Dominación sparse para operadores singulares

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basque center for applied mathematics

Outline

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Basic operators

Definition (Hardy-Littlewood maximal operator)

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|dy \qquad f \in L^1_{loc}(\mathbb{R}^n)$$

Where each Q is a cube of $\mathbb{R}^n$ with sides parallel to the axis.

Definition (Hilbert/Riesz Transforms (Calderón-Zygmund operators))

- $Hf(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}dy \qquad f \in \mathcal{S}(\mathbb{R})$
- $R_jf(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} \frac{x_j-y_j}{|x-y|^{n+1}}f(y)dy \qquad f \in \mathcal{S}(\mathbb{R}^n)$

Some facts

- $f(x) \leq Mf(x)$ a.e $x \in \mathbb{R}^n$
- $\|Gf\|_{L^p} \leq c_{G,p}\|f\|_{L^p} \qquad 1 < p < \infty$
- $|\{x \in \mathbb{R}^n : |Gf| > t\}| \leq c_G \int_{\mathbb{R}^n} \frac{|f|}{t} \qquad t > 0$

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Qualitative weighted estimates

Definition

We say that w is a weight if it is a non-negative locally integrable function.

Theorem (Muckenhoupt)

A_p condition:

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}dx\right)^{\frac{1}{p-1}} < \infty \quad p>1$$

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Provided $w \in A_p$ the same estimates hold for Calderón-Zygmund operators

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A change of perspective

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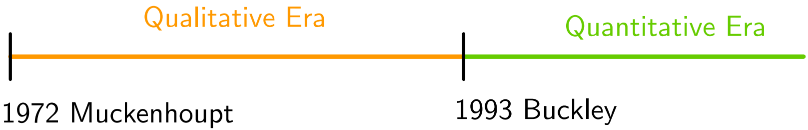

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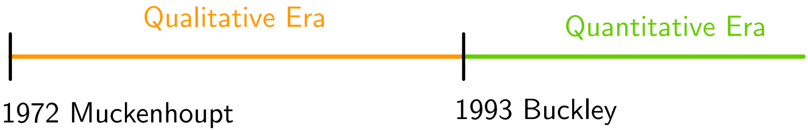

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Let $1 < p < \infty$ *and* $w \in A_p$. *Then*

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Remark

The preceding estimates are sharp, in the sense that if we consider $\Psi(t)$ such that $\frac{\Psi(t)}{\varphi_{T,p}(t)} \to 0$ $t \to \infty$ the estimates don't hold in general.

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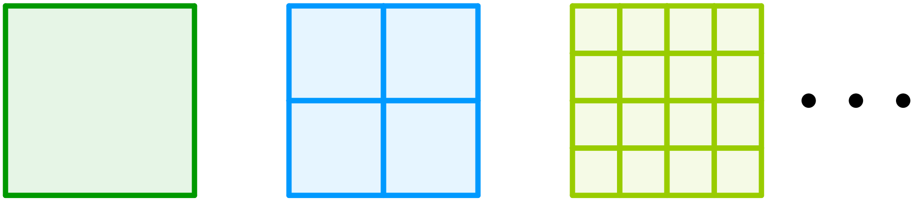
Standard dyadic grids



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Standard dyadic grid $\mathcal{D}(Q)$

- We define $\mathcal{D}(Q) = \bigcup_{i=0}^{\infty} \mathcal{D}_i(Q)$.
- Given $P, R \in \mathcal{D}(Q)$ we have that $P \cap R = P, R, \emptyset$

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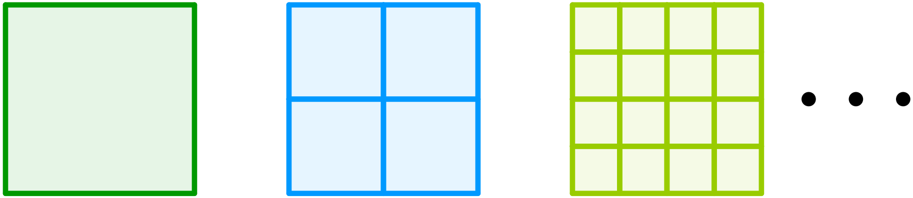

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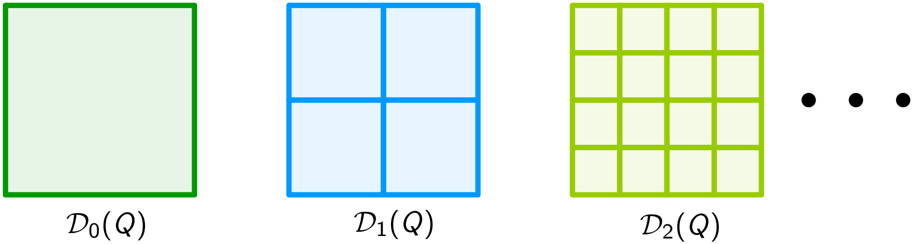

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Dyadic Lattice

Let $\{Q_j\}$ a sequence of dyadic cubes expanding each time from a different vertex. Then $\mathbb{R}^n = \cup_j Q_j$. We will call $\mathcal{D} = \bigcup_j \{Q \in \mathcal{D}(Q_j)\}$ a dyadic lattice.

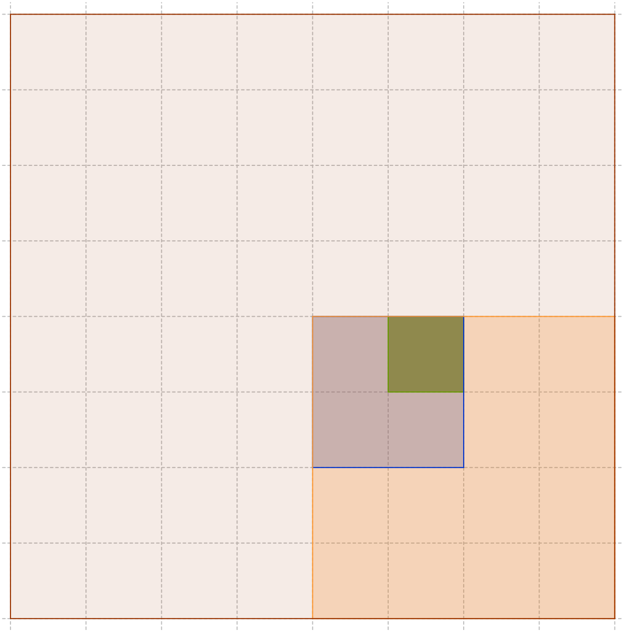

Figure: Construction of a dyadic lattice

Sparse families

Definition

Let $\mathcal{S} \subset \mathcal{D}$. We say that $\mathcal{S}$ is η-sparse $(0 < \eta < 1)$ if for every $Q \in \mathcal{S}$ there exists a measurable subset E_Q such that:

1. The sets E_Q are pairwise disjoint.
2. $\eta|Q| \leq |E_Q|$.

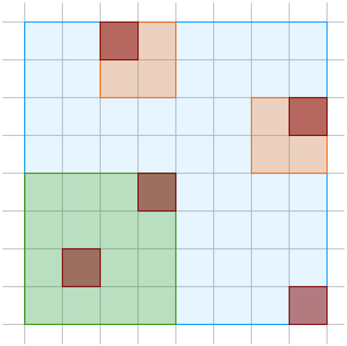

Figure: Example of $\frac{1}{2}$-sparse family

Remark

From the definition of sparse family it follows that, for each cube, the set of points for which there exist infinite cubes in $\mathcal{S}$ containing that point has zero measure.

Sparse domination for Calderón-Zygmund operators

Definition

Let $\mathcal{D}$ a dyadic lattice $\mathcal{S} \subset \mathcal{D}$ and $f \in L^\infty(\mathbb{R}^n)$ with compact support

$$A_{\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q f(y)dy \right) \chi_Q(x).$$

Remarks

- $A_{\mathcal{S}}$ is well defined a.e. (sparseness condition)
- $A_{\mathcal{S}}$ is positive and self-adjoint

Theorem (Lerner, Nazarov, Conde-Alonso, Rey, Lacey, Hytönen, Roncal, Tapiola)

Let T a Calderón-Zygmund operator. For every compactly supported $f \in L^\infty(\mathbb{R}^n)$, there exist 3^n-dyadic lattices $\mathcal{D}^{(j)}$ and $\frac{1}{2 \cdot 9^n}$-sparse families $\mathcal{S}_j \subset \mathcal{D}^{(j)}$ such that for a.e. $x \in \mathbb{R}^n$

$$|Tf(x)| \lesssim \sum_{j=1}^{3^n} \mathcal{A}_{\mathcal{S}_j} |f|(x).$$

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Sparse domination for Calderón-Zygmund operators

Definition

Let $\mathcal{D}$ a dyadic lattice $\mathcal{S} \subset \mathcal{D}$ and $f \in L^\infty(\mathbb{R}^n)$ with compact support

$$A_{\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q f(y)dy \right) \chi_Q(x).$$

Remarks

- $A_{\mathcal{S}}$ is well defined a.e. (sparseness condition)
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A new philosophy

Calderón-Zygmund principle

For every operator G there exists a suitable maximal operator M_G such that for "good weights" w

$$\|Gf\|_{L^p(w)} \leq c_{G,n,w}\|M_G f\|_{L^p(w)}$$

Sparse domination principle

For every operator G there exists a finite family of sparse operators $T_{\mathcal{S}_1}, \dots T_{\mathcal{S}_j}$ such that

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Outline

Commutators

Definition

Let T a Calderón-Zygmund operator and b a locally integrable function. We define the commutator $[b, T]$ as

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x)$$

Usually b is assumed to be in BMO.

Definition

We say a locally integrable function b has bounded mean oscillation, $b \in BMO$ if

$$\|b\|_{BMO} = \sup_Q \frac{1}{|Q|}\int_Q |b(x) - b_Q|dx < \infty \qquad \text{where} \qquad b_Q = \frac{1}{|Q|}\int_Q b(x)dx$$

Theorem (John-Nirenberg)

For every $b \in BMO$ and every cube $Q \subset \mathbb{R}^n$,

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Boundedness of commutators

Theorem (Coifman,Rochberg,Weiss 1976)

Let T a Calderón-Zygmund operator and $b \in BMO$ then $[b, T]$ is bounded on L^p
$\Leftarrow$ *Hilbert, Riesz transforms*

Remark (Pérez 1995)

$[b, T]$ is not of weak type $(1, 1)$.

We have the following replacement:

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Let T a CZO and $b \in BMO$ then

$$|\{x \in \mathbb{R}^n : |[b, T]f(x)| > t\}| \leq c \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO} \frac{|f(x)|}{t}\right) dx$$

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Theorem (Lerner, Ombrosi, R-R - 2016)

Let T a Calderón-Zygmund operator and $b \in L^1_{loc}(\mathbb{R}^n)$. For every compactly supported $f \in L^\infty(\mathbb{R}^n)$, there exist 3^n-dyadic lattices $\mathcal{D}^{(j)}$ and $\frac{1}{2 \cdot 9^n}$-sparse families $\mathcal{S}_j \subset \mathcal{D}^{(j)}$ such that for a.e. $x \in \mathbb{R}^n$

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Consequences of sparse domination for commutators

Theorem (Lerner,Ombrosi,R-R 2016)

Let T a Calderón-Zygmund operator. If $\mu, \lambda \in A_p$, $1 < p < \infty$, $\nu = \left(\frac{\mu}{\lambda}\right)^{\frac{1}{p}}$ and $b \in BMO_\nu(\mathbb{R})$ where BMO_ν is the space of locally integrable functions b such that

$$\|b\|_{BMO_\nu} = \sup_Q \frac{1}{\nu(Q)} \int_Q |b - b_Q| dx < \infty$$

then

$$\|[b, T]f\|_{L^p(\lambda)} \leq c_n C_T \left([\mu]_{A_p}[\lambda]_{A_p}\right)^{\max\left\{1, \frac{1}{p-1}\right\}} \|b\|_{BMO_\nu} \|f\|_{L^p(\mu)}.$$

Remark

For $\mu = \lambda$ the result is due to Chung, Pereyra, Pérez 2010.

Theorem (Lerner,Ombrosi,R-R 2016)

Let T a Calderón-Zygmund operator and $b \in BMO$ then, if $w \in A_1$

$$w\left(\{x \in \mathbb{R}^n : |[b, T]f(x)| > t\}\right) \leq c[w]_{A_1}^2 \log(e+[w]_{A_1}) \int_{\mathbb{R}^n} \Phi\left(\|b\|_{BMO} \frac{|f(x)|}{t}\right) w(x)dx$$

where $t > 0$ and $\Phi(t) = t\log(e + t)$.

Consequences of sparse domination for commutators

Theorem (Lerner,Ombrosi,R-R 2016)

Let T a Calderón-Zygmund operator. If $\mu,\lambda\in A_p$, $1<p<\infty$, $\nu=\left(\frac{\mu}{\lambda}\right)^{\frac{1}{p}}$ and $b\in BMO_\nu(\mathbb{R})$ where BMO_ν is the space of locally integrable functions b such that

$$\|b\|_{BMO_\nu}=\sup_Q\frac{1}{\nu(Q)}\int_Q|b-b_Q|dx<\infty$$

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Consequences of sparse domination for commutators

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THANK
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YAQHANYELAY
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