Extensiones del Teorema de Krasnoselkii de interpolación a operadores estrictamente singulares

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(trabajo conjunto con E. Semenov y P. Tradacete)

1. Extension of Krasnoselskii interpolation theorem to S
2. Properties of strictly singular non-compact operator sets $V_{p,q}$

Theorem (Krasnoselskii 1960)

Let $1 \leq p_0\,,\, p_1\,,\, q_0, q_1 \leq \infty$.

If $T : L_{p_0} \to L_{q_0}$ *compact* , $T : L_{p_1} \to L_{q_1}$ *bounded*

$\implies T : L_{p_\theta} \to L_{q_\theta}$ *compact* , .

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Open problem : Complex interpolation method ?? (Cwikel, ...)

The **characteristic** set of $T : L_\infty \to L_1$ is :

$$L(T) := \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in [0,1] \times [0,1] : \; T : L_p \mapsto L_q \; \textit{bounded} \right\}.$$

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-For integral operators : the set $L(T) \setminus K(T)$ plays an role

[K-Z-P-S 1976] Krasnoselskii, Zabreiko, Pustylnik, Sobolevsky

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- $\mathcal{S}$ is NOT suitable for interpolation

($L_\infty \hookrightarrow L_1$ strictly singular but $L^2 \hookrightarrow L_1$ no)

results for operators defined on the same domain
(*Heinrich*, ..)

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- **extrapolation** property:

If $1 < q, p < \infty$ and $T : L_q \to L_q$, $T : L_p \to L_p$ bounded.Then
$T \in S(L_r)$ for some $r \in (q, p) \Longleftrightarrow T \in K(L_s)$ for some $s \in (q, p)$.

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In particular it follows that $T \in K(L_s)$ for **every** $s \in (q, p)$

The proof uses :

- properties of the sets $S(L_p)\backslash K(L_p)$,

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Theorem (L. Dor, 1975)

Let $1 \leq p \neq 2 < \infty$, $0 < \theta \leq 1$, *and* $(f_i)_{i=1}^{\infty}$ *in* $L_p[0,1]$. *Assume that either:*

1. $1 \leq p < 2$, $\|f_i\| \leq 1$ *and* $\|\sum_{i=1}^n a_i f_i\| \geq \theta(\sum_{i=1}^n |a_i|^p)^{1/p}$ *for scalars* $(a_i)_{i=1}^n$, *and every n,*
   or
2. $2 < p < \infty$, $\|f_i\| \geq 1$ *and* $\|\sum_{i=1}^n a_i f_i\| \leq \theta^{-1}(\sum_{i=1}^n |a_i|^p)^{1/p}$ *for scalars* $(a_i)_{i=1}^n$ *and every n.*

Then there exist disjoint measurable sets $(A_i)_{i=1}^{\infty}$ *s.t.*

$$\|f_i \chi_{A_i}\| \geq \theta^{2/|p-2|}.$$

-interpolation: the general case

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$p_0 < \infty$ is a necessary condition

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-If $q \leq 2 \leq p$.

Assume $T \notin K(L_p, L_q)$, there is (x_i) s.t. $\|x_i\|_p = 1$, $x_i \xrightarrow{w} 0$ and $\|Tx_i\|_q \geq \lambda > 0$. By Kadec-Pelczynski m. there is (x_{i_k}) equivalent to the basis of ℓ_p or ℓ_2.

Now, (x_{i_k}) cannot be equivalent to the basis of ℓ_p,

$$\lambda\, n^{\frac{1}{2}} \leq \int_0^1 \Big\| \sum_{k=1}^n r_k(t) T x_{i_k} \Big\|_q dt \leq \|T\| \int_0^1 \Big\| \sum_{k=1}^n r_k(t) x_{i_k} \Big\|_p dt \leq n^{\frac{1}{p}}.$$

Thus, (x_{i_k}) must be equivalent to the basis of ℓ_2 . Now we can deduce that T is not strictly singular

-Every strictly singular $T : \ell_2 \mapsto F$ is compact if F is Banach lattice with lower 2-estimate (Flores,H.,Kalton, Tradacete 2009)

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- If $q \leq 2 \leq p$, every $T : L_p \mapsto L_q$ is either compact or fixes a ℓ_2-isomorphic copy

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• the case $2 < q \leq p < \infty$.

If $T : L_p \to L_q$ and $T \in V_{p,q} \Longrightarrow$ there exists normalized (y_k) in L_p s.t.:

- (y_k) is equivalent to the basis of ℓ_2 with $(|y_k|)$ equi-measurable,

- (Ty_k) is equivalent to the basis of ℓ_q,

- $[(y_k)]$ is complemented in L_p.

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• There is st. singular $T : L_p \to L_q$ s. t. T^* is not $\Leftrightarrow$ $2 < q < p$.

key result:

Theorem (extrapolation)

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$$V(T) := \left\{ \left(\frac{1}{p}, \frac{1}{q}\right) : T \in V_{p,q} \right\}.$$

It holds that :

$$V(T) \subseteq \partial L(T)$$

strong interpolation property :

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If $T : L_{p_0} \to L_{q_0}$ *strictly singular and* $T : L_{p_1} \to L_{q_1}$ *bounded*

$\Longrightarrow \quad T \in K(L_{p_\theta}, L_{q_\theta})$ *for every* $0 < \theta < 1$.

For *regular* operators :
if $1 < q \leq p < \infty$ and $T : L_p \to L_q$ regular strictly singular then T is compact. (*Caselles – Gonzalez*, 1987)

strong interpolation property :

Proposition

Let $1 < p_0, p_1, q_0, q_1 < \infty$ $(p_0 \neq p_1, \quad q_0 \neq q_1)$ *with*

- $\text{mín}\{\frac{q_0}{p_0}, \frac{q_1}{p_1}\} \leq 1$,

or

- $\text{mín}\{\frac{q_0}{p_0}, \frac{q_1}{p_1}\} > 1$ *and* $\frac{q_1 - q_0}{p_1 - p_0} < 0$.

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Extra conditions are required:

segment $S_\lambda := \{(p,q) : \frac{1}{q} = \frac{1}{p} - \lambda\}, (0 < \lambda < 1)$

Examples: operators $T \in V_{p,q}$ for every $(p,q) \in S_\lambda$

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fractional averaging op. $T_\lambda = T$: let $(A_k)_{k\in\mathbb{N}}$ disjoint

$$T(f) = \sum_{k\in\mathbb{N}} \left(\frac{1}{\mu(A_k)^{1-\lambda}} \int_{A_k} f d\mu \right) \chi_{A_k}.$$

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T factors through $\ell_p \hookrightarrow \ell_q$ (so it is strictly singular) , since

$$\begin{array}{ccc} L_p & \xrightarrow{\quad T \quad} & L_q \\ {\scriptstyle P}\downarrow & & \uparrow{\scriptstyle Q} \\ \ell_p & \xhookrightarrow{\quad i \quad} & \ell_q \end{array}$$

T is not compact ($T(\mu(A_k)^{-\frac{1}{p}}\chi_{A_k}) = \mu(A_k)^{-\frac{1}{q}}\chi_{A_k}$)

Examples : *Riesz potential* operator $T = T_\alpha$, $(0 < \alpha < 1)$

$T : L^p \mapsto L^q$ positive integral operator :

$$Tf(t) = \int_0^1 \frac{f(s)}{|t-s|^\alpha} ds$$

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⋆ **Question:** what is the shape of a $V(T)$ set in general ?

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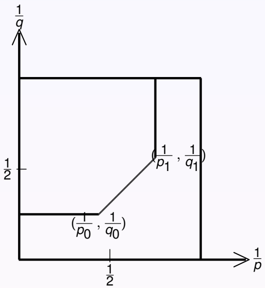
$\star$ **Question:** what is the shape of a $V(T)$ set in general ?

- $V(T) = \emptyset$, (f.e. multiplier operator , $Tf = g\,f$, for some $g \in Ł^r$)

- $V(T)$ is just one point $\{(\frac{1}{p}, \frac{1}{q})\}$ (for $q < p$).

Example: operators $T : L_\infty \mapsto L_1$ with $V(T)$ is



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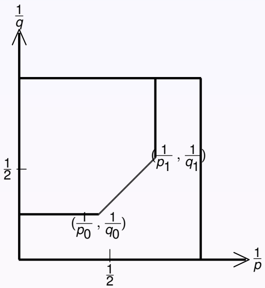

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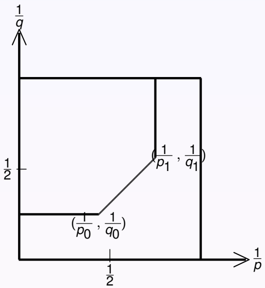

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$$T = T_1 + T_2 + T_3$$

$T_1 : L_p[0, 1/3] \mapsto L_{q_0}[0, 1/3]$ factorizing by $i_{2,q_0} : \ell_2 \to \ell_{q_0}$
$T_2 : L_p[\frac{1}{3}, \frac{2}{3}] \to L_q[\frac{1}{3}, \frac{2}{3}]$ λ-fractional on $[1/3, 2/3]$
$T_3 : L_{p_1}[2/3] \mapsto L_q[2/3, 1]$ factorizing by $i_{p_1,2} : \ell_{p_1} \to \ell_2$,