The Müntz-Szász Theorem and some extensions

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Introduction

Müntz-Szász Theorem

Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots .$$

Then, the collection of finite linear combinations of the functions $t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3}, \dots$, i.e., the set

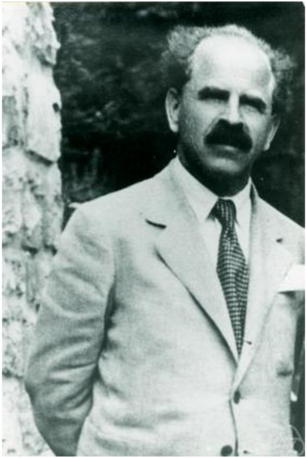
$$\text{span}\{1, t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3}, \dots\}$$

is dense in $C[0,1]$ if and only if

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = +\infty.$$

Introduction

(a) Herman Müntz (1884-1956)



(b) Otto Szász (1884-1952)

1. The Weierstrass Approximation Theorem ([CMOR])

2. Müntz-Szász Theorem ([EMMS, R])

3. The Full Müntz Theorem in $L^2[0,1]$, $C[0,1]$ and $L^1[0,1]$ ([BE])

4. The Full Müntz Theorem in $L^p[0,1]$ ([0])

5. An application of the Müntz-Szász Theorem ([LLPZ])

Bibliography

1. The Weierstrass Approximation Theorem ([CMOR])

Target: To provide a proof of the classical Weierstrass Approximation Theorem (with the $\|\cdot\|_\infty$) on compact sets in the real line.

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Theorem (Korovkin, 1953)

Let $f_0, f_1, f_2 \colon [a,b] \to \mathbb{C}$ defined by

$$f_0(t) = 1, \quad f_1(t) = t, \quad \text{and} \quad f_2(t) = t^2,$$

for $t \in [a,b]$. For $n \geq 1$, let $P_n \colon C[a,b] \to C[a,b]$ a linear operator. Suppose that:

1. Each P_n is positive, i.e., $P_n f \geq 0$ if $f \geq 0$;
2. for $m = 0,1,2$, it satisfies $\lim_{n\to\infty} \|P_n f_m - f_m\|_\infty = 0$.

Then,

$$\lim_{n\to\infty} \|P_n f - f\|_\infty = 0,$$

where $f \in C[a,b]$.

Proof.
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Let $f \in C[a, b]$ a real-valued function and $\alpha > 0$ such that $\|f\|_\infty \le \alpha$.
Let $t, s \in [a, b]$, then,

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$$-2\alpha \leq f(t) - f(s) \leq 2\alpha. \tag{1}$$

Fixed $\varepsilon > 0$. Note that f is uniformly continuous on $[a,b]$. Hence there exists $\delta(\varepsilon) > 0$ such that if $t, s \in [a,b]$ with $|t-s| < \delta$, then

$$-\varepsilon \leq f(t) - f(s) \leq \varepsilon. \tag{2}$$

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Fixed $s \in [a,b]$, define $g_s(t) = (t-s)^2$. If $t, s \in [a,b]$ and $|t-s| \geq \delta$, then $g_s(t) \geq \delta^2$.

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Fixed $s \in [a, b]$, define $g_s(t) = (t - s)^2$. If $t, s \in [a, b]$ and $|t - s| \geq \delta$, then $g_s(t) \geq \delta^2$. Now, combining (1) y (2),

$$-\varepsilon - 2\alpha \frac{g_s(t)}{\delta^2} \leq f(t) - f(s) \leq \varepsilon + 2\alpha \frac{g_s(t)}{\delta^2},$$

for every $t, s \in [a, b]$.

Since P_n is linear and positive,

$$-\varepsilon P_n f_0 - 2\alpha \frac{P_n g_s}{\delta^2} \leq P_n f - f(s) P_n f_0 \leq \varepsilon P_n f_0 + 2\alpha \frac{P_n g_s}{\delta^2}.$$

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By hypothesis, $P_n f_0(s) \to 1$ uniformly in $s \in [a, b]$. Moreover, $P_n g_s(s) \to 0$ uniformly on $[a, b]$.

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By hypothesis, $P_n f_0(s) \to 1$ uniformly in $s \in [a, b]$. Moreover, $P_n g_s(s) \to 0$ uniformly on $[a, b]$. Indeed,

$$g_s = f_2 - 2sf_1 + s^2 f_0$$

and

$$\begin{aligned}\lim_{n\to\infty} P_n g_s(s) &= \lim_{n\to\infty} P_n f_2(s) - 2sP_n f_1(s) + s^2 P_n f_0(s) \\ &= s^2 - 2ss + s^2 1 = 0\end{aligned}$$

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uniformly. Therefore,

$$P_n f(s) \longrightarrow f(s)$$

uniformly in $s \in [a, b]$, as we desired. □

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Firstly, due to the change of variable $t \mapsto a + t(b-a)$, one can suppose, without loss of generality, that $[a,b] = [0,1]$. Consider, for $n \geq 1$, the operator

$$\begin{aligned} B_n\colon C[0,1] &\longrightarrow C[0,1] \\ f &\longmapsto B_n f(t) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} t^k (1-t)^{n-k}. \end{aligned}$$

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Such $B_n f$ is called the $n-$th Berstein's polynomial associated to f.

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$$\begin{aligned} B_n f_0 &= f_0, \\ B_n f_1 &= f_1, \\ B_n f_2 &= \left(1 - \frac{1}{n}\right) f_2 + \frac{1}{n} f_1, \end{aligned} \tag{3}$$

for $n \geq 1$, that implies

$$\lim_{n\to\infty} \|B_n f_m - f_m\|_\infty = 0,$$

for $m = 0, 1, 2$.

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for $n \geq 1$, that implies

$$\lim_{n\to\infty} \|B_n f_m - f_m\|_\infty = 0,$$

for $m = 0, 1, 2$. Finally, we need to proof the truthfulness of (3).

If $t \in [0,1]$, then,

$$B_n f_0(t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} = (t+1-t)^n = 1,$$

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and

$$\begin{aligned} B_n f_2(t) &= \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \binom{n}{k} t^k (1-t)^{n-k} = \sum_{k=1}^{n} \frac{k}{n} \binom{n-1}{k-1} t^k (1-t)^{n-k} \\ &= \sum_{k=1}^{n} \left[\frac{(n-1)(k-1)}{n(n-1)} + \frac{1}{n}\right] \binom{n-1}{k-1} t^k (1-t)^{n-k} \\ &= \left(1 - \frac{1}{n}\right) t^2 + \frac{1}{n} t = \left(1 - \frac{1}{n}\right) f_2(t) + \frac{1}{n} f_1(t). \end{aligned}$$

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This finishes the proof. □

2. Müntz-Szász Theorem ([EMMS, R])

Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a strictly increasing sequence of positive numbers. Then, the collection of finite linear combinations of functions $1, t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3}, ...$, that is $\text{span}\{1, t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3}, ...\}$, is dense in $C[0,1]$ if and only if

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Theorem

Let $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots$ and

$$X = \overline{\operatorname{span}}\{1, t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3}, \ldots\}$$

a) If $\sum_{n=1}^{\infty} 1/\lambda_n = +\infty$, then $X = C[0,1]$.

b) If $\sum_{n=1}^{\infty} 1/\lambda_n < +\infty$ and $\lambda \notin \{\lambda_n\}$, $\lambda \neq 0$, then $t^{\lambda} \notin X$.

Proposition

If $\sum_{n=1}^{\infty} 1/\lambda_n = \infty$, μ is a Borel complex measure on $[0,1]$ and T is the bounded linear functional on $C[0,1]^* \cong M[0,1]$ associated to μ such that

$$T(t^{\lambda_n}) = \int_0^1 t^{\lambda_n} d\mu(t) = 0, \quad n = 1, 2, 3, \ldots \tag{4}$$

then

$$T(t^k) = \int_0^1 t^k d\mu(t) = 0, \quad k = 1, 2, 3, \ldots \tag{5}$$

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Proof.
Since the integrand in (4) and (5) cancels on $t = 0$, we can assume that μ concentrates on $(0, 1]$.
Let's consider the function

$$f(z) = \int_0^1 t^z d\mu(t) = \int_0^1 e^{z \log t} d\mu(t).$$

It is well defined on the right complex semiplane $\mathbb{H}_0$:

$$|f(z)| \leq \int_0^1 |e^{z \ \log t}| d|\mu|(t) = \int_0^1 t^{\Re(z)} d|\mu|(t) \leq ||\mu|| < +\infty. \qquad (6)$$

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In addition we have

$$f(z) - f(z_0) = \int_0^1 (t^z - t^{z_0}) d\mu(t)$$
$$\Rightarrow |f(z) - f(z_0)| \leq \int_0^1 |t^z - t^{z_0}| d|\mu|(t)$$

Then fixed $\varepsilon > 0$, since t^z is continuous on $[0,1] \times \mathbb{H}_0$ (uniformly on t, because $[0,1]$ is compact) exists $\delta(\varepsilon) > 0$ such that if $|z - z_0| < \delta$, then $|t^z - t^{z_0}| < \varepsilon, \forall t \in [0,1]$. Thus,

$$|f(z) - f(z_0)| \leq \varepsilon \int_0^1 d|\mu|(t) = \varepsilon ||\mu||$$

which proves the continuity of f.

Let γ a C^1 closed path on $\mathbb{H}_0$. Then, by Fubini Theorem and since $z \mapsto t^z$ is holomorphic by Cauchy Theorem we have

$$\oint_\gamma f(z)dz = \oint_\gamma \int_0^1 t^z d\mu(t)dz = \int_0^1 \oint_\gamma t^z dz\, d\mu(t) = 0.$$

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On the other hand, on (6) we have proved that f is bounded on $\mathbb{H}_0$.

Let's consider now the composition of f with a Möbius transformation of the disc onto the right semiplane

$$g(z) = f\left(\frac{1+z}{1-z}\right), \quad z \in \mathbb{D}.$$

Notice that $g \in H^\infty$, this is,

- $g \in \mathcal{H}(\mathbb{D})$,
- g is bounded on $\mathbb{D}$, because f is bounded.

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By hypothesis (4) we have $f(\lambda_n) = T(t^{\lambda_n}) = 0,\ n = 1, 2, \ldots$, therefore $g(\alpha_n) = 0$, where $\alpha_n = \dfrac{\lambda_n - 1}{\lambda_n + 1}$.

We claim that $\sum_{n=1}^{\infty} 1/\lambda_n = +\infty \Rightarrow \sum_{n=1}^{\infty} 1 - |\alpha_n| = +\infty$. In fact

$$\sum_{n=1}^{\infty} 1 - \left|\frac{\lambda_n - 1}{\lambda_n + 1}\right| = \sum_{n=1}^{\infty} \frac{\lambda_n + 1 - |\lambda_n - 1|}{\lambda_n + 1}.$$

There are two possible cases:

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There are two possible cases:

- If $0 < \lambda_n < 1, \ \forall n \in \mathbb{N}$, then $\lambda_n + 1 - |\lambda_n - 1| = 2\lambda_n$. Thus

$$\sum_{n=1}^{\infty} 1 - |\alpha_n| = \sum_{n=1}^{\infty} \frac{2\lambda_n}{\lambda_n + 1} = +\infty$$

since $\dfrac{2\lambda_n}{\lambda_n + 1} \nrightarrow 0$, when $n \to \infty$.

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- If $\exists m \in \mathbb{N}$ such that $\lambda_n \geq 1, \forall\ n \geq m$ then $\lambda_n + 1 - |\lambda_n - 1| = 2$. Thus,

$$\sum_{n=1}^{\infty} 1 - |\alpha_n| \geq \sum_{n=m}^{\infty} \frac{2}{\lambda_n + 1} = +\infty.$$

Theorem ([R, Theorem 15.23])

If $f \in H^\infty$ and $\alpha_1, \alpha_2, \ldots$ are the zeros of f in $\mathbb{D}$ and if

$$\sum_{n=1}^{\infty} 1 - |\alpha_n| = +\infty$$

then $f(z) = 0$ for all $z \in \mathbb{D}$.

We deduce that $g(z) = 0$, $\forall z \in \mathbb{D}$. In particular,

$$T(t^k) = \int_0^1 t^k d\mu(t) = f(k) = g\left(\frac{k-1}{k+1}\right) = 0, \quad k = 1, 2, \ldots$$

□

Proof.
Let's proof *a*):
By Weierstrass Approximation Theorem it is enough to see that X contains all the functions t^k, with $k = 1, 2, 3, ...$

Suppose that $\exists\ k_0 \in \mathbb{N}$ such that $t^{k_0} \notin X$. By Hahn-Banach Theorem exists a bounded linear functional $T : C[0,1] \longrightarrow \mathbb{R}$ such that

$$T(t^{k_0}) \neq 0 \text{ and } T|_{\text{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}} \equiv 0.$$

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Riesz Representation Theorem

The space of Borel regular complex measures, $M(I)$, is the dual space of $C(I)$ via

$$M(I) \longrightarrow C(I)^*$$
$$\mu \longmapsto \left(\varphi \mapsto < \varphi, \mu > = \int_0^1 \varphi d\mu\right) = \langle \cdot, \mu \rangle.$$

Since T verifies the hypothesis of Riesz Representation Theorem, exists a Borel complex measure μ such that

$$T(\varphi) = \int_0^1 \varphi(t) d\mu(t), \quad \varphi \in C[0,1],$$

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$$T(\varphi) = \int_0^1 \varphi(t) d\mu(t), \quad \varphi \in C[0,1],$$

satisfying in addition

1. $T(t^{k_0}) = \int_0^1 t^{k_0} d\mu(t) \neq 0;$
2. $T(t^{\lambda_n}) = \int_0^1 t^{\lambda_n} d\mu(t) = 0, \quad n = 1, 2, \ldots.$

Since T verifies the hypothesis of Riesz Representation Theorem, exists a Borel complex measure μ such that

$$T(\varphi) = \int_0^1 \varphi(t) d\mu(t), \quad \varphi \in C[0,1],$$

satisfying in addition

1. $T(t^{k_0}) = \int_0^1 t^{k_0} d\mu(t) \neq 0;$
2. $T(t^{\lambda_n}) = \int_0^1 t^{\lambda_n} d\mu(t) = 0, \quad n = 1, 2, \ldots.$

By the previous proposition we have that $T(t^{k_0}) = 0$ and $T(t^{k_0}) \neq 0$.

Thus $t^k \in X$ for all $k \in \mathbb{N}$. This completes the proof of a).

Let's prove b). We assume

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty.$$

Our goal is to construct a functional $T = \langle \cdot, \mu \rangle \in C[0,1]^*$ such that $T(t^{\lambda_n}) = 0$ for all $n \in \mathbb{N}_0$ $(\lambda_0 = 0)$ that does not vanish on t^{λ} for each positive λ with $\lambda \notin \{\lambda_n\}_{n \in \mathbb{N}_0}$.

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We are looking for a Borel complex measure μ in $[0,1]$ such that

$$\int_0^1 t^z d\mu(t)$$

define a bounded holomorphic function f on $\mathbb{H}_{-1} := \{z \in \mathbb{D} : \Re(z) > -1\}$ with zeros at $\{\lambda_n\}$.

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$\mathbb{H}_{-1} := \{z \in \mathbb{D} : \Re(z) > -1\}$ with zeros at $\{\lambda_n\}$.
We choose

$$f(z) = \frac{z}{(2+z)^3} \prod_{n=1}^{\infty} \frac{\lambda_n - z}{2 + \lambda_n + z}, \quad z \in \mathbb{C} \setminus \{-2 - \lambda_n\}_{n \in \mathbb{N}}.$$

Now we prove that f is a meromorphic function on $\mathbb{C}$ with poles at $\{-2-\lambda_n\}$. It is enough to check that

$$\sum_{n=1}^{\infty}\left|1-\frac{\lambda_n - z}{2+\lambda_n+z}\right| \tag{7}$$

converges uniformly on every compact subset K on $\mathbb{C}\setminus\{-2-\lambda_n\}_{n\in\mathbb{N}}$.

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Fix K compact set. There exists $\alpha > 0$ such that $K \subset \mathbb{H}_{-\alpha} = \{z \in \mathbb{C} : \Re(z) > -\alpha\}$. As $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\lambda_n}$ is a convergent series of positive terms, it is easy to see that there exists $C_K > 0$ and $N \in \mathbb{N}$ such that for all $n > N$

$$\left| \frac{2z+2}{2+\lambda_n+z} \right| \leq \frac{C_K}{2+\lambda_n-\alpha}.$$

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$$\left|\frac{2z+2}{2+\lambda_n+z}\right|\leq\frac{C_K}{2+\lambda_n-\alpha}.$$

Hence, using the Weierstrass criterion and the convergence of the series $\sum_{n=1}^{\infty}\frac{1}{\lambda_n}$ it follows the uniform convergence of (7) on K.

We claim that f is bounded on $\mathbb{H}_{-1}$. We observe all terms in the infinite product and the factor $\frac{z}{2+z}$ are on $\mathbb{D}$, because they are a Möbius transform from $\mathbb{H}_{-1}$ onto the disk. Moreover,

$$\frac{1}{|2+z|^2} \leq 1, \quad \forall z \in \mathbb{H}_{-1},$$

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what proves our claim.

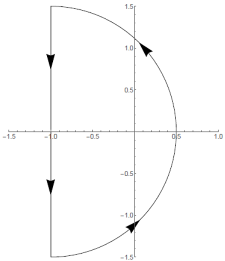
Using the previous bound we deduce that $f \in L^1(\{z \in \mathbb{C} : \Re(z) = -1\})$, since

$$\int_{\mathbb{R}} |f(-1+it)|dt \leq \int_{\mathbb{R}} \frac{dt}{1+t^2} = \pi.$$

Our next step is to represent f using Cauchy Theorem. Given $z_0 \in \mathbb{H}_{-1}$, we will have

$$f(z_0) = \int_C \frac{f(z)}{z - z_0} dz,$$

where C is the semicircumference with center -1 and radium $R > 1 + |z|$, with extreme points $-1 - iR$, $-1 + R$ and closed by the segment that links these points, as we can see in the figure.



If we parameterize the curve, we get

$$f(z_0) = \frac{1}{2\pi}\int_{-R}^{R} \frac{f(-1+is)}{1-is+z_0}ds + \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{f(-1+Re^{i\theta})}{-1+Re^{i\theta}-z_0}Re^{i\theta}d\theta.$$

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It is easy to see using $|f(z)| \leq \left|\frac{z}{2+z^3}\right|$ that if $R \to \infty$, the second term on the sum goes to 0. Therefore, we obtain the following expresion for f:

$$f(z_0) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{f(-1+is)}{1-is+z_0} ds$$

for all $z_0 \in \mathbb{H}_{-1}$.

Due to the identity

$$\frac{1}{z - is + 1} = \int_0^1 t^{z-is} dt = \int_0^1 t^z e^{-is \log t} dt$$

and Fubini Theorem, we can write for each $z \in \mathbb{H}_{-1}$

$$f(z) = \int_0^1 t^z \left[\frac{1}{2\pi} \int_{\mathbb{R}} f(-1 + is) e^{-is \log t} ds \right] dt. \tag{8}$$

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Now, if we define $g(s) = f(-1 + is)$, it is clear that the inner integral at (8) is $\hat{g}(\log t)$, where $\hat{g}$ represents the Fourier transform of g.

Finally, since $\hat{g}$ is a Fourier transform of an integrable function, it follows that is a bounded, continuous function on $(0, 1]$. Then, setting

$$d\mu = \frac{1}{2\pi}\hat{g}(\log t)dt$$

we obtain a Borel complex measure which represents f in the desired way:

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Thus, we get a functional $T = \langle \cdot, \mu \rangle$ that vanishes on $\text{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \cdots\}$, but does not vanish on t^{λ} $(\lambda \notin \{\lambda_n\})$ due to our election of f. Hence, we deduce that $t^{\lambda} \notin X = \overline{\text{span}}\{1, t^{\lambda_1}, t^{\lambda_2}, \cdots\}$, and it finishes the proof. □

3. The Full Müntz Theorem in $L^2[0,1]$, $C[0,1]$ and $L^1[0,1]$ ([BE])

Full Müntz Theorem in $L^2[0,1]$

Let $\{\lambda_i\}_{i=0}^{\infty}$ be a sequence of distinct real numbers greater than $-\frac{1}{2}$. Then, the set

$$\text{span}\left\{t^{\lambda_i} : i \in \mathbb{N}\right\}$$

is dense in $L^2[0,1]$ if and only if

$$\sum_{i=0}^{\infty} \frac{2\lambda_i + 1}{(2\lambda_i + 1)^2 + 1} = +\infty.$$

Proof.
Let m be a positive integer number different of any λ_i. We consider the best approximation in $L^2[0,1]$ of t^m by elements of

$$\mathrm{span}\{t^{\lambda_0}, t^{\lambda_1}, \ldots, t^{\lambda_n}\}.$$

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It is well known that [R]:

$$\min_{b_i \in \mathbb{C}} \left\| t^m - \sum_{i=0}^{\infty} b_i t^{\lambda_i} \right\|_{L^2[0,1]} = \frac{1}{\sqrt{2m+1}} \prod_{i=0}^{n} \left| \frac{m-\lambda_i}{m+\lambda_i+1} \right|.$$

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Then

$$t^m \in \overline{\text{span}} \left\{ t^{\lambda_i} : i \in \mathbb{N} \right\} \Leftrightarrow \limsup_{n\to\infty} \prod_{i=0}^{n} \left| \frac{m-\lambda_i}{m+\lambda_i+1} \right| = 0. \qquad (9)$$

So, condition (9) is equivalent to:

$$\limsup_{n\to\infty} \prod_{i=0,\lambda_i<m}^{n} \left(1 - \frac{2\lambda_i + 1}{m + \lambda_i + 1}\right) = 0,$$

or

$$\limsup_{n\to\infty} \prod_{i=0,\lambda_i>m}^{n} \left(1 - \frac{2m + 1}{m + \lambda_i + 1}\right) = 0.$$

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$$\limsup_{n\to\infty} \prod_{i=0,\lambda_i>m}^{n} \left(1 - \frac{2m + 1}{m + \lambda_i + 1}\right) = 0.$$

And that holds if and only if:

$$\sum_{i=0,\lambda_i<m}^{\infty} (2\lambda_i + 1) = +\infty \quad \text{or} \quad \sum_{i=0,\lambda_i>m}^{\infty} \left(\frac{1}{2\lambda_i + 1}\right) = +\infty$$

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In summary, we have proved that

$$\sum_{i=0}^{\infty} \frac{2\lambda_i + 1}{(2\lambda_i + 1)^2 + 1} = +\infty \Leftrightarrow t^m \in \overline{\mathrm{span}} \left\{ t^{\lambda_i} : i \in \mathbb{N} \right\}, \quad m \in \mathbb{N},$$

and by the Weierstrass Approximation Theorem the proof is finished. □

The Full Müntz Theorem in $C[0,1]$

Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of distinct, positive real numbers. Then

$$\operatorname{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \dots\},$$

is dense in $C[0,1]$ if and only if

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda_n^2 + 1} = +\infty.$$

Proof.

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CASE III: $\{\lambda_n\} = \{\alpha_n\} \cup \{\beta_n\}$, with $\alpha_n \to 0$ and $\beta_n \to \infty$.

CASE IV: $\{\lambda_n\}$ has a cluster point in $(0, \infty)$.

Case I is proved.

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Let's prove II. It follows from $\lim\limits_{n\to+\infty} \lambda_n = 0$, that

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Case I is proved.
Let's prove II. It follows from $\lim_{n\to+\infty} \lambda_n = 0$, that

$$\sum_{n=1}^{\infty} \lambda_n = +\infty \Leftrightarrow \sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda_n^2 + 1} = +\infty.$$

$\boxed{\Longrightarrow}$ Assume that $\sum_{n=1}^{\infty} \lambda_n = +\infty$.

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$\boxed{\Longrightarrow}$ Assume that $\sum_{n=1}^{\infty} \lambda_n = +\infty$.
Then, $\lambda_n \to 0$ implies that

$$\sum_{n=1}^{\infty} \left(1 - \left|\frac{\lambda_n - 1}{\lambda_n + 1}\right|\right) = +\infty;$$

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$$\sum_{n=1}^{\infty} \left(1 - \left|\frac{\lambda_n - 1}{\lambda_n + 1}\right|\right) = +\infty;$$

so we have that

$$\text{span}\{1, t^{\lambda_1}, t^{\lambda_2}, ...\}$$

is dense in $C[0,1]$ by the same argument of case $\inf \lambda_n > 0$.

$\boxed{\Longleftarrow}$ Let's suppose now that $\sum_{n=1}^{\infty} \lambda_n < +\infty$ and we show span$\{1, t^{\lambda_1}, t^{\lambda_2}, \cdots\}$ is not dense in $C[0,1]$.

$\boxed{\Longleftarrow}$ Let's suppose now that $\sum_{n=1}^{\infty} \lambda_n < +\infty$ and we show span$\{1, t^{\lambda_1}, t^{\lambda_2}, \cdots\}$ is not dense in $C[0,1]$.

We need the following inequality:

Newman's inequality

The inequality

$$||tp'(t)||_\infty \leq 11 \left(\sum_{i=1}^{n} \lambda_i \right) ||p||_\infty$$

holds for every $p \in \text{span}\{1, t^{\lambda_1}, \cdots, t^{\lambda_n}\}$.

Then, if $\eta = \sum_{n=1}^{\infty} \lambda_n < +\infty$, we have that

$$||tp'(t)||_\infty \leq 11\eta||p||_\infty \tag{10}$$

for every $p \in \text{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \cdots\}$.

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Let us suppose that it is dense. If we set $f(t) = \sqrt{1-t}$, for every $m \in \mathbb{N}$ there exists $p \in \text{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \cdots\}$ such that $||p - f||_\infty < 1/m^2$.

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It follows from this fact and Mean Value Theorem that

$$||tp'(t)||_\infty \geq \frac{m-2}{2}$$

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It follows from this fact and Mean Value Theorem that

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and this clearly contradicts (10) (this counterexample is shown in [A]).

Theorem: Existence of Chebyshev Polynomials.

Let A be a compact subset of $[0, \infty)$ containing at least $n+1$ points. Then there exists a unique (extended) Chebyshev polynomial

$$T_n := T_n\{\lambda_0, \lambda_1, \ldots, \lambda_n; A\},$$

for $\text{span}\{t^{\lambda_0}, \ldots, t^{\lambda_n}\}$ on A defined by

$$T_n(t) = c\left(t^{\lambda_n} - \sum_{i=0}^{n-1} a_i t^{\lambda_i}\right),$$

where the numbers $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$ are chosen to minimize

$$\left\|t^{\lambda_n} - \sum_{i=0}^{n-1} a_i t^{\lambda_i}\right\|_\infty,$$

and where $c \in \mathbb{R}$ is a normalization constant chosen so that $||T_n||_\infty = 1$, and the sign of c is determined by $T_n(\max A) > 0$.

Theorem: Alternation Characterization.

The Chebyshev polynomial

$$T_n := T_n\{\lambda_0, \lambda_1, \ldots, \lambda_n; A\} \in \operatorname{span}\{t^{\lambda_0}, \ldots, t^{\lambda_n}\},$$

is uniquely characterized by the existence of an alternation set

$$\{t_0 < t_1 < \cdots < t_n\} \subset A$$

for which

$$T_n(t_j) = (-1)^{n-j} = (-1)^{n-j}||T_n||_\infty, \quad j = 0, 1, \ldots, n.$$

Theorem [BE, Theorem 3.4]

Suppose $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence of nonnegative real numbers satisfying $\lambda_0 = 0$, $\lambda_i \geq 1$ for $i = 1, 2, \dots$, and

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} < +\infty.$$

Let $\varepsilon \in (0, 1)$. Then there exists a constant c depending only on $\{\lambda_i\}_{i=1}^{\infty}$ and ε so that

$$||p'||_{[0,1-\varepsilon]} \leq c||p||_{[0,1]}$$

for every $p \in \text{span}\{t^{\lambda_0}, t^{\lambda_1}, \dots\}$.

CASE III: $\{\lambda_i : i \in \mathbb{N}\} = \{\alpha_i : i \in \mathbb{N}\} \cup \{\beta_i : i \in \mathbb{N}\}$ with

$$\lim_{i\to\infty} \alpha_i = 0 \quad \text{and} \quad \lim_{i\to\infty} \beta_i = +\infty.$$

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$$\lim_{i \to \infty} \alpha_i = 0 \quad \text{and} \quad \lim_{i \to \infty} \beta_i = +\infty.$$

Note that $\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2 + 1} = \infty$ is equivalent to

$$\sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \frac{1}{\beta_i} = +\infty. \tag{11}$$

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Note that $\sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2+1} = \infty$ is equivalent to

$$\sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \frac{1}{\beta_i} = +\infty. \tag{11}$$

$\boxed{\Longleftarrow}$ If (11) holds, then the already examined cases yield the denseness of $\text{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \dots\}$ in $C[0,1]$.

$\boxed{\Longrightarrow}$ If (11) does not hold, then

$$\sum_{i=1}^{\infty} \alpha_i < \infty \quad \textit{and} \quad \sum_{i=1}^{\infty} \frac{1}{\beta_i} < \infty.$$

$\boxed{\Longrightarrow}$ If (11) does not hold, then

$$\sum_{i=1}^{\infty} \alpha_i < \infty \quad \textit{and} \quad \sum_{i=1}^{\infty} \frac{1}{\beta_i} < \infty.$$

Let

$$T_{n,\alpha} := T_n\{1, t^{\alpha_1}, \ldots, t^{\alpha_n} : [0,1]\},$$
$$T_{n,\beta} := T_n\{1, t^{\beta_1}, \ldots, t^{\beta_n} : [0,1]\},$$
$$T_{2n,\alpha,\beta} := T_n\{1, t^{\alpha_1}, \ldots, t^{\alpha_n}, t^{\beta_1}, \ldots, t^{\beta_n} : [0,1]\}.$$

Newman's inequality and the **Mean Value Theorem** imply that for each $\varepsilon > 0$ exists a $k_1(\varepsilon) \in \mathbb{N}$ depending only on $\{\alpha_i\}_{i=1}^{\infty}$ and ε such that $T_{n,\alpha}$ has at most $k_1(\varepsilon)$ zeros in $[\varepsilon, 1)$ and at least $n - k_1(\varepsilon)$ zeros in $(0, \varepsilon)$.

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[BE, Theorem 3.4] and the **Mean Value Theorem** imply that for every $\varepsilon > 0$ exists a $k_2(\varepsilon) \in \mathbb{N}$ depending only on $\{\beta_i\}_{i=1}^{\infty}$ and ε such that $T_{n,\beta}$ has at most $k_2(\varepsilon)$ zeros in $(0, 1 - \varepsilon]$ and at least $n - k_2(\varepsilon)$ zeros in $(1 - \varepsilon, 1)$.

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Now, counting the zeros of $T_{n,\alpha} - T_{2n,\alpha,\beta}$ and $T_{n,\beta} - T_{2n,\alpha,\beta}$, we can deduce that for every $\varepsilon > 0$ exists $k(\varepsilon) \in \mathbb{N}$ depending only on $\{\lambda_i\}_{i=1}^{\infty}$ and ε, such that $T_{2n,\alpha,\beta}$ has at most $k(\varepsilon)$ zeros in $[\varepsilon, 1-\varepsilon]$.

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and a function $f \in C[0,1]$ so that $f(t) = 0$ for all $t \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$, while

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Assume that there exists a $p \in span\{1, t^{\lambda_1}, t^{\lambda_2}, \dots\}$ so that

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Assume that there exists a $p \in span\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$ so that

$$||f - p||_{[0,1]} < 1.$$

Then $p - T_{2n,\alpha,\beta}$ has at least $2n+1$ zeros in $(0,1)$.

However for sufficiently large n,

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However for sufficiently large n,

$$p - T_{2n,\alpha,\beta} \in \text{span}\{1, t^{\lambda_1}, \ldots, t^{\lambda_{2n}}\},$$

which can have at most $2n$ zeros in $[0,1]$.This contradiction shows that

$$\text{span}\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\},$$

is not dense in $C[0,1]$.

□

CASE IV: Assume that $\{\lambda_n\}$ has a cluster point in $(0, \infty)$.

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Then there exists a subsequence $\{\lambda_{n_k}\}$ such that $\inf_{k \in \mathbb{N}} \lambda_{n_k} > 0$, and it follows from case II.

$\square$

Full Müntz Theorem in $L^1[0,1]$

Suppose $\{\lambda_i\}_{i=0}^{\infty}$ is a sequence of distinct real numbers greater than -1. Then

$$\operatorname{span}\left\{t^{\lambda_i} : i \in \mathbb{N} \cup \{0\}\right\},$$

is dense in $L^1[0,1]$ if and only if

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1}{(\lambda_i + 1)^2 + 1} = +\infty.$$

Proof.

$\boxed{\Longrightarrow}$ Assume that

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is dense in $L^1[0,1]$.

Let $m \in \mathbb{Z}^+$ be fixed. Let $\varepsilon > 0$. Choose a

$$p \in \text{span}\{t^{\lambda_0}, t^{\lambda_1}, \dots\},$$

such that

$$||t^m - p||_\infty < \varepsilon.$$

Now let define

$$q(t) := \int_0^t p(s)ds \in \mathrm{span}\{t^{\lambda_0+1}, t^{\lambda_1+1}, \dots\}.$$

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So the Weierstrass Approximation Theorem yields that

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So the Weierstrass Approximation Theorem yields that

$$\mathrm{span}\{1, t^{\lambda_0+1}, t^{\lambda_0+1}, \dots\},$$

is dense in $C[0,1]$. Using the Full Müntz Theorem in $C[0,1]$,

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1}{(\lambda_i + 1)^2 + 1} = +\infty.$$

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By the Hahn-Banach Theorem and the Riesz Representation Theorem

$$\text{span}\{t^{\lambda_0}, t^{\lambda_1}, \dots\}$$

is not dense in $L^1[0,1]$ if and only if exists a $0 \neq h \in L^\infty[0,1]$ satisfying

$$\int_0^1 t^{\lambda_i} h(t) dt = 0; \quad i = 0, 1, \dots$$

Suppose there exists such function. Let

$$f(z) := \int_0^1 t^z h(t) dt.$$

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Note that $\sum_{i=0}^{\infty} \frac{\lambda_i+1}{(\lambda_i+1)^2+1} = +\infty$. implies

$$\sum_{n=1}^{\infty}\left(1 - \left|\frac{\lambda_n}{\lambda_n+2}\right|\right) = +\infty.$$

Hence Blaschke's Theorem ([R, Theorem 15.23]) yields that $g = 0$ on the open unit disk.

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$$f(n) = \int_0^1 t^n h(t) dt = 0; \quad n = 0, 1, \ldots$$

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for every $u \in C[0,1]$. which contradicts the fact that $h \neq 0$.

So span$\{t^{\lambda_0}, t^{\lambda_1}, \dots\}$ is dense in $L^1[0,1]$. $\square$

4. The Full Müntz Theorem in $L^p[0,1]$ ([0])

The Full Müntz Theorem in $L^p[0,1]$

Let $1 < p < \infty$ and $\{\lambda_i\}_{i=0}^{\infty}$ be a sequence of distinct real numbers greater than $-1/p$. Then, the collection of finite linear combinations of functions $\{t^{\lambda_0},\ t^{\lambda_1},\ t^{\lambda_2},\dots\}$ is dense in $L^p[0,1]$ if and only if

$$\sum_{n=0}^{\infty} \frac{\lambda_i + 1/p}{(\lambda_i + 1/p)^2 + 1} = +\infty. \tag{12}$$

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$$\sum_{n=0}^{\infty} \frac{\lambda_i + 1/p}{(\lambda_i + 1/p)^2 + 1} = +\infty. \tag{12}$$

To prove this theorem we will use the following lemma:

Lemma

Suppose $\{\mu_i\}_{i=0}^{\infty}$ is a sequence of distinct positive real numbers such that $\text{span}\{t^{\mu_i - 1/r}\}_{i=0}^{\infty}$ is dense in $L^r[0,1]$. Then, $\text{span}\{t^{\mu_i - 1/s}\}_{i=0}^{\infty}$ is dense in $L^s[0,1]$ for every $s > r$ and $\text{span}\{1, t^{\mu_0}, t^{\mu_1}, \dots\}$ is dense in $C[0,1]$.

Proof.
Let $X = L^r[0,1],\ Y = L^s[0,1],\ A = \mathrm{span}\{t^{\mu_i - 1/r}\}_{i=0}^{\infty}$.

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Let $X = L^r[0,1]$, $Y = L^s[0,1]$, $A = \text{span}\{t^{\mu_i - 1/r}\}_{i=0}^{\infty}$.
For the first part, we consider the operator $J : L^r[0,1] \to L^s[0,1]$ defined by:

$$(J\varphi)(t) = t^{-(1/r'+1/s)} \int_0^t \varphi(s)ds, \quad (t \in [0,1],\ \varphi \in L^r[0,1])$$

where $\frac{1}{r} + \frac{1}{r'} = 1$.

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Let $X = L^r[0,1]$, $Y = L^s[0,1]$, $A = \text{span}\{t^{\mu_i - 1/r}\}_{i=0}^{\infty}$.
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where $\frac{1}{r} + \frac{1}{r'} = 1$.
We have for every $n \in \mathbb{N}$ that:

$$(J\psi_n)(t) = t^n, \quad \psi_n(t) = (n + 1/r' + 1/s)t^{n+1/s-1/r},$$

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$$(J\psi_n)(t) = t^n, \quad \psi_n(t) = (n + 1/r' + 1/s)t^{n+1/s-1/r},$$

then, by the Weierstrass Approximation Theorem, $J(X)$ is dense in Y and consequently, $J(A) = \text{span}\{t^{\mu_i - 1/s}\}_{i=0}^{\infty}$ is dense in $L^r[0,1]$.

For the second part, we consider the operator $J : L^r[0,1] \to L^s[0,1]$ defined by:

$$(J\varphi)(t) = t^{-1/r'} \int_0^t \varphi(s)ds, \quad \forall t \in (0,1], \qquad (J\varphi)(0) = 0,$$

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where $\frac{1}{r} + \frac{1}{r'} = 1$. A similar argument implies that $\text{span}\{1, t^{\mu_0}, t^{\mu_1}, \dots\}$ is dense in $C[0,1]$. □

Proof of the Theorem.
Firstly, let $\{\lambda_i\}_{i=0}^{\infty}$ be a sequence of distinct real numbers greater than $-1/p$ satisfying (12).

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We consider $\{v_i = \lambda_i - 1/p'\}_{i=0}^{\infty}$, where $\frac{1}{p} + \frac{1}{p'} = 1$, is a sequence of real numbers greater than -1 and satisfying:

$$\sum_{i=0}^{\infty} \frac{v_i + 1}{(v_i + 1)^2 + 1}.$$

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We consider $\{v_i = \lambda_i - 1/p'\}_{i=0}^{\infty}$, where $\frac{1}{p} + \frac{1}{p'} = 1$, is a sequence of real numbers greater than -1 and satisfying:

$$\sum_{i=0}^{\infty} \frac{v_i + 1}{(v_i + 1)^2 + 1}.$$

By the Full Müntz Theorem in $L^1[0,1]$, the set

$$\text{span}\{t^{v_i}\}_{i=0}^{\infty} = \text{span}\{t^{\lambda_i - 1/p'}\}_{i=0}^{\infty}$$

is dense in $L^1[0,1]$. Choosing $\mu_i = \lambda_i + 1/p$ and applying the lemma we will have that

$$\text{span}\{t^{\mu_i - 1/p}\}_{i=0}^{\infty} = \text{span}\{t^{\lambda_i}\}_{i=0}^{\infty}$$

is dense in $L^p[0,1]$ for $p > 1$.

For the reciprocal, suppose that $\text{span}\{t^{\lambda_i}\}_{i=0}^{\infty}$ is dense in $L^p[0, 1]$.

For the reciprocal, suppose that $\mathrm{span}\{t^{\lambda_i}\}_{i=0}^{\infty}$ is dense in $L^p[0,1]$. Defining $\mu_i = \lambda_i + 1/p$, the set

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is dense in $L^p[0,1]$, and by the lemma $\mathrm{span}\{1, t^{\mu_i}\}_{i=0}^{\infty}$ is dense in $C[0,1]$.

For the reciprocal, suppose that $\text{span}\{t^{\lambda_i}\}_{i=0}^{\infty}$ is dense in $L^p[0,1]$. Defining $\mu_i = \lambda_i + 1/p$, the set

$$\text{span}\{t^{\mu_i - 1/p}\}_{i=0}^{\infty}$$

is dense in $L^p[0,1]$, and by the lemma $\text{span}\{1, t^{\mu_i}\}_{i=0}^{\infty}$ is dense in $C[0,1]$. It is enough to apply The Full Müntz Theorem in $C[0,1]$ to obtain:

$$\sum_{i=0}^{\infty} \frac{\lambda_i + 1/p}{(\lambda_i + 1/p)^2 + 1} = \sum_{i=0}^{\infty} \frac{\mu_i}{\mu_i^2 + 1} = +\infty.$$

$\square$

5. An application of the Müntz-Szász Theorem ([LLPZ])

Definition

We define the *finite continuous Cesàro operator* C_1 on the complex Banach space $L^p[0,1]$ for $1 < p < \infty$ by the expression:

$$(C_1 f)(t) := \frac{1}{t}\int_0^t f(s)\,ds \qquad (t \in [0,1],\; f \in L^p[0,1]).$$

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Definition

Let T be an operator on a complex Banach space X.

- The *point spectrum* of T is the set of those $\lambda \in \mathbb{C}$ for which there exists a nonzero vector $x \in X$ such that $Tx = \lambda x$.
- We say that T has *rich point spectrum* provided that int $\sigma_p(T) \neq \emptyset$, and that for every open disc $D \subset \sigma_p(T)$, the family of eigenvectors

$$\bigcup_{z \in D} \ker(T - z)$$

is a total set.

Lemma

Let T be a bounded linear operator on a complex Banach space X and let us suppose that there is an analytic mapping $h : \text{int}\ \sigma_p(T) \to X$ verifying:

(*i*) $h(z) \in \ker(T - z) \setminus \{0\}$ for all $z \in \text{int}\ \sigma_p(T)$,

(*ii*) $\{h(z)\ :\ z \in \text{int}\ \sigma_p(T)\}$ is a total subset of X.

Then T has rich point spectrum.

Lemma

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(*ii*) $\{h(z) \ : \ z \in \text{int } \sigma_p(T)\}$ is a total subset of X.

Then T has rich point spectrum.

Using this we will prove the following result:

Theorem

The finite continuous Cesàro operator C_1 on $L^p[0,1]$ has rich point spectrum.

Proof.

It is known that $\sigma_p(C_1) = D(p'/2, p'/2)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, each $z \in D(p'/2, p'/2)$ is a simple eigenvalue of C_1 and a corresponding eigenfunction is given by $h_z(t) = t^{(1-z)/z}, \forall t \in [0, 1]$.

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So $h_{(\cdot)} : \sigma_p(C_1) \to L^p[0, 1]$ is analytic and $h_z \in \ker(C_1 - z)\backslash\{0\}$.

Proof.
It is known that $\sigma_p(C_1) = D(p'/2, p'/2)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, each $z \in D(p'/2, p'/2)$ is a simple eigenvalue of C_1 and a corresponding eigenfunction is given by $h_z(t) = t^{(1-z)/z}, \forall t \in [0,1]$.
So $h_{(\cdot)} : \sigma_p(C_1) \to L^p[0,1]$ is analytic and $h_z \in \ker(C_1 - z)\backslash\{0\}$. It suffices to consider the sequence $\{z_i\}$ defined by:

$$z_i = \frac{i+1}{i+2}p', \qquad \forall\, i \in \mathbb{N} \cup \{0\}.$$

We have that the sequence $\lambda_i = (1 - z_i)/z_i$ is greater than $-1/p$ and satisfies condition (12) and therefore $\text{span}\{t^{\lambda_i}\}_{i=0}^{\infty}$ is dense in $L^p[0,1]$ and, consequently,

$$\{h_z \ : \ z \in D(p'/2, p'/2)\}$$

is total in $L^p[0,1]$.

Proof.
It is known that $\sigma_p(C_1) = D(p'/2, p'/2)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, each $z \in D(p'/2, p'/2)$ is a simple eigenvalue of C_1 and a corresponding eigenfunction is given by $h_z(t) = t^{(1-z)/z}, \forall t \in [0,1]$.
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is total in $L^p[0,1]$. The result now follows from the previous lemma.

□

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