On the $\mathcal{H}_p$-convergence of Dirichlet series

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joint work with Andreas Defant

ICMAT

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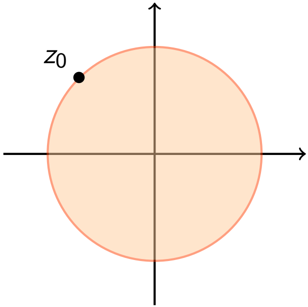
Dirichlet series

$$\sum_{n \in \mathbb{N}} a_n n^{-s}$$

Dirichlet series vs Power series

$$\sum_{n\in\mathbb{N}} a_n n^{-s}$$

$$\sum_{k\geq 0} c_k z^k$$



$$\mathbb{D}(0,|z_0|) := \{\,|z| < |z_0|\,\}$$

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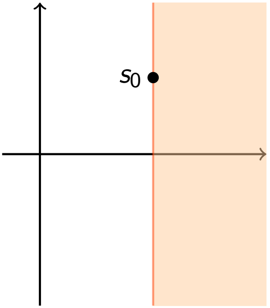
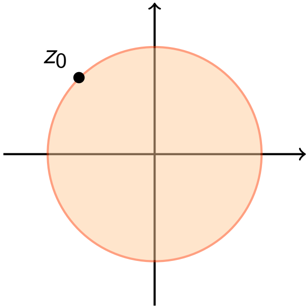


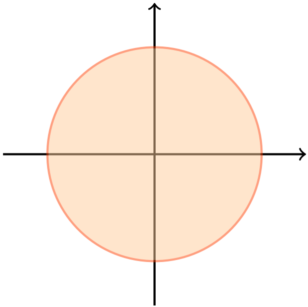


$$\mathbb{C}_{s_0} := \{\, \mathrm{Re}\, s > \mathrm{Re}\, s_0 \,\}$$

$$\mathbb{D}(0, |z_0|) := \{\, |z| < |z_0| \,\}$$

Radius of convergence

$f = \sum_{k \geq 0} c_k z^k$



$$\begin{aligned} R(f) :&= \sup\{r > 0\colon f \text{ converges pointwise on } D(0,r)\} \\ &= \sup\{r > 0\colon f \text{ converges uniformly on } D(0,r)\} \\ &= \sup\{r > 0\colon f \text{ converges absolutely on } D(0,r)\} \end{aligned}$$

Abscissas of convergence

$D = \sum_n a_n n^{-s}$

$$\sigma_c(D) := \inf\{\sigma \colon D \text{ converges pointwise on } \mathbb{C}_\sigma\}$$

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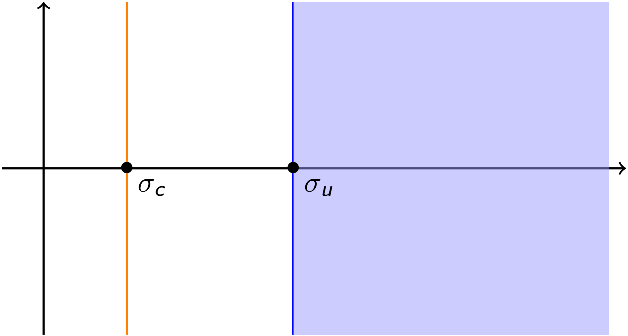

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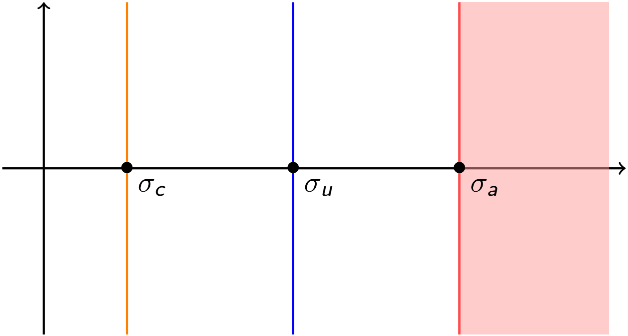

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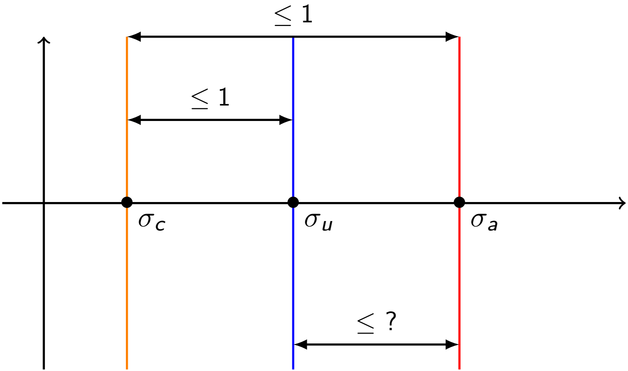

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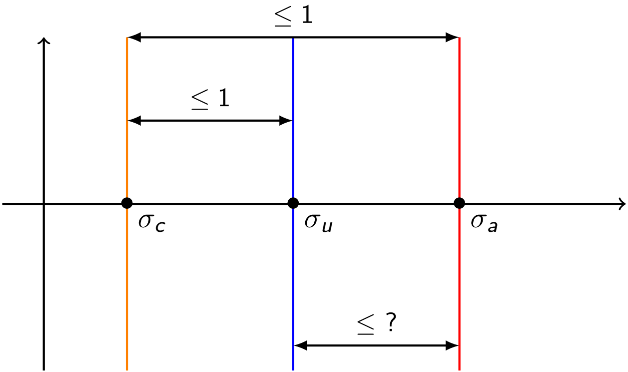

Theorem (Bohr 1914, Bohnenblust, Hille, 1931)

$$\sup_D \{\sigma_a(D) - \sigma_u(D)\} = \frac{1}{2}$$

Absolute vs uniform convergence: a deeper insight

For each $x \geq 1$, let $\mathcal{S}(x)$ be the smallest constant satisfying

$$\sum_{n\leq x} |a_n| \leq \mathcal{S}(x) \sup_{\mathrm{Re}\,(s)>0} \Big| \sum_{n\leq x} \frac{a_n}{n^s} \Big| \qquad \forall\ (a_n)_{n\leq x} \text{ in } \mathbb{C}$$

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Queffélec (1995); Konyagin, Queffélec (2002); Balasubramanian, Calado, Queffélec (2006); de la Bretèche (2007); Maurizi, Queffélec (2010); Defant, Frerick, Ortega-Cerdá, Ounaïes, Seip (2012); Brevig (2014)

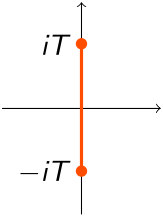
Theorem (Defant, Frerick, Ortega-Cerdá, Ounaïes, Seip, 2012)

$$\mathcal{S}(x) = \sqrt{x} \cdot \exp\left[\left(-\frac{1}{\sqrt{2}} + o(1)\right) \sqrt{\log x \, \log\log x}\right]$$

$\mathcal{H}_p$-norm

Let $D(s) = \sum_n a_n n^{-s}$ be a Dirichlet polynomial

$$\|D\|_{\mathcal{H}_p} := \lim_{T\to+\infty} \left(\frac{1}{2T} \int_{-T}^{T} \left| D(it) \right|^p dt \right)^{\frac{1}{p}} \qquad (1 \le p < +\infty)$$



The notation $\mathcal{H}_p$ refers to the so-called Hardy spaces of Dirichlet series introduced by Hedenmalm, Lindqvist, Seip (1997) and Bayart (2002).

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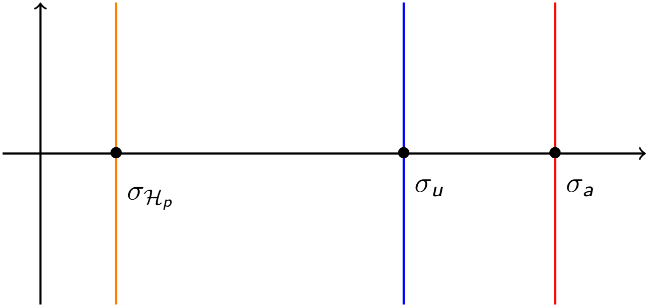

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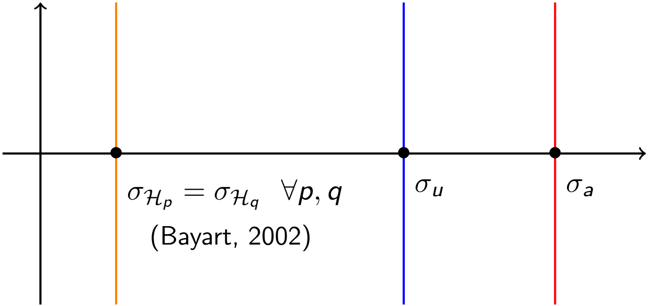

Let $1 \leq p < q < \infty$ and $x \geq 1$.

Define $\mho(q,p,x)$ as the best (i.e. smallest) constant satisfying

$$\Big\| \sum_{n \leq x} \frac{a_n}{n^s} \Big\|_{\mathcal{H}_q} \leq \mho(q,p,x) \cdot \Big\| \sum_{n \leq x} \frac{a_n}{n^s} \Big\|_{\mathcal{H}_p}$$

for every $(a_n)_{n \leq x}$ in $\mathbb{C}$.

Problem

Estimate $\mho(q,p,x)$ asymptotically on $x \to +\infty$.

Particular case: $q = 2$, $p = 1$

$$\Big\| \sum_{n \leq x} \frac{a_n}{n^s} \Big\|_{\mathcal{H}_2} \leq \quad ???? \quad \cdot \Big\| \sum_{n \leq x} \frac{a_n}{n^s} \Big\|_{\mathcal{H}_1}$$

Particular case: $q = 2$, $p = 1$

- $\Big\| \sum_{n \le x} \frac{a_n}{n^s} \Big\|_{\mathcal{H}_2} = \Big(\sum_{n \le x} |a_n|^2 \Big)^{\frac{1}{2}}$ (Carlson, 1922)

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$$\left\| \sum_{n\le x} \frac{a_n}{n^s} \right\|_{\mathcal{H}_2} \le \sqrt{\max_{n\le x} d(n)} \cdot \left\| \sum_{n\le x} \frac{a_n}{n^s} \right\|_{\mathcal{H}_1}$$

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- $$\max_{n\le x} d(n) \le \exp\left[\frac{\log x}{\log\log x} \big(\log 2 + o(1) \big) \right]$$ (Wigert, 1907)

$$\Big\| \sum_{n\le x} \frac{a_n}{n^s} \Big\|_{\mathcal{H}_2} \le \exp\left[\frac{\log x}{\log\log x} \big(\log \sqrt{2} + o(1) \big) \right] \cdot \Big\| \sum_{n\le x} \frac{a_n}{n^s} \Big\|_{\mathcal{H}_1}$$

Bohr's point of view

{Dirichlet series}

$$\sum_n a_n n^{-s}$$

Prime factorization: $n = \mathfrak{p}^{\alpha} := \mathfrak{p}_1^{\alpha_1} \mathfrak{p}_2^{\alpha_2} \dots$

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{DIRICHLET SERIES}

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For every $1 \leq p < +\infty$

$$\Big\| \sum_{n \leq x} a_n n^{-s} \Big\|_{\mathcal{H}_p} = \Big\| \sum_{\mathfrak{p}^\alpha \leq x} a_{\mathfrak{p}^\alpha} z^\alpha \Big\|_{L_p(\mathbb{T}^\mathbb{N})}$$

Reformulation using Bohr's correspondence

Let $1 \leq p < q < \infty$ and $x \geq 1$.

Then, $\mho(q, p, x)$ is the smallest constant satisfying

$$\Big\| \sum_{\mathfrak{p}^\alpha \leq x} c_\alpha z^\alpha \Big\|_{L_q(\mathbb{T}^\mathbb{N})} \leq \mho(q, p, x) \Big\| \sum_{\mathfrak{p}^\alpha \leq x} c_\alpha z^\alpha \Big\|_{L_p(\mathbb{T}^\mathbb{N})}$$

for every $(c_\alpha)_\alpha$ in $\mathbb{C}$.

Comparing L_p and L_q norms

Theorem (Bayart, 2002)

For every $m \in \mathbb{N}$ and every finitely supported family $(c_\alpha)_\alpha$ in $\mathbb{C}$

$$\Big\| \sum_{|\alpha|=m} c_\alpha z^\alpha \Big\|_{L_q} \leq \exp\left[m \cdot \log \sqrt{\frac{q}{p}} \right] \cdot \Big\| \sum_{|\alpha|=m} c_\alpha z^\alpha \Big\|_{L_p}$$

where $|\alpha| := \alpha_1 + \alpha_2 + \dots$

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Theorem (Bayart, 2002 - non-homogenous version)

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where $|\alpha| := \alpha_1 + \alpha_2 + \dots$

Consequence:

$$\Big\| \sum_{n \leq x} \frac{a_n}{n^s} \Big\|_{\mathcal{H}_q} \leq \exp\left(\frac{\log x}{\log 2} \log \sqrt{\frac{q}{p}} \right) \cdot \Big\| \sum_{n \leq x} \frac{a_n}{n^s} \Big\|_{\mathcal{H}_p}$$

Not good enough!! For $p = 1, q = 2$ we got a better estimation!

Decomposition method of Konyagin-Queffélec

Fixed $2 \leq y \leq x$ consider

$$L(x,y) := \{n \leq x \colon \mathfrak{p} | n \Rightarrow \mathfrak{p} > y\}$$
$$S(x,y) := \{n \leq x \colon \mathfrak{p} | n \Rightarrow \mathfrak{p} \leq y\}$$

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Each $n \leq x$ is uniquely written as

$$n = j \cdot k \quad \text{for some} \quad j \in S(x,y), k \in L(x,y)$$

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so that

$$\sum_{n \leq x} \frac{a_n}{n^s} = \sum_{j \in S(x,y)} \Big(\sum_{k \in L(x,y)} \frac{a_{jk}}{k^s} \Big) \frac{1}{j^s}$$

Combining Bayart's result and Konyagin-Queffélec decomposition, we get that for every $2 \leq y \leq x$

$$\mho(q, p, x) \leq \exp\left(\frac{\log x}{\log y} \log\sqrt{\frac{q}{p}}\right) \cdot |S(x, y)|$$

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Theorem (de Bruijn, 1966)

We have, uniformly for $2 \leq y \leq x$,

$$\log|S(x,y)| = Z\left(1 + O\Big(\frac{1}{\log y} + \frac{1}{\log\log 2x}\Big)\right)$$

where

$$Z := \frac{\log x}{\log y}\log\left(1 + \frac{y}{\log x}\right) + \frac{y}{\log y}\log\left(1 + \frac{\log x}{y}\right).$$

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The choice

$$y := \exp\left(\frac{(\log\log x)^2}{\log\log x + \log\log\log x}\right)$$

leads to

$$\mho(q,p,x) \leq \exp\left[\frac{\log x}{\log\log x}\left(\log\sqrt{\frac{q}{p}} + o(1)\right)\right]$$

Lower estimation

There is a family of Dirichlet polynomials

$$D_x(s) = \sum_{n \le x} \frac{a_{n,x}}{n^s}, \quad x \ge 1$$

satisfying

$$\frac{\|D_x\|_{\mathcal{H}_q}}{\|D_x\|_{\mathcal{H}_p}} \ge \exp\left[\frac{\log x}{\log\log x}\left(\log\sqrt{\frac{q}{p}} + o(1)\right)\right]$$

We consider polynomials of the type:

$$D_{k,m}(s) = \Big(\frac{1}{\sqrt{k}} \sum_{j=1}^{k} \mathfrak{p}_j^{-s} \Big)^m \qquad \longleftrightarrow \qquad Q_{k,m}(z) = \Big(\frac{1}{\sqrt{k}} \sum_{j=1}^{k} z_j \Big)^m$$

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Theorem

If $k, m \in \mathbb{N}$ satisfy that $1 < \left[\frac{mp}{2}\right] + 1 < \left[\frac{mq}{2}\right] + 1 < k$, then

$$\frac{\|Q_{k,m}\|_{L_q(\mathbb{T}^{\mathbb{N}})}}{\|Q_{k,m}\|_{L_p(\mathbb{T}^{\mathbb{N}})}} \geq C(p,q)\, m^{\frac{1}{2q}-\frac{1}{2p}} \left(\frac{q}{p}\right)^{m/2} e^{-\frac{4m^2}{k}}$$

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