Operadores integrales de Fourier con una singularidad de tipo Hölder en la fase

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trabajo conjunto con

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Bilbao

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1 Introduction

2 Continuity in L^1

3 Continuity in L^2

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$$K(x,y) = \int_{\mathbb{R}^d} \Phi(u)e^{-2\pi i(\beta(|u|)u\cdot y - u\cdot x)}du, \tag{1}$$

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This integral operator can be seen as a Fourier integral operator, FIO, of type II,

$$T_{II,\varphi,\sigma}f(x) = \int_{\mathbb{R}^d} e^{-2\pi i(\varphi(y,u) - u\cdot x)}\sigma(y,u)f(y)dy\,du,$$

with $\varphi(y,u) = \beta(|u|)u \cdot y$ and $\sigma(y,u) = \Phi(u)$.

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This estimate would imply the corresponding operator A is bounded on $L^1(\mathbb{R}^d)$.

Usually, the function $\Phi(u)$ has a good decay at infinity but could be not smooth at the origin $u = 0$. A typical example is given by radial functions

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with large real m. In particular, this $\Phi(u)$ belongs to $W(\mathcal{F}L^1, L^1)(\mathbb{R}^d)$, when $m > (d+1)/2$.

Definition

The **Wiener amalgam space** $W(\mathcal{F}L^1, L^1)(\mathbb{R}^d)$ with local component $\mathcal{F}L^1(\mathbb{R}^d)$ and global component $L^1(\mathbb{R}^d)$ is defined as the space of all functions f such that

$$\|f\|_{W(\mathcal{F}L^1, L^1)} = \| \|fT_x g\|_{\mathcal{F}L^1} \|_{L^1} < \infty,$$

where $g \in C_0^\infty(\mathbb{R}^d)$.

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As an oversimplified model we can consider the case

$$\beta(r) = a + br^{\gamma}, \quad 0 < r \leq 1,$$

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and the estimate

$$\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} |K(x, y)|\, dx < \infty$$

holds if, and only if, $\Phi \in \mathcal{F}L^1(\mathbb{R}^d)$, i.e. , Φ has Fourier transform in $L^1(\mathbb{R}^d)$.

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$$Af = \mathcal{F}^{-1}\Phi * \mathcal{F}^{-1}(\mathcal{F}f \circ \tilde{\varphi}), \quad \text{with} \quad \tilde{\varphi}(u) := \beta(|u|)u.$$

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The Beurling-Helson phenomenon, roughly speaking, states that the change-of-variable operator $f \mapsto f \circ \psi$ is not bounded on $\mathcal{F}L^1(\mathbb{R}^d)$ except in the case that $\psi : \mathbb{R}^d \to \mathbb{R}^d$ is an affine mapping.

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Then, the kernel K is well-defined for every $x, y \in \mathbb{R}^d$. Indeed, since $W(\mathcal{F}L^1, L^1)(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^d)$, the integral in the kernel is absolutely convergent.

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$$Af(x) = \int_{\mathbb{R}^{2d}} e^{-2\pi i(\beta(|u|)u\cdot y - x\cdot u)}\Phi(u)f(y)\, dy\, du.$$

Theorem

Consider functions $\Phi \in W(\mathcal{F}L^1, L^1)(\mathbb{R}^d)$ *and* $\beta : (0, +\infty) \to \mathbb{R}$. *Moreover, assume that for some exponent* $\gamma \in (-1, 1]$, *with* $l = \lfloor d/2 \rfloor + 1$,

$$|\partial^\alpha \beta(|u|)u| \leq C_\alpha |u|^{\gamma+1-|\alpha|}, \qquad \textit{for} \;\; 0 \neq |u| \leq 1, \; |\alpha| \leq l,$$

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with $C'_\alpha > 0$. *Then, the integral kernel in* (1) *satisfies*

$$\int_{\mathbb{R}^d} |K(x, y)| dx \leq C(1 + |y|)^{d/(\gamma+1)},$$

for a suitable constant $C > 0$ *independent of* y.

In our oversimplified model,

$$\beta(r) = a + br^{\gamma}, \quad 0 < r \leq 1,$$

for some $a, b \in \mathbb{R}$, with $\gamma \in (-1, 1]$. And assume β has at most linear growth as $r \to +\infty$.

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Consider functions $\Phi \in W(\mathcal{F}L^1, L^1)(\mathbb{R}^d)$ *and* $\beta : (0, +\infty) \to \mathbb{R}$. *Moreover, setting* $l = \lfloor d/2 \rfloor + 1$, *assume that the function* $\beta(|u|)$ *extends to a* $\mathcal{C}^{2l}$ *function on* $\mathbb{R}^d$ *and satisfies*

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Then, the integral kernel in (1) *satisfies*

$$\int_{\mathbb{R}^d} |K(x,y)|dx \leq C(1+|y|)^{\frac{d}{2}}.$$

Suppose that $\tilde{\varphi}(u) := \beta(|u|)u$ extends to a smooth function in $\mathbb{R}^d$, with an at most quadratic growth at infinity. Let

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Then the integral operator A *with kernel* K *is bounded,*

$$A : L^1_v(\mathbb{R}^d) \to L^1(\mathbb{R}^d).$$

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We now show that A is not bounded on $L^2(\mathbb{R}^d)$.
For $f \in \mathcal{S}(\mathbb{R})$,

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Then, by Parseval's Theorem,

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Then, by Parseval's Theorem,

$$\|Af\|_2^2 = \|h(|u|)\mathcal{F}f(\beta(|u|)u)\|_2^2 = \int_{\mathbb{R}} |h(|u|)|^2|\mathcal{F}f(\beta(|u|)u)|^2du.$$

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so that

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and $du = \frac{1}{1+\gamma}|\tilde{u}|^{\frac{1}{1+\gamma}-1} d\tilde{u}$.

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$$u = \begin{cases} \tilde{u}^{\frac{1}{\gamma+1}}, & \tilde{u} \geq 0, \\ -(-\tilde{u})^{\frac{1}{\gamma+1}}, & \tilde{u} < 0, \end{cases}$$

and $du = \frac{1}{1+\gamma}|\tilde{u}|^{\frac{1}{1+\gamma}-1}d\tilde{u}$. In this way, we obtain

$$\|Af\|_2^2 = \int_{\mathbb{R}} |h(|u|)|^2 |\mathcal{F}f(\beta(|u|)u)|^2 du$$
$$= \frac{1}{1+\gamma}\int_{\mathbb{R}} |\tilde{u}|^{\frac{1}{1+\gamma}-1} |h(|\tilde{u}|^{\frac{1}{1+\gamma}})|^2 |\mathcal{F}f(\tilde{u})|^2 d\tilde{u}.$$

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Now, the last expression is controlled by $C\|f\|_{L^2}^2$, for a suitable constant $C > 0$ and for every $f \in \mathcal{S}(\mathbb{R})$, if and only if

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$h(|\tilde{u}|^{\frac{1}{1+\gamma}})$ has compact support, and this fails because $-\gamma/(1+\gamma) < 0$ and $|h(|u|)| \geq \delta > 0$ in a neighbourhood of 0.

Theorem

Consider $\Phi \in L^{\infty}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. *Let* $\beta : (0,\infty) \to \mathbb{R}$ *satisfy the following assumptions:*
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Then the integral operator A *with kernel* K *in* (1) *is bounded on* $L^2(\mathbb{R}^d)$.

In our simplified model,

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Then, if $a(a + (\gamma + 1)b) > 0$, the operator A is bounded in $L^2(\mathbb{R}^d)$.

Remark

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