En busca de la linealidad en Matemáticas

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1. Motivation
2. Everywhere differentiable nowhere monotone functions
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Definition (Aron, Gurariy, Seoane, 2004)

1. We say that a subset M of a linear space E is λ-**lineable** if there exists a λ-dimensional subspace V of E such that $V \subset M \cup \{0\}$. If V is infinite-dimensional, we simply say that M is **lineable**.
2. A subset M of functions on $\mathbb{R}$ is said to be **spaceable** if $M \cup \{0\}$ contains a *closed* infinite dimensional subspace.

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Theorem (Aron, Gurariy, Seoane, 2004)

The set $\mathcal{DNM}(\mathbb{R})$ of differentiable functions on $\mathbb{R}$ which are nowhere monotone is lineable in $\mathcal{C}(\mathbb{R})$.

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a positive function which is integrable on each finite subinterval. We say that f is $H-$**fat** $(0 < H < \infty)$ if for each $a < b$,

$$\frac{1}{b-a} \cdot \int_a^b f(t)dt \leq H \cdot \min\{f(a), f(b)\} \quad (1)$$

$H_f = \inf(H)$ in (1) will be called the **fatness** of f. We say that f is **fat** if it is H-fat for some $H \in (0, \infty)$. A family $\mathcal{F}$ of such functions $\{f\}$ will be called **uniformly fat** if $H_{\mathcal{F}} = \sup_{f \in \mathcal{F}}(H_f) < \infty$.

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Definition

A positive continuous even function φ on $\mathbb{R}$ that is decreasing on $\mathbb{R}^+$ is called a **scaling function**.

Proposition

Given a scaling function φ, if for each $b > 0$

$$\frac{1}{b} \cdot \int_0^b \varphi(t)dt \leq K \cdot \varphi(b),$$

then φ is fat and $H_\varphi \leq 2K$.

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Proof.

1. $-b < a \leq 0$. Then

$$\frac{1}{b-a} \cdot \int_a^b \varphi(t)dt \leq \frac{2}{b} \cdot \int_0^b \varphi(t)dt \leq 2K \cdot \varphi(b).$$

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2. $0 < a < b$. Making a proper linear substitution we have $t(x) \geq x$ on $[0, b]$. Since φ is decreasing, $\varphi(t(x)) \leq \varphi(x)$. Therefore

$$\frac{1}{b-a} \cdot \int_a^b \varphi(t)dt = \frac{1}{b} \cdot \int_0^b \varphi(t(x))dx \leq \frac{1}{b} \cdot \int_0^b \varphi(x)dx \leq K \cdot \varphi(b).$$

Example

The scaling function

$$\varphi(t) = \frac{1}{\sqrt{1+|t|}}$$

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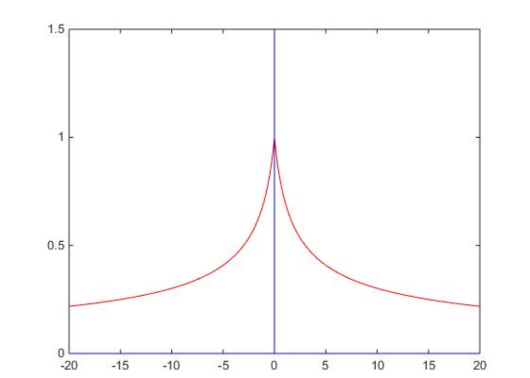

Definition (φ-wavelet)

Given a scaling function φ, let $L(\varphi)$ denote the set of functions of the form

$$\Psi(x) = \sum_{j=1}^{n} c_j \cdot \varphi(\lambda_j(x - \alpha_j)) \quad \text{where} \quad c_j, \lambda_j > 0, \quad \text{and} \quad \alpha_j \in \mathbb{R}.$$

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Proposition

If a scaling function φ is fat, then $L(\varphi)$ is uniformly fat. Moreover, $H_{L(\varphi)} = H_\varphi$.

Proposition (flexibility of $L(\varphi)$)

Choose an arbitrary scaling function φ, $n \in \mathbb{N}$, n distinct real numbers $\{\alpha_j\}_{j=1}^n$, and intervals $\{I_j = (y_j, \tilde{y}_j)\}_{j=1}^n$, where $0 < y_j < \tilde{y}_j$ for each $j = 1, 2, \ldots, n$. Then there exists $\psi \in L(\varphi)$ such that the following two conditions are satisfied:

1. $\psi(\alpha_j) \in I_j$ for $j = 1, 2, \ldots, n$.
2. $\psi(x) < \max_{1 \le j \le n} \tilde{y}_j$ for all $x \in \mathbb{R}$.

Proposition

Let $\sum_{n=1}^{\infty} \Psi_n(x)$ be a formal series of $\mathcal{C}^1$-functions on $\mathbb{R}$, such that for some $x_0 \in \mathbb{R}$, $\sum_{n=1}^{\infty} \Psi_n(x_0)$ converges. For each n, let $\Psi_n' = \psi_n$ and suppose that $\{\psi_n \ : \ n \in \mathbb{N}\}$ is a uniformly fat sequence of positive functions, with $\sum_{n=1}^{\infty} \psi_n(a)$ converging to s, say, for some a. Then

1. $F(x) \equiv \sum_{n=1}^{\infty} \Psi_n(x)$ is uniformly convergent on each bounded subset of $\mathbb{R}$.
2. $F'(a)$ exists and $F'(a) = s$.

Theorem

Let $0 = y_0 < y_1 < y_2 < \cdots < y_n < \cdots \to 1$. *Let* $S_0 = \{\alpha_j^0\}_{j=1}^{\infty}$ *be a countable set of distinct real numbers and, for each* $i \in \mathbb{N}$, *let* $S_i = \{\alpha_j^{(i)}\}_{j=1}^{m_i}$ *be a finite set of distinct real numbers. Suppose further that the sets* $\{S_i\}_{i=0}^{\infty}$ *are pairwise disjoint. Then, there exists a differentiable function* F *on* $\mathbb{R}$ *such that*

1. $F'(\alpha_j^{(i)}) = y_j$ *for all* $j = 1, 2, \ldots, m_i$ *and* $i = 1, 2, \ldots$
2. $F'(\alpha_j^{(0)}) = 1$ *for all* $j \in \mathbb{N}$.
3. $0 < F'(x) \leq 1$, *for all* $x \in \mathbb{R}$.

Proof. For each i and each interval $I_i = (y_{i-1}, y_i)$, consider a strictly increasing sequence $(y_{i,j})$ such that $(y_{i,j}) \in I_i$ and $\lim_{j\to\infty} y_{i,j} = y_i$. Let φ be a fat scaling function on $\mathbb{R}$.

Proof. For each i and each interval $I_i = (y_{i-1}, y_i)$, consider a strictly increasing sequence $(y_{i,j})$ such that $(y_{i,j}) \in I_i$ and $\lim_{j\to\infty} y_{i,j} = y_i$. Let φ be a fat scaling function on $\mathbb{R}$. By the previous proposition there exists $f_1 = \psi_1 \in L(\varphi)$ such that:

I1. $\psi_1(\alpha_j^{(1)}) \in (y_{1,0}, y_{1,1})$ for $j = 1, 2, \ldots, m_1$.

I2. $\psi_1(\alpha_1^0) \in (y_{1,0}, y_{1,1})$, and

I3. $\psi_1(x) < y_{1,1}$ for all $x \in \mathbb{R}$.

Proof. For each i and each interval $I_i = (y_{i-1}, y_i)$, consider a strictly increasing sequence $(y_{i,j})$ such that $(y_{i,j}) \in I_i$ and $\lim_{j\to\infty} y_{i,j} = y_i$. Let φ be a fat scaling function on $\mathbb{R}$. By the previous proposition there exists $f_1 = \psi_1 \in L(\varphi)$ such that:

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I2. $\psi_1(\alpha_1^0) \in (y_{1,0}, y_{1,1})$, and

I3. $\psi_1(x) < y_{1,1}$ for all $x \in \mathbb{R}$.

By the same argument, we can choose $\psi_2 \in L(\varphi)$ such that if $f_2 = \psi_1 + \psi_2$, then the following hold:

II1. $f_2(\alpha_j^{(1)}) \in (y_{1,1}, y_{1,2})$, for $j = 1, 2, \ldots, m_1$.
$f_2(\alpha_j^{(2)}) \in (y_{2,1}, y_{2,2})$, for $j = 1, 2, \ldots, m_2$.

II2. $f_2(\alpha_j^0) \in (y_{2,1}, y_{2,2})$ for $j = 1, 2$, and

II3. $f_2(x) < y_{2,2}$, for all $x \in \mathbb{R}$.

Continuing in this fashion we obtain a sequence (f_n), where $f_n = \sum_{i=1}^{n} \psi_i, n = 1, 2, \ldots,$ and where each $\psi_i \in L(\varphi)$ is such that the following conditions hold:

Continuing in this fashion we obtain a sequence (f_n), where $f_n = \sum_{i=1}^{n} \psi_i, n = 1, 2, \ldots,$ and where each $\psi_i \in L(\varphi)$ is such that the following conditions hold:

N1. $f_n(\alpha_j^{(1)}) \in (y_{1,n-1}, y_{1,n})$, for $j = 1, 2, \ldots, m_1$.
$f_n(\alpha_j^{(2)}) \in (y_{2,n-1}, y_{2,n})$, for $j = 1, 2, \ldots, m_2$.
$\vdots$
$f_n(\alpha_j^{(n)}) \in (y_{n,n-1}, y_{n,n})$, for $j = 1, 2, \ldots, m_n$.

N2. $f_n(\alpha_j^0) \in (y_{n,n-1}, y_{n,n})$ for $j = 1, 2, \ldots, n$.

N3. $f_n(x) < y_{n,n}$, for all $x \in \mathbb{R}$.

Continuing in this fashion we obtain a sequence (f_n), where $f_n = \sum_{i=1}^{n} \psi_i, n = 1, 2, \ldots,$ and where each $\psi_i \in L(\varphi)$ is such that the following conditions hold:

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N2. $f_n(\alpha_j^{0}) \in (y_{n,n-1}, y_{n,n})$ for $j = 1, 2, \ldots, n$.

N3. $f_n(x) < y_{n,n}$, for all $x \in \mathbb{R}$.

Since $f_n(x) \leq 1$ and $\psi_n(x) > 0$ for all x, the series $\psi(x) = \sum_{n=1}^{\infty} \psi_n(x)$ converges for all $x \in \mathbb{R}$.
It follows from the previous theorem that the function $F(x) = \int_0^{\infty} \psi(x)dx$ satisfies all the assertions in the statement of the theorem.

Theorem

Let A^+, A^-, A^0 be pairwise disjoint countable sets in $\mathbb{R}$. There exists a differentiable function F on $\mathbb{R}$ such that $F'(x) \leq 1$ for all $x \in \mathbb{R}$ and such that:

1. $F'(x) > 0$, $x \in A^+$.
2. $F'(x) < 0$, $x \in A^-$.
3. $F'(x) = 0$, $x \in A^0$.

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Proof.

1. $H'(x) = 1$ for $x \in A^+ \cup A^0$, $H'(x) < 1$ for $x \in A^-$, and $0 < H'(x) \leq 1$ for $x \in \mathbb{R}$.

Theorem

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Proof.

1. $H'(x) = 1$ for $x \in A^+ \cup A^0$, $H'(x) < 1$ for $x \in A^-$, and $0 < H'(x) \leq 1$ for $x \in \mathbb{R}$.
2. $G'(x) = 1$ for $x \in A^- \cup A^0$, $G'(x) < 1$ for $x \in A^+$, and $0 < G'(x) \leq 1$ for $x \in \mathbb{R}$.

Theorem

Let A^+, A^-, A^0 be pairwise disjoint countable sets in $\mathbb{R}$. There exists a differentiable function F on $\mathbb{R}$ such that $F'(x) \leq 1$ for all $x \in \mathbb{R}$ and such that:

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2. $G'(x) = 1$ for $x \in A^- \cup A^0$, $G'(x) < 1$ for $x \in A^+$, and $0 < G'(x) \leq 1$ for $x \in \mathbb{R}$.

The function $F(x) = H(x) - G(x)$ satisfies the conditions of the theorem.

Theorem (Aron, Gurariy, Seoane, 2004)

The set $\mathcal{DNM}(\mathbb{R})$ is lineable in $C(\mathbb{R})$.

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Proof. Let's consider the sequence on triples of pairwise disjoint sets $\{A_k^+, A_k^-, A_k^0\}$ with the following properties:

1. Each of the three sets in each triple is dense in $\mathbb{R}$.
2. Each of the three sets in the triple $\{A_k^+, A_k^-, A_k^0\}$ is a subset of A_{k-1}^0.

By the previous theorem , for each k there exists an everywhere differentiable function $f_k(x)$ on $\mathbb{R}$ such that

1. $f_k'(x) > 0,\ x \in A_k^+$.
2. $f_k'(x) < 0,\ x \in A_k^-$.
3. $f_k'(x) = 0,\ x \in A_k^0$.

Obviously each f_k is nowhere monotone and the sequence $\{f_k\}_1^\infty$ is linearly independent.

Let us show that if $f = \sum_{k=1}^{n} \alpha_k f_k$, with $\{\alpha_k\}_1^n$ not all zero, then f is nowhere monotone. Without loss, we may suppose that $\alpha_n \neq 0$. On A_n^+ all f_k' vanish for $k < n$, and so $f' = \alpha_n f_n'$, which implies that f is nowhere monotone. This proves the lineability of $\mathcal{DNM}(\mathbb{R})$.

Theorem (Gurariy,1966)

If all elements of a subspace E of $C[0,1]$ are differentiable on $[0,1]$, then E is finite dimensional.

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If all elements of a subspace E of $C\,[0,1]$ are differentiable on $[0,1]$, then E is finite dimensional.

Proposition

For finite a, b the set $\mathcal{DNM}[a,b]$ is lineable and not spaceable in $C[a,b]$.

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Definition

Given a function $f : \mathbb{R} \to \mathbb{R}$, we say that f is **everywhere surjective** (denoted $f \in ES$) if $f(I) = \mathbb{R}$ for every non-trivial interval $I \subset \mathbb{R}$.

Proposition

There exists a vector space $\Lambda \subset \mathbb{R}^{\mathbb{R}}$ enjoying the following two properties:

(i) Every non-zero element of Λ is an onto function, and

(ii) $\dim(\Lambda) = 2^{\mathfrak{c}}$.

Proof. Given any non-empty subset $C \subset \mathbb{R}$, let us define $H_C : \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}$ as follows:

$$H_C(y, x_1, x_2, x_3, \dots) = y \cdot \prod_{i=1}^{\infty} \chi_c(x_i).$$

Proof. Given any non-empty subset $C \subset \mathbb{R}$, let us define $H_C : \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}$ as follows:

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The strategy is to show the following:

1. $\forall C \subset \mathbb{R}$, H_C is onto.

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2. The family $\{H_C : C \subset \mathbb{R}\}$ is linearly independent.

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The strategy is to show the following:

1. $\forall C \subset \mathbb{R}$, H_C is onto.
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2. The family $\{H_C : C \subset \mathbb{R}\}$ is linearly independent.
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Thus, we have that $dim(span\{H_C : C \subset \mathbb{R}\}) = 2^{\mathfrak{c}}$. Since there exists a bijection between $\mathbb{R}$ and $\mathbb{R}^{\mathbb{N}}$, we can construct the vector space that we are looking for.

Theorem (Aron, Gurariy, Seoane, 2004)

The set

$$\{f \in \mathbb{R}^{\mathbb{R}} : f(I) = \mathbb{R} \text{ for every } I \subset \mathbb{R}\}$$

is $2^{\mathfrak{c}}$-lineable.

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Definition

Let $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$. The **additivity** of $\mathcal{F}$ is defined as the following cardinal number:

$$\mathcal{A}(\mathcal{F}) = \min(\{card(F) : F \subset \mathbb{R}^{\mathbb{R}}, \varphi + F \not\subseteq \mathcal{F}, \forall \varphi \in \mathbb{R}^{\mathbb{R}}\} \cup \{(2^{\mathfrak{c}})^{+}\})$$

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Proposition

Let $\mathcal{F}, \mathcal{G} \subset \mathbb{R}^{\mathbb{R}}$. The additivity verifies the following properties:

1. $1 \leq \mathcal{A}(\mathcal{F}) \leq (2^{\mathfrak{c}})^{+}$,
2. If $\mathcal{F} \subset \mathcal{G}$ then $\mathcal{A}(\mathcal{F}) \leq \mathcal{A}(\mathcal{G})$,
3. $\mathcal{A}(\mathcal{F}) = 1$ if and only if $\mathcal{F} = \emptyset$,
4. $\mathcal{A}(\mathcal{F}) = (2^{\mathfrak{c}})^{+}$ if and only if $\mathcal{F} = \mathbb{R}^{\mathbb{R}}$,
5. $\mathcal{A}(\mathcal{F}) = 2$ if and only if $\mathcal{F} - \mathcal{F} \neq \mathbb{R}^{\mathbb{R}}$.

Definition

Given a function $f : \mathbb{R} \to \mathbb{R}$, we say that:

1. f is **perfectly everywhere surjective** ($f \in PES$) if $f(P) = \mathbb{R}$ for every perfect set $P \subset \mathbb{R}$.
2. f is a **Jones function** ($f \in J$) if $C \cap f \neq \emptyset$ for every closed $C \subset \mathbb{R}^2$ with $\pi_x(C)$ (i.e., projection of C on the first coordinate) has cardinality continuum $\mathfrak{c}$.

Theorem (Ciesielski, Gámez, Natkaniec, Seoane, 2018)

$\mathcal{A}(PES \setminus J) \leq \mathfrak{c}$.

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Proof. Let $F = C(\mathbb{R})$. Since $|C(\mathbb{R})| = \mathfrak{c}$, we shall see that $h + C(\mathbb{R}) \not\subset PES \setminus J$ for every $h \in \mathbb{R}^{\mathbb{R}}$.
Suppose $h + C(\mathbb{R}) \subset PES \setminus J$ for some $h \in \mathbb{R}^{\mathbb{R}}$.

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Suppose $h + C(\mathbb{R}) \subset PES \setminus J$ for some $h \in \mathbb{R}^{\mathbb{R}}$.
Then, clearly, $h \in h + C(\mathbb{R}) \subset PES \setminus J$ and, thus, $h \notin J$.
Therefore, $\exists C \subset \mathbb{R}^2$ closed such that $|\pi_x(C)| = \mathfrak{c}$ and $C \cap h = \emptyset$.

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Then, clearly, $h \in h + C(\mathbb{R}) \subset PES \setminus J$ and, thus, $h \notin J$.
Therefore, $\exists C \subset \mathbb{R}^2$ closed such that $|\pi_x(C)| = \mathfrak{c}$ and $C \cap h = \emptyset$.
The function $\gamma : \pi_x(C) \to \mathbb{R}$ given by $\gamma(x) = \inf\{y : (x, y) \in C\}$ is Borel. Thus, $\exists P \subset \pi_x(C)$ compact perfect such that $\gamma \restriction P$ is continuous [M. Morayne, 1985].

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Then, clearly, $h \in h + C(\mathbb{R}) \subset PES \setminus J$ and, thus, $h \notin J$.
Therefore, $\exists C \subset \mathbb{R}^2$ closed such that $|\pi_x(C)| = \mathfrak{c}$ and $C \cap h = \emptyset$.
The function $\gamma : \pi_x(C) \to \mathbb{R}$ given by $\gamma(x) = \inf\{y : (x, y) \in C\}$ is Borel. Thus, $\exists P \subset \pi_x(C)$ compact perfect such that $\gamma \restriction P$ is continuous [M. Morayne, 1985].
By Tietze's Extension Theorem, $\exists f \in C(\mathbb{R})$ extension of $\gamma \restriction P$.
However, $0 \notin (h - f)(P)$, since h is disjoint with $C \supset \gamma \restriction P$.
Therefore $h - f \notin PES$ and we are done.

Lemma

Let $\mathcal{F}, G \subsetneq \mathbb{R}^{\mathbb{R}}$ such that $G - G \subset G$ and $\aleph_0 < card(G) < \mathcal{A}(\mathcal{F})$ then there exists $z \in \mathcal{F} \setminus G$ such that $z + G \subset \mathcal{F}$.

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Theorem (Gámez, Muñoz, Seoane, 2010)

Let $\mathcal{F} \subsetneq \mathbb{R}^{\mathbb{R}}$ star-like, that is, $\alpha\mathcal{F} \subset \mathcal{F}$ for all $\alpha \in \mathbb{R}$.
If $\mathfrak{c} < \mathcal{A}(\mathcal{F}) \leq 2^{\mathfrak{c}}$, then $\mathcal{F}$ is $\mathcal{A}(\mathcal{F})$-lineable.

Lemma

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Corollary

Let $\mathcal{F} \subsetneq \mathbb{R}^{\mathbb{R}}$ be a star-like with $\mathcal{A}(\mathcal{F}) > \mathfrak{c}$. Then $\mathcal{F}$ is lineable.

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$\mathcal{A}(J) = e_{\mathfrak{c}}$ where
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Definition

Let M be a subset of some vector space W. The **lineability cardinality number** of M is defined as

$\mathcal{L}(M) = \min\{\kappa : M \cup \{0\}$ contains no vector space of dimension $\kappa\}$.

If $\lambda(M)$ exists, then $\mathcal{L}(M) = (\lambda(M))^+$.

Theorem (Bartoszewicz, Głab, 2013)

Let $2 \leq \kappa \leq \mu$ and let $\mathbb{K}$ be a field with $|\mathbb{K}| = \mu$. Also, let V be a $\mathbb{K}$-vector space with $dim(V) = 2^{\mu}$ and $1 < \lambda \leq (2^{\mu})^{+}$. There exists a star-like family $\mathcal{F} \subset V$ such that:

1. $\kappa \leq \mathcal{A}(\mathcal{F}) \leq \kappa^{+}$.
2. $\mathcal{L}(\mathcal{F}) = \lambda$.

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THANK YOU

FOR YOUR ATTENTION!!!

Supervised by Juan Seoane