A straight way to twisted sums

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1. Setting off: twisted sums and diagrams
2. On our way: constructing twisted sums
3. Getting there: twisted sums of ℓ_1 and c_0

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In this case, we say Z is the *trivial sum* of Y and X.

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called *exact sequence.*

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In particular, every twisted sum of c_0 and ℓ_1 is trivial.

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- ...but that is all we know.

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Examples of quasi-linear maps:

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- A bounded homogeneous map $B : X \to Y$.

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$$\|u + v\| \leq \Delta\big(\|u\| + \|v\|\big) \quad , \quad \Delta \geq 1$$

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According to this:

- $\mathrm{Ext}(X, Y) = 0 \iff$ every twisted sum of Y and X is trivial.
- $\mathrm{Ext}(X, Y) \neq 0 \iff$ there are non-trivial twisted sums of Y and X.

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is *strictly singular*. That means its restrictions to infinite dimensional subspaces are never trivial.

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This twisted sum is *strictly non-singular*. That means
in every infinite dimensional subspace there is a further subspace
on which it is trivial.

Composition of quasi-linear maps with operators

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If Ω is a quasi-linear map and T is an operator, then ΩT and $T\Omega$ are quasi-linear maps.

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How can we represent this situation using diagrams?

Learning Japanese

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mathematics $\Rightarrow$ 数学 $\Rightarrow$ *suugaku*

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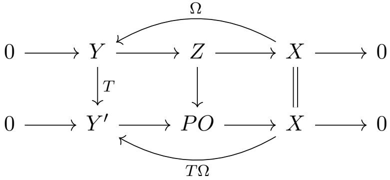

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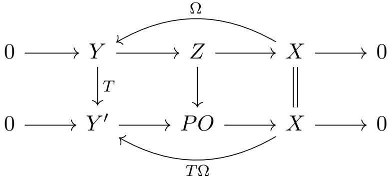

$\Rightarrow$ *push-out*

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 & & & & \overset{\Omega}{\curvearrowleft} & & & & \\
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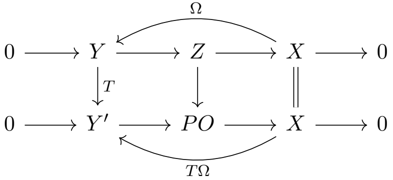

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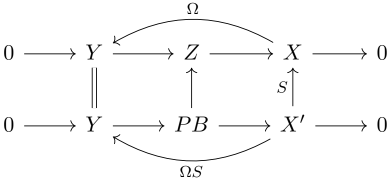

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Theorem 2 (*Cabello, Castillo, Kalton and Yost (2004)*)

$$\mathrm{Ext}(L_\infty, L_1) \neq 0 \quad \Rightarrow \quad \mathrm{Ext}(c_0, \ell_1) \neq 0$$

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If $\Phi : C(K) \to L_1(\mu)$ is non-trivial, there is a copy of c_0 in $C(K)$ so that $\Phi|_{c_0}$ is non-trivial.

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4. Apply Theorem 3 to Ψ^*, finally obtaining

$$\phi : c_0 \to \ell_1$$

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- Other methods to obtain them?

Thank you for your attention