Linear Dynamics: Somewhere dense orbits are everywhere dense!

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Objective

Theorem of Bourdon-Feldman, 2003

Let T be an operator on a topological vector space X and $x \in X$. If $\mathrm{Orb}(x, T)$ is somewhere dense in X, then it is dense in X.

Topological vector space

Definition

A topological vector space (t.v.s) is a vector space X over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ endowed with a Hausdorff topology such that

$$
\begin{aligned}
+ : X \times X &\to X \\
(x, y) &\mapsto x + y
\end{aligned}
$$

$$
\begin{aligned}
\cdot : \mathbb{K} \times X &\to X \\
(\lambda, x) &\mapsto \lambda \cdot x
\end{aligned}
$$

are continuous maps.

Definitions and notation

Notation

Let $T : X \to X$ an operator (i.e, a continuous and linear map), and let $x \in X$. We denote the orbit of x under T the set
$\mathrm{Orb}(x, T) = \{x, Tx, T^2x, T^3x, ...\} = \{T^k x \; ; \; k \in \mathbb{N}_0\}$

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Definition

Let X be a topological vector space, and let $T : X \to X$ be an operator. We say that T is hypercyclic if there exists some $x \in X$ whose orbit under T is dense in X, i.e, $\overline{\mathrm{Orb}(x, T)} = X$.
We denote $HC(T)$ the set of all vectors $x \in X$ such that $\overline{\mathrm{Orb}(x, T)} = X$.

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It is clear that if T is hypercyclic, then X is separable.

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Obviously, if $x \in HC(T)$, then $\mathrm{Orb}(x, T)$ is somewhere dense.
Our goal is to show that under very few conditions (X is a t.v.s and $T : X \to X$ is an operator), these two notions are equivalent.

Some properties

Property 1

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Proof: We will prove it using a stronger result. That is, given $\lambda \in \mathbb{K} \setminus \{0\}$ and $y \in X$, the maps

$$\begin{aligned} M_\lambda : X &\to X \\ x &\mapsto \lambda x \end{aligned}$$

$$\begin{aligned} T_y : X &\to X \\ x &\mapsto x + y \end{aligned}$$

are homeomorphisms.

Some properties

The result follows from the facts that the maps $+$ and $\cdot$ are continuous, and so M_λ, $M_{\frac{1}{\lambda}}$, T_y and T_{-y} are continuous, bijective and $M_\lambda^{-1} = M_{\frac{1}{\lambda}}$, $T_y^{-1} = T_{-y}$.
From this, we obtain the property because a set S is open if and only if $x + S$ is open ($\forall x \in X$).

Some properties

Property 2

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Proof: It follows from the continuity of the scalar product map. For every $x \in X$, $\cdot_x : \mathbb{K} \to X$ is a continuous map. Then we find $\mu \in \mathbb{K} \setminus \{0\}$ small enough with $\cdot_x(\mu) \in W$. We just need to take $\lambda = \frac{1}{\mu}$ to obtain $x \in \lambda W$.

Some properties

Property 3

Let $L \subset X$ be a closed subspace. Then, the quotient space $X/L = \{x + L : x \in X\} = \{[x] : x \in X\}$ (where $x \sim y \Leftrightarrow x - y \in L$) is a t.v.s with the quotient topology (so it is Hausdorff).

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Proof: We see that X/L is Hausdorff by showing that [0] is a closed set in X/L. This is because

$$[0] = \bigcap_{W \in \mathcal{U}_0} q(W) = \bigcap_{W \in \mathcal{U}_0} (W + L) = \overline{L} = L,$$

where the $\mathcal{U}_0$ is an arbitrary basis of 0-neigbourhoods in X.

Theorem (Ansari, 1995)

Let X be a topological vector space and let T be an operator on X. Then $HC(T) = HC(T^p)$, for any $p \in \mathbb{N}$.

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Theorem (Costakis-Peris, 2000)

Let X be a topological vector space, T an operator on X and $x_1, \ldots, x_n \in X$. If $\bigcup_{j=1}^{n} \mathrm{Orb}(x_j, T)$ is dense in X, then there is some $i \in \{x_1, \ldots x_n\}$ such that $\mathrm{Orb}(x_i, T)$ is dense in X

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Problem (Peris, 2001)

If an operator T admits a somewhere dense orbit in X, should it be everywhere dense?

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$$X = \overline{\bigcup_{i=1}^{n} \mathrm{Orb}(x_j, T)} = \bigcup_{i=1}^{n} \overline{\mathrm{Orb}(x_j, T)}$$

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$$\mathrm{Orb}(x, T) = \bigcup_{j=0}^{p-1} \mathrm{Orb}(T^j x, T^p)$$

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$$T(\overline{\text{Orb}(x, T)}) \subset \overline{T(\text{Orb}(x, T))} \subset \overline{\text{Orb}(x, T)}$$

- $y \in D(x) \Rightarrow \text{Orb}(y, T) \subset \overline{\text{Orb}(x, T)} \Rightarrow \overline{\text{Orb}(y, T)} \subset \overline{\text{Orb}(x, T)}$

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$U(x) = U(T^k x), \forall k \in \mathbb{N}$

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$$\overline{\mathrm{Orb}(x, T)} = \{x, Tx, ..., T^{k-1}x\} \cup \overline{\mathrm{Orb}(T^k x, T)}$$

- We write

$$\overline{\mathrm{Orb}(x, T)} = \{x_1, \ldots, x_m\} \cup \overline{\mathrm{Orb}(T^k x, T)}$$

  with $x_i \notin \overline{\mathrm{Orb}(T^k x, T)}$, for any $i = 1, \ldots, m$.
- Thus we have, by taking interiors,

$$U(x) = int(\overline{\mathrm{Orb}(x, T)}) = int(\{x_1, \ldots, x_m\} \cup \overline{\mathrm{Orb}(T^k x, T)})$$

$$= int(\overline{\mathrm{Orb}(T^k x, T)}) = U(T^k x).$$

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$$R(\overline{\mathrm{Orb}(x,T)}) \subseteq \overline{R(\mathrm{Orb}(x,T))} = \overline{\{Rx, RTx, RT^2x, \dots\}}$$
$$= \overline{\{Rx, TRx, T^2Rx, \dots\}} = \overline{\mathrm{Orb}(Rx,T)}$$

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Complex case:

- $\exists a, \lambda_1, \ldots, \lambda_d \in \mathbb{C} : p(T) = a(T - \lambda_1 I) \cdots (T - \lambda_d I)$.

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- Let $q : X \to X/L$ be the quotient map.
- Let $S : X/L \to X/L,\ S[x] = \lambda[x]\ \forall x \in X$.

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- $\text{Orb}([x], S) = \{S^n[x] : n \in \mathbb{N}_0\} = \{[T^n x] : n \in \mathbb{N}_0\} = q(\text{Orb}(x, T))$ $\Rightarrow q(\overline{\text{Orb}(x, T)}) \subset \overline{q(\text{Orb}(x, T))} = \overline{\text{Orb}([x], S)}$.

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- There exists $z \in \mathbb{C}$ such that $\{\lambda^n z : n \in \mathbb{N}_0\}$ is somewhere dense.
- Contradiction.

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- $\exists x \in X$ such that $\{T^n x : n \in \mathbb{N}_0\}$ is somewhere dense in $X \Rightarrow \{T^n x + iT^m x : n, m \in \mathbb{N}_0\}$ is somewhere dense in $\tilde{X}$.

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- $\exists x \in X$ such that $\{T^n x : n \in \mathbb{N}_0\}$ is somewhere dense in $X \Rightarrow$ $\{T^n x + iT^m x : n, m \in \mathbb{N}_0\}$ is somewhere dense in $\tilde{X}$.
- $q(\{T^n x + iT^m x : n, m \in \mathbb{N}_0\}) = \{q(T^n x) + iq(T^m x) : n, m \in \mathbb{N}_0\}$ $= \{\lambda^n [x] + i\lambda^m [x] : n, m \in \mathbb{N}_0\} = \{(\lambda^n + i\lambda^m)[x] : n, m \in \mathbb{N}_0\}$ is somewhere dense in $\tilde{X}/L \Rightarrow \tilde{X}/L \simeq \mathbb{C}$.

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- $q(\{T^n x + iT^m x : n, m \in \mathbb{N}_0\}) = \{q(T^n x) + iq(T^m x) : n, m \in \mathbb{N}_0\}$ $= \{\lambda^n [x] + i\lambda^m [x] : n, m \in \mathbb{N}_0\} = \{(\lambda^n + i\lambda^m)[x] : n, m \in \mathbb{N}_0\}$ is somewhere dense in $\tilde{X}/L \Rightarrow \tilde{X}/L \simeq \mathbb{C}$.
- We consider $|q| : X \to \mathbb{R}_0{}^+$. Since $\{T^n x : n \in \mathbb{N}_0\}$ is somewhere dense in X, $|q| (\{T^n x : n \in \mathbb{N}_0\}) = \{|\lambda|^n \, |[x]| : n \in \mathbb{N}_0\}$ is somewhere dense in $\mathbb{R}_0{}^+$.

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Proof: The set $A := \{p(T)x; p \neq 0 \text{ a polynomial}\}$ is path connected. Take p, q nonzero polynomials such that p is not multiple of q then the path

$$tp(T)x + (1-t)q(T)x, \quad t \in [0,1],$$

is contained in A. If p is multiple of q, we take a polynomial r that is not multiple of q.

Lemma 2

... Observe that $\overline{A}$ is a subspace of X. The element 0 is in $\overline{A}$ since for every 0-neighbourhood W we can find a polynomial p_W such that

$$p_W(T)x \in W.$$

Since $A \cup \{0\}$ is a subspace of X we have that $\overline{A}$ is a subspace of X since

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Moreover $\overline{\mathrm{Orb}(x, T)} \subseteq \overline{A}$.

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Thus, for any $y \in X$, there is a scalar λ with $y \in \lambda W$.
Then, since $\lambda x_0 \in \overline{A}$ and $\lambda(x_0 + W) \subset \overline{A}$ we have that

$$y \in \lambda(x_0 + W) - \lambda x_0 \subset \overline{A}.$$

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Then, since $\lambda x_0 \in \overline{A}$ and $\lambda(x_0 + W) \subset \overline{A}$ we have that

$$y \in \lambda(x_0 + W) - \lambda x_0 \subset \overline{A}.$$

Therefore $\overline{A} = X$.

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- *Step 2.* For any $z \in X \setminus U(x)$, $D(z) \subset X \setminus U(x)$.

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- *Step 1.* $T(X \setminus U(x)) \subset X \setminus U(x)$.
- *Step 2.* For any $z \in X \setminus U(x)$, $D(z) \subset X \setminus U(x)$.
- *Step 3.* For any polynomial $p \neq 0$, $p(T)x \in X \setminus \partial D(x)$.

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- *Step 1.* $T(X \setminus U(x)) \subset X \setminus U(x)$.
- *Step 2.* For any $z \in X \setminus U(x)$, $D(z) \subset X \setminus U(x)$.
- *Step 3.* For any polynomial $p \neq 0$, $p(T)x \in X \setminus \partial D(x)$.
- *Step 4.* $D(x) = X$.

Proof

Step 1

$$T(X \backslash U(x)) \subset X \backslash U(x)$$

We show,equivalently, that $T^{-1}(U(x)) \subset U(x)$.

- There exists $m \in \mathbb{N}_0$ such that $x_m := T^m x \in U(x)$.
- Let $y \in T^{-1}(U(x))$, and let V be an arbitrary neighbourhood of y.
- $A := \{q(T)x_m : q \neq 0 \text{ a polynomial}\}$ is dense in X, which implies that there is $p \neq 0$ polynomial such that $p(T)x_m \in V \cap T^{-1}(U(x))$.

Proof

We will show $p(T)x_m \in V \cap D(x)$. Note that

$$p(T)x_m \in p(T)(U(x)) = p(T)(U(T^{m+1}x)) \subset p(T)(D(T^{m+1}x)), \quad (1)$$

and

$$p(T)(D(T^{m+1}x)) \subset D(p(T)T^{m+1}x) = D(Tp(T)x_m). \quad (2)$$

By (1) and (2), we have

$$p(T)x_m \in D(Tp(T)x_m).$$

Since

$$p(T)x_m \in T^{-1}(U(x)) \Rightarrow Tp(T)x_m \in U(x) \subset D(x),$$

which implies

$$p(T)x_m \in D(x) \Rightarrow p(T)x_m \in V \cap D(x).$$

Proof

Since V is an arbitrary neighbourhood of y and $D(x)$ is closed, we deduce that $y \in D(x)$. Also, $U(x)$ is open and T is continuous, which yields $T^{-1}(U(x)) \subset U(x)$.

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Step 2

For any $z \in X \backslash U(x)$, $D(z) \subset X \backslash U(x)$.

By Step 1, $X \backslash U(x)$ is T-invariant, and we have that $T^m z \in X \backslash U(x)$ for all $m \in \mathbb{N}_0$. Therefore $\mathrm{Orb}(z, T) \subset X \backslash U(x)$. Since $X \backslash U(x)$ is closed, we deduce that

$$D(z) \subset X \backslash U(x).$$

Proof

Step 3

For any polynomial $p \neq 0, \quad p(T)x \in X \setminus \partial D(x)$

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Suppose that $p(T)x \in \partial D(x)$ for some polynomial $p \neq 0$.

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Suppose that $p(T)x \in \partial D(x)$ for some polynomial $p \neq 0$.

- There exists some $y \in X$ such that $p(T)y \in U(x)$.

Proof

Step 3

For any polynomial $p \neq 0, \;\; p(T)x \in X \setminus \partial D(x)$

Suppose that $p(T)x \in \partial D(x)$ for some polynomial $p \neq 0$.

- There exists some $y \in X$ such that $p(T)y \in U(x)$.
- $p(T)(D(x)) \subseteq D(p(T)x) \subseteq X \setminus U(x)$.

Proof

Step 3

For any polynomial $p \neq 0, \quad p(T)x \in X \setminus \partial D(x)$

Suppose that $p(T)x \in \partial D(x)$ for some polynomial $p \neq 0$.

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Proof

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For any polynomial $p \neq 0$, $p(T)x \in X \setminus \partial D(x)$

Suppose that $p(T)x \in \partial D(x)$ for some polynomial $p \neq 0$.

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- $p(T)(D(x)) \subseteq D(p(T)x) \subseteq X \setminus U(x)$.
- $y \in X \setminus D(x)$.
- There exists some polynomial $q \neq 0$ such that

$$q(T)x \in X \setminus D(x) \quad \& \quad p(T)q(T)x \in U(x)$$

Proof

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- There exists some polynomial $q \neq 0$ such that

$$q(T)x \in X \setminus D(x) \quad \& \quad p(T)q(T)x \in U(x)$$

- $q(T)p(T)x \in q(T)(D(x)) \subseteq D(q(T)x) \subseteq X \setminus U(x)$.

Which is a contradiction.

Proof

Step 4

$$D(x) = X$$

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We recall that the set

$$A = \{p(T)x : p \neq 0 \text{ is a polynomial}\}$$

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$$A \subseteq U(x) \cup (X \setminus D(x)),$$

where $U(x)$ and $X \setminus D(x)$ are open disjoint sets.

Proof

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- $A \cap U(x) \neq \emptyset$.
- $X \setminus D(x) = \emptyset$.

Which implies that $D(x) = X$, as desired.

Bibliography

S.I. Ansari, Hypercyclic and cyclic vectors, J. Funct. Anal. **128** (1995), 374–383.

P.S. Bourdon and N.S. Feldman, Somewhere dense orbits are everywhere dense, Indiana Univ. Math. J. **52** (2003), no. 3, 811–819.

G. Costakis, On a conjecture of D. Herrero concerning hypercyclic operators, C.R. Acad. Sci. Paris **330** (2000), 179–182.

K.-G. Grosse-Erdmann and A. Peris Manguillot, *Linear Chaos*. Universitext. Springer, London, 2011.
http://dx.doi.org/10.1007/978-1-4471-2170-1

A. Peris, Multihypercyclic operators are hypercyclic, Math. Z. **236** (2001), 779–786.

J. Wengenroth, Hypercyclic operators on non-locally convex spaces, Proc. Amer. Math. Soc. **131** (2003), no. 6, 1759–1761.