Geometric characterisations of $\ell^1(\Gamma)$

Antonio Zarauz Moreno

XIV Encuentro de la Red de Análisis Funcional y Aplicaciones

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Outline

Notation

- $\mathbb{K}$: $\mathbb{R}$ or $\mathbb{C}$.
- $\mathbb{T} = \{\lambda \in \mathbb{K}\colon |\lambda| = 1\}$.
- $(X, \|\cdot\|_X)$: Banach space.
- $\mathsf{B}_{(X,\|\cdot\|_X)}$: Closed unit ball of $(X, \|\cdot\|_X)$.
- $\mathsf{E}_{(X,\|\cdot\|_X)}$: Extreme points of $\mathsf{B}_{(X,\|\cdot\|_X)}$.

First definitions

First we introduce some definitions to deal with the problem.

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Particular subsets of the previous ones lie in the next definition.

Definition

A subset F of a convex set A is called a **face** *of A if:*

1. *F is convex.*
2. *For every $x \in F$ and $y, z \in A$ with $x = \frac{1}{2}(y+z)$, we have that $y, z \in F$. In particular, zero-dimensional faces are* **extreme points**.

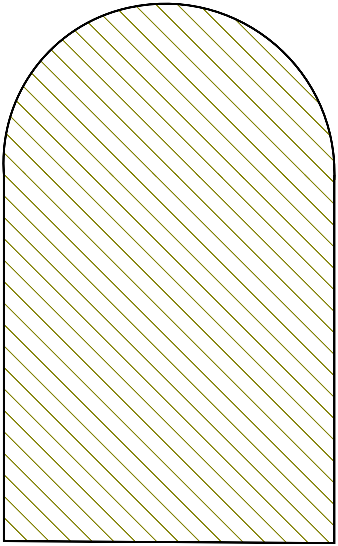

Figure: Example of convex set

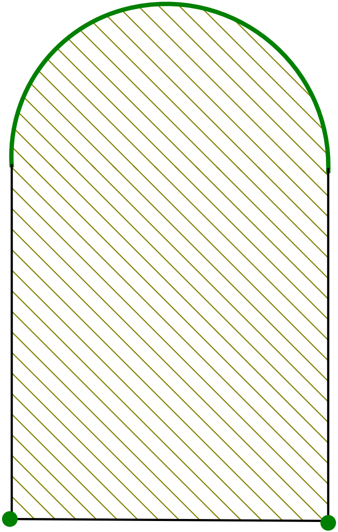

Figure: Extreme points

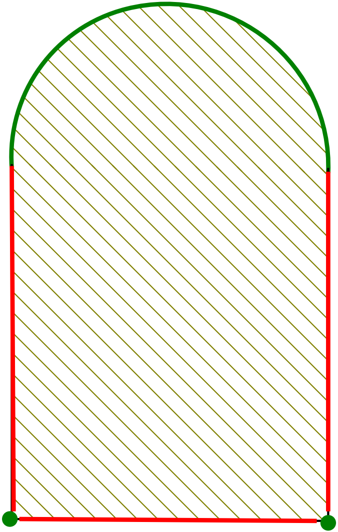

Figure: One-dimensional faces

Finite dimensional spaces

Let X be a finite-dimensional normed space ($\dim(X) = n$). Thanks to Minkowski's theorem,

$$\mathrm{B}_X = \mathrm{co}(\mathrm{E}_X)$$

If $x \in \mathrm{E}_X$, it is easy to show that $\mathbb{T}x \subset \mathrm{E}_x$, hence the group $\mathbb{T}$ acts transitively on E_X. Since the unit ball of a normed space is absorbing, in particular $|\mathrm{E}_X/\sim| \geq n$.

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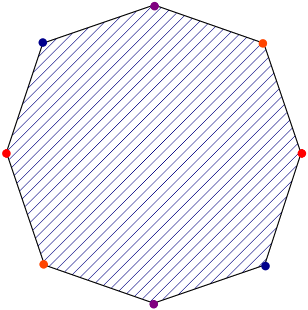

Infinite-dimensional spaces: Preserving the equation

Theorem (Krein-Milman)

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Theorem (Namioka, Phelps)

Let A be a norm-closed convex bounded set in a **separable dual space** *X^*. Then, A is the norm-closed convex hull of its strongly exposed points.*

Infinite-dimensional spaces: Preserving the coordinates

$a \in A$

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Theorem (Krein-Milman revisited)

Suppose that A is a metrizable compact convex subset of a locally convex space X, and that $x_0 \in A$. Then there is a probability measure on A which represents x_0 and is supported by the **closure of the extreme points** *of A.*

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Minkowski's functional

Theorem

Let X be a locally convex space. The map $p_A : X \to \mathbb{R}_0^+$ given by

$$p_A(x) = \inf \{\lambda \in \mathbb{R}_0^+ : x \in \lambda A\}$$

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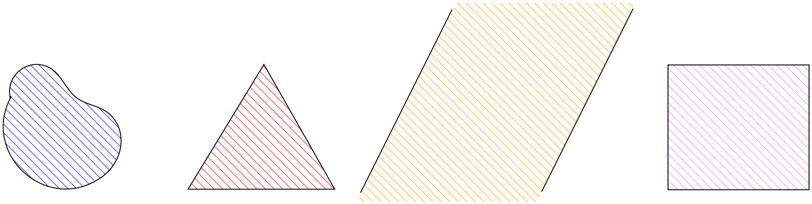

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Problem (Infinite-dimensional setting)

Given a vector space X, is there any norm for which B_X has a "minimum number" of extreme points such that $B_X = \overline{co}(E_X)$?

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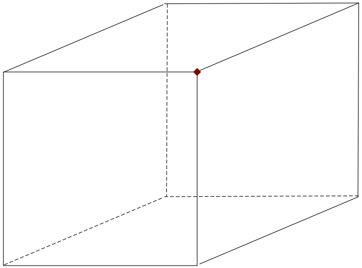
- $H^1(\mathbb{D})$, $\ell^\infty_{\mathbb{C}}(\Gamma)$.

Lemma

Let X be a normed space. Then, the $(*)$-property is equivalent to the linear independence of the set $E_X/\sim$.



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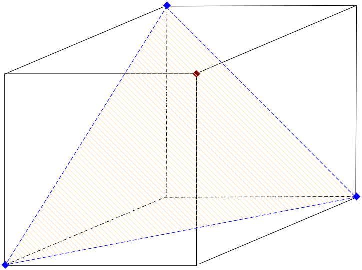

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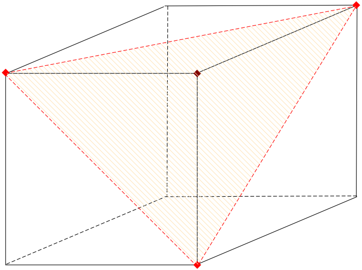

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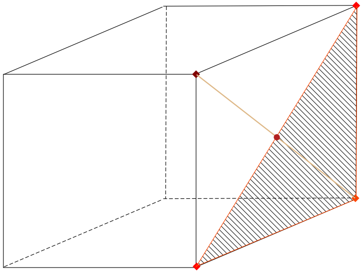

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Under the hypothesis of the previous theorem, if $E_X/\sim$ is also countable, then $X \cong \ell^1(\mathbb{N})$.

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Under the hypothesis of the previous theorem, if $E_X/\sim$ is also countable, then $X \cong \ell^1(\mathbb{N})$.

Corollary

Let X be a finite-dimensional normed space ($\dim(X) = n$) with the $(*)$-property. Then, $X \cong \ell^1_n$.

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Definition

Let $(X, \|\cdot\|_X)$ *be a normed space.* $\|\cdot\|_X$ *has the minimal vertices property if the following conditions hold:*

- $B_{(X,\|\cdot\|_X)} = \overline{co}\left(E_{(X,\|\cdot\|_X)}\right)$.
- *If* $|||\cdot|||$ *is an equivalent norm with* $B_{(X,|||\cdot|||)} = \overline{co}\left(E_{(X,|||\cdot|||)}\right)$ *and* $E_{(X,|||\cdot|||)} \subset E_{(X,\|\cdot\|_X)}$*, then* $E_{(X,|||\cdot|||)} = E_{(X,\|\cdot\|_X)}$.

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- Using that $\mathsf{B}_{(\ell^1(\Gamma),|||\cdot|||)}$ is a balanced set, we get that

$$\mathbb{T}x \cap \mathsf{B}_{(\ell^1(\Gamma),|||\cdot|||)} = \emptyset$$

which means that $\mathbb{D}x \cap \mathsf{B}_{(\ell^1(\Gamma),|||\cdot|||)} = \{0\}$ in light of the linear independence of $\mathsf{E}_{\ell^1(\Gamma)}/\sim$.

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- Thanks to the fact that $\mathbb{D}x \cap \mathsf{B}_{(X,|||\cdot|||)} = \{0_X\}$, we have that $x = 0_X$.

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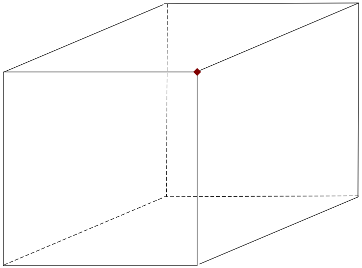

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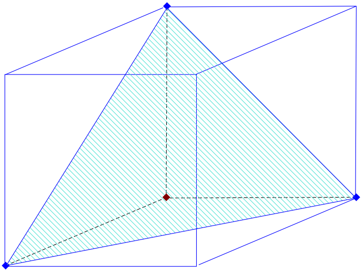

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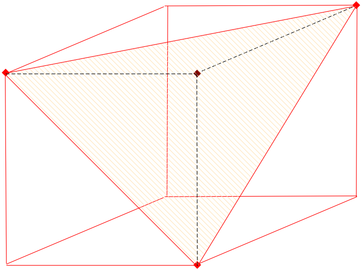

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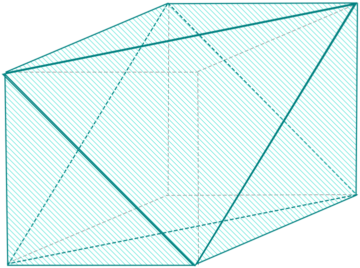

Thanks for your attention!