Teoría de pesos para operadores multilineales, extensiones vectoriales y extrapolación

Encuentro Análisis Funcional

Sheldy Ombrosi

Universidad Nacional del Sur y CONICET

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if the last inequality holds we say that w belongs to A_p.
If $\frac{Mw}{w} \in L^\infty$ we say $w \in A_1$

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Calderón-Zygmund operators

Definition

A Calderón-Zygmund operator T (CZO) is an operator bounded on $L^2(\mathbb{R}^n)$ that admits the following representation

$$Tf(x) = \int K(x,y)f(y)dy$$

with $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $x \notin \operatorname{supp} f$ and where $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) \,:\, x \in \mathbb{R}^n\} \longrightarrow \mathbb{R}$ has the following properties

Size condition: $|K(x,y)| \leq C_2 \frac{1}{|x-y|^n} \qquad x \neq 0.$

Smoothness condition (Hölder-Lipschitz):

$$|K(x,y) - K(x,z)| \leq C_1 \frac{|y-z|^\delta}{|x-y|^{n+\delta}} \qquad \frac{1}{2}|x-y| > |y-z|$$

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where $C_1 > 0$ and $C_2 > 0$ are constants independent of x, y, z.

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Coifman-Fefferman estimate, if $0 < r < \infty$ and v is a "good" weight, then

$$\int |Tf|^r v(x)dx \leq C_{p,v} \int Mf^r v(x)dx$$

Rubio de Francia's extrapolation theorem

Theorem (Rubio de Francia, 1984)

Fixed $1 \leq p_0 < \infty$, *if* T *is a bounded operator on* $L^{p_0}(w)$ *for every* $w \in A_{p_0}$.
Then for every $1 < p < \infty$ *and for all* $w \in A_p$; T *is bounded on* $L^p(w)$.

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- It can be consider a pair of functions (f, g), where, in particular, g could be Tf...

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Given $g \geq 0$ with $\|g\|_{L^{p'}(\sigma)} = 1$, define

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$$\begin{aligned}\int |Tf|g &\leq \int |Tf|\mathcal{R}g \leq C\|f\|_{L^1(\mathcal{R}g)} \\ &\leq C\|f\|_{L^p(w)}\|\mathcal{R}g\|_{L^{p'}(\sigma)} \\ &\leq 2C\|f\|_{L^p(w)}.\end{aligned}$$

Bilinear Calderón-Zygmund operators

let $T : S(\mathbb{R}^n) \times S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$. T is an bilinear Calderón-Zygmund operator if, for some $1 \leq q_1, q_2 < \infty$ and $\frac{1}{2} \leq q < \infty$ satisfying $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, it extends to a bounded bilinear operator from $L^{q_1} \times L^{q_2}$ to L^q, and if there exists K defined off the diagonal $x = y_1 = y_2$ in $(\mathbb{R}^n)^3$ satisfying

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$$T(f_1, f_2)(x) = \int_{(\mathbb{R}^n)^2} K(x, y_1, y_2) f_1(y_1) f_2(y_2)\, dy_1 dy_2$$

for all $x \notin \cap_{j=1}^2 \mathrm{supp} f_j$;

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$$|K(y_0, y_1, y_2) - K(y_0, y_1', y_2)| \leq \frac{A|y_1 - y_1'|^{\epsilon}}{\left(\sum_{k,l=0}^{2} |y_k - y_l|\right)^{2n+\epsilon}},$$

for some $\epsilon > 0$ and all $0 \leq j \leq m$, whenever $|y_1 - y_1'| \leq \frac{1}{2} max_{0 \leq k \leq 2} |y_j - y_k|$.

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As a consequence of a "control" of the way...

$$T(f_1, f_2) \preceq Mf_1 Mf_2$$

Theorem (Grafakos and Martell, 2004)

Let $1 < r_1, r_2 < \infty$ *and* $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. *Assume that*

$$\|T(f_1, f_2)\|_{L^r(\nu_{\vec{w}})} \leq C \prod_{i=1}^{2} \|f_i\|_{L^{r_i}(w_i)}$$

holds for all $(w_1, w_2) \in (A_{r_1}, A_{r_2})$. *Then*

$$\|T(f_1, f_2)\|_{L^p(\nu_{\vec{w}})} \leq C \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(w_i)}$$

holds for all $(w_1, w_2) \in (A_{p_1}, A_{p_2})$ *with* $1 < p_1, p_2 < \infty$ *and* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

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Multilinear Muckenhoupt weights

Let $\nu_{\vec{w}} = w_1^{q/q_1} w_2^{q/q_2}$. Let $1 \leq q_1, q_2 < \infty$ and q such that $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. We say that $\vec{w} = (w_1, w_2)$ satisfies the *multilinear* $A_{\vec{q}}$ *condition* if

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Then, if $\vec{w}$ satisfies $A_{\vec{q}}$ a bilinear Calderón-Zygmund operator T also maps $L^{q_1}(w_1) \times L^{q_2}(w_2)$ into $L^q(\nu_{\vec{w}})$

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-Moreover other general properties as monotonicity and (reasonable) factorization are not true for the clases $A_{\vec{q}}$.
-All these facts kept open the *extrapolation theorem* related to multiple $A_{\vec{q}}$ weights...

Extrapolation for multiple $A_{\vec{p}}$ weights

Theorem (K. Li, J. M. Martell, O., 2018)

Let $\mathcal{F}$ be a collection of 3-tuples of non-negative functions. Let $\vec{p} = (p_1, p_2)$, with $1 \leq p_1, p_2 < \infty$, such that given any $\vec{w} \in A_{\vec{p}}$ the inequality

$$\|f\|_{L^p(w)} \leq C([\vec{w}]_{A_{\vec{p}}}) \prod_{i=1}^{2} \|f_i\|_{L^{p_i}(w_i)}$$

holds for every $(f, f_1, f_2) \in \mathcal{F}$, where $\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2}$ and $w := \prod_{i=1}^{2} w_i^{\frac{p}{p_i}}$.

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holds for every $(f, f_1, f_2) \in \mathcal{F}$, where $\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2}$ and $w := \prod_{i=1}^{2} w_i^{\frac{p}{p_i}}$. Then for all exponents $\vec{q} = (q_1, q_2)$, with $q_i > 1$, $i = 1, 2$, and for all weights $\vec{v} \in A_{\vec{q}}$ the inequality

$$\|f\|_{L^q(v)} \leq C([\vec{v}]_{A_{\vec{q}}}) \prod_{i=1}^{2} \|f_i\|_{L^{q_i}(v_i)}$$

holds for every $(f, f_1, f_2) \in \mathcal{F}$, $\frac{1}{q} := \frac{1}{q_1} + \frac{1}{q_2}$ and $v := \prod_{i=1}^{2} v_i^{\frac{q}{q_i}}$.

Moreover, for the same family of exponents and weights, and for all exponents $\vec{s} = (s_1, s_2)$ *with* $s_i > 1$, $i = 1, 2$,

$$\Big\| \Big(\sum_j (f^j)^s \Big)^{\frac{1}{s}} \Big\|_{L^q(v)} \leq C([\vec{v}]_{A_{\vec{q}}}) \prod_{i=1}^{2} \Big\| \Big(\sum_j (f_i^j)^{s_i} \Big)^{\frac{1}{s_i}} \Big\|_{L^{q_i}(v_i)}$$

for all $\{(f^j, f_1^j, f_2^j)\}_j \subset \mathcal{F}$, *where* $\frac{1}{s} := \frac{1}{s_1} + \frac{1}{s_2}$.

New starting point...

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$$w_1^{\frac{1}{2}} \in A_1 \qquad w_2^{\frac{1}{2}} \in A_1 \quad \text{and} \quad w_1^{\frac{1}{2}} w_2^{\frac{1}{2}} \in A_1.$$

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Since p_1 and w_1 are fixed, we can seek for some characterization of w_2 when assuming $\vec{w} \in A_{(p_1,p_2)}$.....

Theorem (D. Carando, M. Mazzitelli, S.O., 2016)

Let T be a bilinear Calderón-Zygmund operator. Let $1 < r \leq 2$ and let $1 < q_1, q_2 < \infty$ if $r = 2$ or $1 < q_1, q_2 < r$ if $1 < r < 2$. Then for $\vec{w} = (w_1, w_2) \in A_{\vec{q}}$ there holds

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bilinear Marcinkiewicz-Zygmund inequalities

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Corollary

Let T be bilinear Calderón-Zygmund operator. Given $1 < r \leq 2$ and $1 < q_1, q_2 < \infty$, then previous inequality holds for all $\vec{w} = (w_1, w_2) \in A_{\vec{q}}$.

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-From A. Culiuc, F. Di Plinio and Y. Ou (2016) we can go to the quasi-Banach range and to recover several recent results of Benea-Muscalu.

¡Muchas gracias!

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y al Madrid también!!!