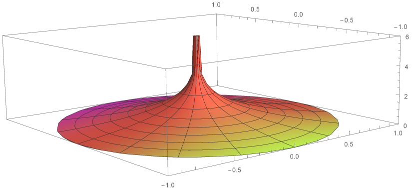
- The solution is given by $u(x) = (N - 1 - \lambda)\log\left(\dfrac{|x|}{R}\right)$.



Approximate problems for $1 < p < \infty$

$$\begin{cases} -\text{div}\left(|\nabla u|^{p-2}\nabla u\right) + |\nabla u|^p = \lambda \dfrac{u^{p-1}}{|x|^p} & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

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There is a solution for $\lambda > 0$ small enough.

A general gradient term

Problem

$$\begin{cases} -\text{div}\left(\dfrac{Du}{|Du|}\right) + g(u)|Du| = f & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

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- First case: there exists $m > 0$ such that $\mathbf{g(s) > m > 0}$.

Definition

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Theorem

There is a **unique solution** to this problem when g is a continuous function such that there exists $m > 0$ with $g(s) > m$ for all $s > 0$.

g(s) touches the s–axis

Problem

$$\begin{cases} -\text{div}\left(\dfrac{Du}{|Du|}\right) + g(u)|Du| = f & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

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Theorem

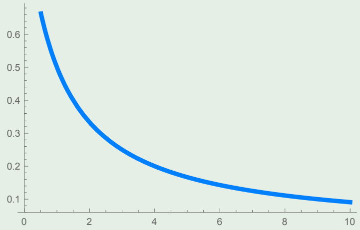
With the function g defined as above, there exists a **unique solution** to our problem.

Example

Let $\Omega = B_1(0)$. Consider problem

$$\begin{cases} -\text{div}\left(\dfrac{Du}{|Du|}\right) + g(u)|Du| = \dfrac{\lambda}{|x|} & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

with $g(s) = \dfrac{1}{1+s}$.



- The solution for $\lambda > N-1$ is

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- Nevertheless, for $\lambda > 2(N-1)$ we have $u \notin BV(\Omega)$ because

$$|Du| = (N - 1 - \lambda)|x|^{N-2-\lambda}$$

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Assume that the function g is **continuous**, and **no integrable** with $\mathbf{g(s) > 0}$ **a.e.** in $[0, \infty[$.

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  - we cannot assure that the boundary condition holds.

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Let $\Omega = B_1(0)$. Consider problem

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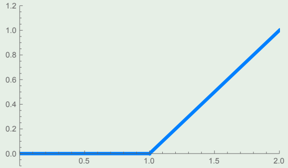

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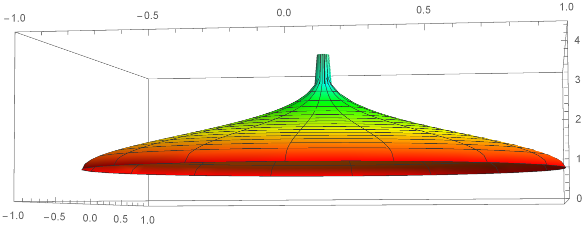

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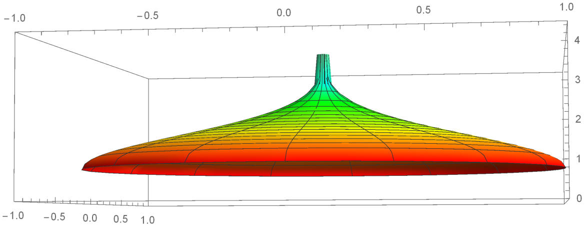

- Although $u\big|_{\partial\Omega} = 1$, the solution achieves the boundary weakly:

$$[\mathbf{z}, \nu] = -\frac{x}{|x|}\frac{x}{|x|} = -1 = \text{sign}\,(-u)\,.$$

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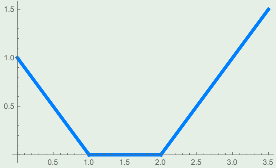

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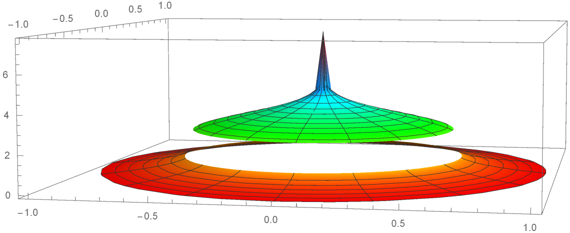

which has jump part.

Function g	Existence and uniqueness	Regularity
$\mathbf{0 < m \leq g(s)}$	For all data	$D^j u = 0$, $u \in L^q(\Omega)$ for $1 \leq q < \infty$
$\mathbf{0 < m < g(s)}$ **for** $\mathbf{s > t_0}$	For all data	$D^j u = 0$
$\mathbf{g(s) > 0}$ **a.e. in** $\mathbf{[0, \infty[}$	For all data, with other concept of solution	$D^j u = 0$
$\mathbf{g \in L^1([0, \infty[)}$	For data small enough	$D^j u = 0$
$\mathbf{g(s) = 0}$ **in an interval**	No uniqueness	$D^j u \neq 0, \quad u\big\vert_{\partial\Omega} \neq 0$

Thank you for your attention