Four definitions for the fractional Laplacian

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[1] International Women's Day

Laplace fractional operator: several points of view

- Functional analysis: M. Riesz, S. Bochner, W. Feller, E. Hille, R. S. Phillips, A. V. Balakrishnan, T. Kato, Martínez–Carracedo y Sanz–Alix, K. Yosida
- Potencial theory for fractional laplacian: N. S. Landkof
- Lévy's processes: K. Bogdan e.a.
- Partial Derivative Ecuations: L. Caffarelli y L. Silvestre
- Scattering theory: C. R. Graham y M. Zworski, S-Y. A. Chang y M.d.M. González
- Kato's square root (solved by P. Auscher e.a.)

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Basic example of fractional operator : **fractional Laplacian**

A pointwise definition of the fractional Laplacian

The working space

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$\mathscr{S}(\mathbb{R}^n)$ is the space $C^\infty(\mathbb{R}^n)$ of functions that

$$\|f\|_p = \sup_{|\alpha|\leq p} \sup_{x\in\mathbb{R}^n} (1+|x|^2)^{p/2} |\partial^\alpha f(x)| < \infty \quad p \in \mathbb{N} \cup \{0\}$$

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This space endowed with the metric topology

$$d(f,g) = \sum_{p=0}^{\infty} 2^{-p} \frac{\|f-g\|_p}{1 + \|f-g\|_p}$$

First definition motivation

Let $f \in C^2(a,b)$, then for every $x \in (a,b)$ one has

$$-f''(x) = \lim_{y \to 0} \frac{2f(x) - f(x+y) - f(x-y)}{y^2}$$

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If we introduce the spherical and solid averaging operators

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then we can reformulate $-f''(x)$ like this

$$-f''(x) = 2 \lim_{y \to 0} \frac{f(x) - \mathscr{M}_y f(x)}{y^2} = 6 \lim_{y \to 0} \frac{f(x) - \mathscr{A}_y f(x)}{y^2}$$

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Thus, if we bear in mind that $-\Delta f = -\sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k^2}$ and making an extension of the spherical and solid averaging operators

$$\mathscr{M}_y f(x) = \frac{1}{\sigma_{(n-1)} r^{(n-1)}} \int_{S(x,r)} f(y)\, d\sigma(y) \quad \mathscr{A}_y f(x) = \frac{1}{\omega_n r^n} \int_{B(x,r)} f(y)\, dy$$

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then we have

$$-\Delta f(x) = 2n \lim_{y \to 0} \frac{f(x) - \mathscr{M}_y f(x)}{y^2} = 2(n+2) \lim_{y \to 0} \frac{f(x) - \mathscr{A}_y f(x)}{y^2}$$

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Observation 3: since as $s \to 1^-$ the fractional Laplacean tends (at least, formally right now) to $(-\Delta)$, one might surmise that in the regime $1/2 < s < 1$ the operator $(-\Delta)^s$ should display properties closer to those of the classical Laplacian, whereas since $(-\Delta)^s \to I \;\; as \;\; s \to 0^+$, the stronger discrepancies might present themselves in the range $0 < s < 1/2$.

$(-\Delta)^s u$ is well-defined

It is important to observe that the integral in the right-hand side is convergent. In order to see this, it suffices to write:

$$\int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}}\, dy = \int_{|y|\leq 1} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}}\, dy +$$

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Using Taylor $2u(x) - u(x+y) - u(x-y) = -\langle \nabla^2 u(x) y, y\rangle + o(|x|^3)$

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$$= \int_{|y| \leq 1} \frac{|\langle \nabla^2 u(x) y, y \rangle + o(|x|^3)|}{|y|^{n+2s}} \, dy$$

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Using Cauchy-Schwarz inequality

$$\leq \int_{|y|\leq 1} \frac{|\nabla^2 u(x) y||y| + |o(|x|^3)|}{|y|^{n+2s}} \, dy \leq \int_{|y|\leq 1} \frac{|\nabla^2 u(x)||y|^2 + |o(|x|^3)|}{|y|^{n+2s}} \, dy$$

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Now, for (b):

$$\left| \int_{|y|\geq 1} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}}\, dy \right| \leq 4\|u\|_{L_\infty(\mathbb{R}^n)} \int_{|y|\geq 1} \frac{1}{|y|^{n+2s}}\, dy < \infty$$

Translations and dilations

Let $h \in \mathbb{R}^n$ and $\lambda > 0$, the translation and dilation operators are defined, respectively, by

$$\tau_h f(x) = f(x+h); \quad \delta_\lambda f(x) = f(\lambda x)$$

for every $f \colon \mathbb{R}^n \to \mathbb{R}$ and every $x \in \mathbb{R}^n$

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Proposition 1

Let $u \in \mathcal{S}(\mathbb{R}^n)$, then for every $h \in \mathbb{R}^n$ and $\lambda > 0$ we have

$$(-\Delta)^s(\tau_h u) = \tau_h((-\Delta)^s u)$$

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In particular, $(-\Delta)^s$ is a homogeneous operator of order $2s$.

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Recall that the orthogonal group is

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We say that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ has spherical symmetry if $f(x) = f^*(|x|)$ for some $f^*\colon \mathbb{R}^n \to \mathbb{R}$ or, equivalently, if $f(Tx) = f(x)$ for every $T \in \mathbb{O}(n)$ and every $x \in \mathbb{R}$

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Proposition 2

Let $u \in \mathcal{S}(\mathbb{R}^n)$ (actually, it is enough that $u \in \mathcal{C}^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$) be a function with spherical symmetry. Then, $(-\Delta)^s$ has spherical symmetry.

Orthogonal group (Cont.)

PROOF.-

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Let's see that $(-\Delta)^s u(Tx) = (-\Delta)^s u(x)$ for each $T \in \mathbb{O}(n)$ and each $x \in \mathbb{R}^n$

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$$(-\Delta)^s u(Tx) = \frac{\gamma(n,s)}{2} \int_{\mathbb{R}^n} \frac{2u^*(|Tx|) - u^*(|Tx+y|) - u^*(|Tx-y|)}{|y|^{2n+s}}\, dy$$

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Alternative expression for the fractional Laplacian

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Theorem

Let $u \in \mathcal{S}(\mathbb{R}^n)$, then

$$(-\Delta)^s u(x) = \gamma(n,s) P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy$$

where P.V. means the Cauchy's principal value, i.e.

$$P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy = \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy$$

Alternative expression for the fractional Laplacian (Cont.)

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Changes of variables:

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Another two definitions of the fractional Laplacian

Fourier transform

We recall the definition of the Fourier transform, $\mathcal{F}$, of a function $f \in \mathcal{S}(\mathbb{R}^n)$:

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx, \qquad \xi \in \mathbb{R}^n,$$

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whose inverse function is given by

$$\mathcal{F}^{-1}(f)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\xi) e^{ix\cdot\xi} d\xi, \qquad x \in \mathbb{R}^n,$$

Fourier transform

We recall the definition of the Fourier transform, $\mathcal{F}$, of a function $f \in \mathcal{S}(\mathbb{R}^n)$:

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$$\mathcal{F}^{-1}(f)(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x \cdot \xi} d\xi, \qquad x \in \mathbb{R}^n,$$

so that

$$f(x) = \mathcal{F}^{-1} \circ \mathcal{F}(f)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i x \cdot \xi} d\xi, \quad x \in \mathbb{R}^n.$$

Definition of $(-\Delta)^s$ via the heat semigroup $e^{t\Delta}$

We will define $(-\Delta)^s f$ in terms of the heat semigroup $e^{t\Delta}$, which is nothing but an operator such that maps every function $f \in \mathcal{S}(\mathbb{R}^n)$ to the solution of the heat equation with initial data given by f:

$$\begin{cases} v_t = \Delta v, & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ v(x,0) = f(x), & x \in \mathbb{R}^n. \end{cases}$$

Using Fourier transform and its inverse and with a bit of magic, we can write

$$e^{t\Delta} f(x) := v(x,t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-t|\xi|^2} \hat{f}(\xi) e^{ix\cdot\xi} d\xi = \int_{\mathbb{R}^n} W_t(x-z) f(z) dz,$$

where

$$W_t(x) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n,$$

is the Gauss-Weierstrass kernel.

Definition of $(-\Delta)^s$ via the heat semigroup $e^{t\Delta}$

Inspired by the following numerical identity: for $\lambda > 0$,

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\lambda} - 1) \frac{dt}{t^{1+s}}, \qquad 0 < s < 1,$$

where

$$\Gamma(-s) = \int_0^\infty (e^{-r} - 1) \frac{dr}{r^{1+s}} < 0;$$

we can think of $(-\Delta)^s$ as the following operator

$$(-\Delta)^s f(x) \sim \frac{1}{\Gamma(-s)} \int_0^\infty (e^{t\Delta} f(x) - f(x)) \frac{dt}{t^{1+s}}, \qquad 0 < s < 1.$$

Definition of $(-\Delta)^s$ via the Fourier Transform

By the well-known properties of $\mathcal{F}$ with respect to derivatives, we have that, for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\mathcal{F}[-\Delta f](\xi) = |\xi|^2 \mathcal{F}(f)(\xi), \qquad \xi \in \mathbb{R}^n,$$

so it is reasonable to write something like

$$(-\Delta)^s f(x) \sim \mathcal{F}^{-1}[|\cdot|^{2s}\mathcal{F}(f)](x), \qquad x \in \mathbb{R}^n,\ 0 < s < 1.$$

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By the well-known properties of $\mathcal{F}$ with respect to derivatives, we have that, for $f \in \mathcal{S}(\mathbb{R}^n)$,

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so it is reasonable to write something like

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$\sim$ is $=$

Theorem (Lemma 2.1. P. Stinga's PhD thesis)

Given $f \in \mathcal{S}(\mathbb{R}^n)$ and $0 < s < 1$,

$$\mathcal{F}^{-1}[|\cdot|^{2s}\mathcal{F}(f)](x) = \frac{1}{\Gamma(-s)}\int_0^\infty (e^{t\Delta}f(x) - f(x))\frac{dt}{t^{1+s}}, \quad x \in \mathbb{R}^n$$

and this two functions coincide in a pointwise way with $(-\Delta)^s f(x)$ when the constant $\gamma(n,s)$ in its definition is given by

$$\gamma(n,s) = \frac{4^s\Gamma(n/2+s)}{-\pi^{n/2}\Gamma(-s)} > 0.$$

Everybody wants to be the fractional Laplacian

Let $x \in \mathbb{R}^n$. By Fubini's theorem and inverse Fourier formula,

$$\begin{aligned}\frac{1}{\Gamma(-s)}\int_0^\infty (e^{t\Delta}f(x) - f(x))\frac{dt}{t^{1+s}} &= \frac{1}{\Gamma(-s)}\int_{\mathbb{R}^n}\int_0^\infty (e^{-t|\xi|^2} - 1)\frac{dt}{t^{1+s}}\hat{f}(\xi)e^{ix\cdot\xi}d\xi \\ &= \frac{1}{\Gamma(-s)}\int_{\mathbb{R}^n}\int_0^\infty (e^{-r} - 1)\frac{dr}{r^{1+s}}|\xi|^{2s}\hat{f}(y)e^{ix\cdot\xi}dy \\ &= \int_{\mathbb{R}^n}|\xi|^{2s}\hat{f}(\xi)e^{ix\cdot\xi}d\xi = \mathcal{F}^{-1}[|\cdot|^{2s}\mathcal{F}(f)](x).\end{aligned}$$

Since $f \in \mathcal{S}(\mathbb{R}^n)$, we have that

$$\int_{\mathbb{R}^n}\int_0^\infty |e^{-t|\xi|^2} - 1||\hat{f}(\xi)|\frac{dt}{t^{1+s}}d\xi < \infty,$$

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$$= \frac{1}{\Gamma(-s)} \int_{\mathbb{R}^n} \int_0^\infty (e^{-r} - 1) \frac{dr}{r^{1+s}} |\xi|^{2s} \hat{f}(y) e^{ix\cdot\xi} dy$$

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$$\frac{1}{\Gamma(-s)}\int_0^\infty (e^{t\Delta}f(x)-f(x))\frac{dt}{t^{1+s}} = \frac{4^s\Gamma(n/2+s)}{-\pi^{n/2}\Gamma(-s)}\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{f(x)-f(z)}{|x-z|^{n+2s}}dz, \quad x\in\mathbb{R}^n.$$

Let $\varepsilon>0$. Using that $\|W_t(x-\cdot)\|_{L^1(\mathbb{R}^n)}=1$ for any $x\in\mathbb{R}^n$, and Fubini's theorem (that can be legally applied),

$$\begin{aligned}\int_0^\infty (e^{t\Delta}f(x)-f(x))\frac{dt}{t^{1+s}} &= \int_0^\infty\int_{\mathbb{R}^n} W_t(x-z)(f(z)-f(x))dz\frac{dt}{t^{1+s}} \\ &= \int_{\mathbb{R}^n}\int_0^\infty W_t(x-z)(f(z)-f(x))\frac{dt}{t^{1+s}}dz \\ &= I_\varepsilon + II_\varepsilon.\end{aligned}$$

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I_\varepsilon &= \int_{|x-z|>\varepsilon} \int_0^\infty (4\pi t)^{-n/2} e^{-\frac{|x-z|^2}{4t}} (f(z) - f(x)) \frac{dt}{t^{1+s}} dz \\
&= \int_{|x-z|>\varepsilon} (f(z) - f(x)) \int_0^\infty (4\pi t)^{-n/2} e^{-\frac{|x-z|^2}{4t}} \frac{dt}{t^{1+s}} dz \\
&= \int_{|x-z|>\varepsilon} (f(x) - f(z)) \frac{4^s \Gamma(n/2+s)}{-\pi^{n/2}} \frac{1}{|x-z|^{n+2s}} dz
\end{aligned}
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where we used the change of variables $r = \frac{|x-z|^2}{4t}$.

Observe that I_ε converges absolutely for any $\varepsilon > 0$ since f is bounded.

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Using polar coordinates,

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\begin{aligned}
II_\varepsilon &= \int_0^\infty \int_{|x-z|<\varepsilon} W_t(x-z)(f(z)-f(x))dz \frac{dt}{t^{1+s}} \\
&= \int_0^\infty (4\pi t)^{-n/2} \int_0^\varepsilon e^{-\frac{r^2}{4t}} r^{n-1} \int_{|z'|=1} (f(x+rz')-f(z))dS(z')dr \frac{dt}{t^{1+s}}.
\end{aligned}
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By Taylor's theorem, using the symmetry of the sphere,

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\int_{|z'|=1} (f(x+rz')-f(z))dS(z') = C_n r^2 \Delta f(x) + O(r^3),
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thus

$$
\begin{aligned}
|II_\varepsilon| &\leq C_{n,\Delta f(x)} \int_0^\varepsilon r^{n+1} \int_0^\infty \frac{e^{-\frac{r^2}{4t}}}{t^{n/2+s}} \frac{dt}{t} \\
&= C_{n,\Delta f(x)} \int_0^\varepsilon r^{n+1} C_{n,s} r^{-n-2s} dr = C_{n,\Delta f(x),s} \varepsilon^{2(1-s)}.
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Everybody wants to be the fractional Laplacian

Using polar coordinates,

$$\begin{aligned} II_\varepsilon &= \int_0^\infty \int_{|x-z|<\varepsilon} W_t(x-z)(f(z)-f(x))dz \frac{dt}{t^{1+s}} \\ &= \int_0^\infty (4\pi t)^{-n/2} \int_0^\varepsilon e^{-\frac{r^2}{4t}} r^{n-1} \int_{|z'|=1} (f(x+rz')-f(z))dS(z')dr \frac{dt}{t^{1+s}}. \end{aligned}$$

By Taylor's theorem, using the symmetry of the sphere,

$$\int_{|z'|=1} (f(x+rz')-f(z))dS(z') = C_n r^2 \Delta f(x) + O(r^3),$$

thus

$$\begin{aligned} |II_\varepsilon| &\le C_{n,\Delta f(x)} \int_0^\varepsilon r^{n+1} \int_0^\infty \frac{e^{-\frac{r^2}{4t}}}{t^{n/2+s}} \frac{dt}{t} \\ &= C_{n,\Delta f(x)} \int_0^\varepsilon r^{n+1} C_{n,s} r^{-n-2s} dr = C_{n,\Delta f(x),s} \varepsilon^{2(1-s)}. \end{aligned}$$

Everybody IS the fractional Laplacian

This proves that $II_\varepsilon \to 0$ as $\varepsilon \to 0$, so

$$\int_0^\infty (e^{t\Delta}f(x) - f(x))\frac{dt}{t^{1+s}} = \lim_{\varepsilon\to 0} I_\varepsilon + II_\varepsilon$$
$$= \frac{4^s\Gamma(n/2+s)}{-\pi^{n/2}}\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{f(x)-f(z)}{|x-z|^{n+2s}}dz$$

This kind of computations (bearing in mind the exact expression of the constant $\gamma(n,s)$) also prove the following pointwise convergence

$$(-\Delta)^s f(x) \to -\Delta f(x), \qquad x \in \mathbb{R}^n \text{ as } s \to 0^+,$$

when $f \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ (observe that, in $\mathcal{S}(\mathbb{R}^n)$ this is obvious by the definition via Fourier transform).

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And last but not least

Extension Problem

Let $s \in (0,1)$ and consider $a = 1-2s$. We want to solve the extension problem

$$\begin{cases} L_a U(x,y) = \mathrm{div}_{x,y}(y^a \nabla_{x,y} U) = 0, & x \in \mathbb{R}^n_+, y > 0, \\ U(x,0) = u(x), \\ U(x,y) \to 0 \text{ as } y \to \infty. \end{cases}$$

The previous system can be written as

$$\begin{cases} -\Delta_x U(x,y) = \left(\partial_{yy} + \frac{a}{y}\partial_y\right) U(x,y), & x \in \mathbb{R}^n_+, y > 0, \\ U(x,0) = u(x), \\ U(x,y) \to 0 \text{ as } y \to \infty. \end{cases} \quad (1)$$

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Theorem 1 (Extension Theorem)

Let $u \in \mathcal{S}(\mathbb{R}^n)$. Then, the solution U to the extension problem (1) is given by

$$U(x,y) = (P_s(\cdot,y) \star u)(x) = \int_{\mathbb{R}^n} P_s(x-z,y)u(z)\,dz, \tag{2}$$

where

$$P_s(x,y) = \frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)} \frac{y^{2s}}{(y^2+|x|^2)^{(n+2s)/2}} \tag{3}$$

is the Poisson Kernel for the extension problem in the half-space $\mathbb{R}^{n+1}_+$. For U as in (2) one has

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Extension Problem

PROOF

If we take a partial Fourier transform of (1)

$$\begin{cases} \partial_{yy}\hat{U}(\xi,y) + \frac{1-2s}{y}\partial_y \hat{U}(\xi,y) - 4\pi^2|\xi|^2\hat{U}(\xi,y) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \hat{U}(\xi,0) = \hat{u}(\xi), \quad \hat{U}(\xi,y) \to 0 \text{ as } y \to \infty, & x \in \mathbb{R}^n. \end{cases}$$

If we fix $\xi \in \mathbb{R}^n \setminus \{0\}$ and $Y(y) = Y_\xi(y) = \hat{U}(\xi,y)$,

$$\begin{cases} y^2 Y''(y) + (1-2s)yY'(y) - 4\pi^2|\xi|^2 y^2 Y(y) = 0 & y \text{ in } \mathbb{R}_+, \\ Y(0) = \hat{u}(\xi), \quad y(y) \to 0 \text{ as } y \to \infty, & \end{cases}$$

then it can be compared with the generalized modified Bessel equation:

$$y^2Y'' + (1-2\alpha)yY'(y) + [\beta^2\gamma^2 y^{2\gamma} + (\alpha - \nu^2\gamma^2)]Y(y) = 0 \qquad (5)$$

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The general solutions of (5) are given by

$$\hat{U}(\xi, y) = Ay^s I_s(2\pi|\xi|y) + By^s K_s(2\pi|\xi|y)$$

where I_s and K_s are the Bessel functions of second and third kind, both independent solutions of the modified Bessel equation of order s

$$z^2\phi^{''} + z\phi^{'} - (z^2 + s^2)\phi = 0 \qquad (6)$$

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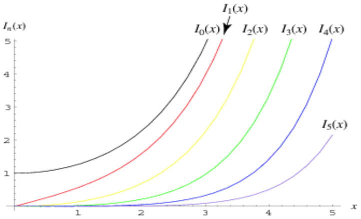

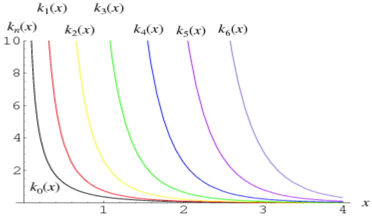

$$J_s(z) = \sum_{k=0}^{\infty}(-1)^k \frac{(z/2)^{s+2k}}{\Gamma(k+1)\Gamma(k+s+1)}, \quad |z| < \infty, |arg(z)| < \pi,$$

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$$K_s(z) = \frac{\pi}{2}\frac{I_{-s}(z) - I_s(z)}{\sin \pi s}, \quad |arg(z)| < \pi.$$

Extension Problem

The condition $\hat{U}(\xi, y) \to 0$ as $y \to \infty$ forces $A = 0$. Using I_s asymptotic behavior,

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By^s K_s(2\pi|\xi|y) &= B\frac{\pi}{2}\frac{y^s I_{-s}(2\pi|\xi|y) - y^s I_s(2\pi|\xi|y)}{\sin \pi s} \\
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In order to fulfill the condition $\hat{U}(\xi, 0) = \hat{u}(\xi)$, we impose

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We want to prove

$$U(x,y) = (P_s(\cdot, y) \star u)(x).$$

Taking inverse Fourier transform and using (7), we have to show that

$$\mathcal{F}^{-1}_{\xi \to x}\left(\frac{(2\pi|\xi|)^s}{2^{s-1}\Gamma(s)} y^s K_s(2\pi|\xi|y)\right) = \frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)} \frac{y^{2s}}{(y^2+|x|^2)^{(n+2s)/2}}.$$

Since the function in the left hand-side of (33) is **spherically symmetric**, proving (33) is equivalent to establishing the follow identity

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Theorem 2 (Fourier-Bessel Representation)

Let $u(x) = f(|x|)$, *and suppose that*

$$t \mapsto t^{n/2} f(t) J_{n/2-1}(2\pi|\xi|t) \in L^1(\mathbb{R}^n).$$

Then,

$$\hat{u}(\xi) = 2\pi|\xi|^{-n/2+1} \int_0^\infty t^{n/2} f(t) J_{n/2-1}(2\pi|\xi|t)\, dt.$$

Then, the latter identity (33) is equivalent to

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$$y^{1-2s}\partial_y \hat{U}(\xi,y) = \frac{(2\pi|\xi|)^{s+1}\hat{u}(\xi)}{2^{2s-1}\Gamma(s)} y^{1-s}K_{1-s}(2\pi|\xi|y).$$

As before, we have

$$\lim_{y\to 0^+} y^{1-s} K_{1-s}(2\pi|\xi|y) = 2^{-s}\Gamma(1-s)(2\pi|\xi|)^{s-1}.$$

We finally reach the conclusion

$$\lim_{y\to 0^+} y^{1-2s}\partial_y \hat{U}(\xi, y) = -\frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}(2\pi|\xi|)^{2s}\hat{u}(\xi).$$

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Remark 1 (Alternative proof of (4))

Using that

$$\int_{\mathbb{R}^n} P_s(x, y)\, dx = 1, \qquad y > 0,$$

we will show another proof.

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Let $u \in \mathcal{S}(\mathbb{R}^n)$ and consider the solution $U(x,y) = (P_s(\cdot, y) \star u)(x)$ to the extension problem (1). We can write

$$U(x,y) = \frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)} \int_{\mathbb{R}^n} \frac{u(z)-u(x)}{(y^2+|z-x|^2)^{(n+2s)/2}}\,dz + u(x).$$

Differentiating both sides respect to y we obtain

$$y^{1-2s}\partial_y U(x,y) = 2s\frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)} \int_{\mathbb{R}^n} \frac{u(z)-u(x)}{(y^2+|z-x|^2)^{(n+2s)/2}}\,dz + O(y^2).$$

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Now, letting $y \to 0^+$ and using the Lebesgue dominated convergence theorem, we thus find

$$\begin{aligned}\lim_{y\to 0^+} y^{1-2s}\partial_y U(x,y) &= 2s\frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)}\text{P.V.}\int_{\mathbb{R}^n}\frac{u(z)-u(x)}{(|z-x|^2)^{(n+2s)/2}}\,dz \\ &= -2s\frac{\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)}\gamma(n,s)^{-1}(-\Delta)^s u(x).\end{aligned}$$

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La exposición está basada en:

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Thanks for your attention
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