Algoritmos avariciosos y bases bidemocráticas

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Joint work with F. Albiac, J. L. Ansorena, M. Berasategui and S. Lassalle

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Notation:

- Let $\mathbb{X}$ be a Banach (or quasi-Banach) space over $\mathbb{F}$.
- Let $\mathcal{B} := \{\mathbf{x}_n\}_{n=1}^{\infty}$ a basis for $\mathbb{X}$:
  - $\overline{span(\mathbf{x}_n : n \in \mathbb{N})} = \mathbb{X}$ (fundamental system).
  - There is a sequence $\mathcal{B}^* = (\mathbf{x}_n^*)_{n=1}^{\infty}$ such that $\mathbf{x}_n^*(\mathbf{x}_m) = \delta_{n,m}$.
  - $\sup_n \max\{\|\mathbf{x}_n\|, \|\mathbf{x}_n^*\|\} < \infty$.

If $f \in \mathbb{X}$, $f \sim \sum_n \mathbf{x}_n^*(f)\mathbf{x}_n$, with $(\mathbf{x}_n^*(f))_n \in c_0$.

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If $f \in \mathbb{X}$, $f \sim \sum_n \mathbf{x}_n^*(f)\mathbf{x}_n$, with $(\mathbf{x}_n^*(f))_n \in c_0$.

Also, if the basis is total, that is

$$\mathbf{x}_j^*(f) = 0 \text{ for all } j \in \mathbb{N} \Rightarrow f = 0,$$

the basis $\mathcal{B}$ is Markushevich.

Let A be a finite set of indices and we define $\mathcal{E}_A$ the set of the signs:

$$\mathcal{E}_A = \{\varepsilon = (\varepsilon_n)_{n\in A} : |\varepsilon_n| = 1\}.$$

"The indicator sum on A with signs":

$$\mathbf{1}_{\varepsilon A} = \sum_{n\in A} \varepsilon_n \mathbf{x}_n, \ \varepsilon \in \mathcal{E}_A.$$

If $\varepsilon \equiv 1$, we use $\mathbf{1}_A$.

- The projection operator: if A is a finite set,

$$P_A(f) = \sum_{n \in A} \mathbf{x}_n^*(f)\mathbf{x}_n.$$

- $\mathcal{B}$ is $\mathbf{K}$-unconditional if for every finite set A and $f \in \mathbb{X}$,

$$\|P_A(f)\| \leq \mathbf{K}\|f\|.$$

- $\mathcal{B}$ is Schauder if there is a constant C such that

$$\|P_{\{1,\ldots,m\}}(f)\| \leq C\|f\|,$$

for every $m \in \mathbb{N}$ and $f \in \mathbb{X}$.

Thresholding greedy algorithm

Let $f \in \mathbb{X}$. The m**th greedy sum** of f is the sum

$$\mathcal{G}_m(f) = \sum_{j \in A_m(f)} \mathbf{x}_j^*(f)\mathbf{x}_j,$$

$$\min_{j \in A_m(f)} |\mathbf{x}_j^*(f)| \geq \max_{j \notin A_m(f)} |\mathbf{x}_j^*(f)|.$$

The set $A_m(f)$ is called the mth **greedy set** and the collection $\{\mathcal{G}_m\}_m$ is the **Greedy Algorithm**.

Quasi-greedy bases

Definition

A basis $\mathcal{B}$ is **quasi-greedy** if there exists a constant C such that for any $f \in \mathbb{X}$ and $m \in \mathbb{N}$ we have

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Theorem (Wojtasczyk; 2000)

A basis is quasi-greedy if and only if

$$\lim_{m\to\infty} \|f - \mathcal{G}_m(f)\| = 0.$$

Remark: every quasi-greedy basis is total, so if $\mathcal{B}$ is quasi-greedy, then $\mathcal{B}$ is Markushevich.

P. Wojtaszczyk, *Greedy algorithm for general biorthogonal systems*, J.Approx.Theory **107** (2000), no.2, 293-314.

Greedy Bases

$$\sigma_m(f) := \inf\{\|f - \sum_{n\in A} c_n \mathbf{x}_n\| \,:\, |A| = m,\ c_n \in \mathbb{F}\}.$$

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Definition

A basis $\mathcal{B}$ is **greedy** if there is a positive constant C such that

$$\sigma_m(f) \leq \|f - \mathcal{G}_m(f)\| \leq C\sigma_m(f),\ \ \forall m \in \mathbb{N},\ \ \forall f \in \mathbb{X}.$$

We denote by $C_g = C_g[\mathcal{B}, \mathbb{X}]$ the least constant verifying the definition.

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Theorem (Konyagin, Temlyakov; 1999), (AABW; 2021)

A basis is greedy if and only if the basis is unconditional and democratic.

We say that a basis $\mathcal{B}$ is Δ_d-**democratic** with $\Delta_d > 0$ if

$$\|1_A\| \leq \Delta_d \|1_B\|,$$

for any $|A| \leq |B|$.

S.V.Konyagin, V.N.Temlyakov, *A remark on greedy approximation in Banach spaces*, East J. Approx. **5** (1999), 365-379.

Almost greedy Bases

$$\tilde{\sigma}_m(f) := \inf\{\|f - P_A(f)\| : |A| = m\}.$$

Definition

A basis $\mathcal{B}$ is **almost-greedy** if there exists an absolute constant $C \geq 1$ such that

$$\tilde{\sigma}_m(f) \leq \|f - \mathcal{G}_m(f)\| \leq C\tilde{\sigma}_m(f), \ \ \forall m \in \mathbb{N}, \ \ \forall x \in \mathbb{X}.$$

Theorem (Dilworth, Kutzarova, Kalton Temlyakov; 2003), (AABW; 2021)

A basis is almost-greedy if and only if the basis is quasi-greedy and democratic.

S.J. Dilworth, N.J. Kalton, Denka Kutzarova, V.N. Temlyakov, *The thresholding greedy algorithm, greedy bases, and duality*, Constr.Approx.**19** (2003), no.4, 575-597.

What about duality?

If $\mathcal{B}$ is greedy, is the dual basis $\mathcal{B}^*$ also greedy?

Greediness

Consider $\mathcal{H}_1 = (h_n^1)_{n=1}^\infty$ the Haar basis normalized in $L_1[0,1]$ and consider $\mathbb{X}$ the space of all sequences of scalaras $(a_n)_{n=1}^\infty$ such that

$$\|(a_n)_n\| = \int_0^1 \left(\sum_{n=1}^\infty (a_n h_n^1(t))^2 \right)^{1/2} dt < \infty.$$

The unit vector basis $\mathcal{B} = (e_n)_n$ in $(\mathbb{X}, \|\cdot\|)$ is a normalized greedy basis but $\mathcal{B}^*$ is not greedy.

F. ALBIAC, N.J. KALTON, *Topics in Banach space Theory*, Springer.

Almost-greediness

Let $(e_n)_n$ be the canonical basis in $\ell_1(\mathbb{N})$ and define the vectors

$$\mathbf{x}_n = e_n - \frac{1}{2}e_{2n+1} - \frac{1}{2}e_{2n+2},\ n = 1, 2, ...$$

The system $\mathcal{L} = (\mathbf{x}_n)_n$ was introduced by Lindestrauss and it is an almost-greedy basis, but $\mathcal{L}^*$ is not almost-greedy.

S.J. DILWORTH, D. MITRA, *A conditional quasi-greedy basis of ℓ_1*. Studia Math. 144 (2001), 95-100.

P. M. BERNÁ, Ó. BLASCO, G. GARRIGÓS, E. HERNÁNDEZ, T. OIKHBERG, *Embeddings and Lebesgue-type inequalities for the greedy algorithm in Banach spaces*. Constr. Approx. **48** (3) (2018), 415–451.

The fundamental functions of $\mathbb{X}$ and $\mathbb{X}^*$:

$$\varphi(m) = \varphi[\mathcal{B}, \mathbb{X}](m) := \sup_{\substack{\varepsilon \in \mathcal{E}_A \\ |A| \leq m}} \|\mathbf{1}_{\varepsilon A}\|,$$

$$\varphi^*(m) = \varphi[\mathcal{B}^*, \mathbb{Y}](m) = \sup_{\substack{\varepsilon \in \mathcal{E}_A \\ |A| \leq m}} \|\mathbf{1}^*_{\varepsilon A}\|,$$

where $\mathbf{1}^*_{\varepsilon A} = \sum_{n \in A} \varepsilon_n \mathbf{x}_n^*$ and $\mathbb{Y}$ is the subspace of $\mathbb{X}^*$ spanned by $\mathcal{B}^*$.

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where $\mathbf{1}^*_{\varepsilon A} = \sum_{n \in A} \varepsilon_n \mathbf{x}^*_n$ and $\mathbb{Y}$ is the subspace of $\mathbb{X}^*$ spanned by $\mathcal{B}^*$.

Definition

A basis $\mathcal{B}$ is **bidemocratic** if there is $C > 0$ such that

$$\varphi(m)\varphi^*(m) \leq C\, m, \ \forall m \in \mathbb{N}.$$

We denote by $\Delta = \Delta[\mathcal{B}, \mathbb{X}]$ the least constant verifying the definition.

Remark: if $|A| = m$,

$$m = \mathbf{1}^*_A(\mathbf{1}_A) \leq \|\mathbf{1}^*_A\|_* \|\mathbf{1}_A\| \leq \varphi(m)\varphi^*(m).$$

Duality results

Theorem (DKKT, 2003)

Let $\mathcal{B}$ a quasi-greedy Schauder basis in a Banach space. The following are equivalent:

- $\mathcal{B}$ is bidemocratic.
- $\mathcal{B}$ and $\mathcal{B}^*$ are almost-greedy.

Theorem (DKKT, 2003)

Let $\mathcal{B}$ an unconditional basis in a Banach space. The following are equivalent:

- $\mathcal{B}$ is bidemocratic.
- $\mathcal{B}$ and $\mathcal{B}^*$ are greedy.

S.J. Dilworth, N.J. Kalton, D. Kutzarova, V.N. Temlyakov, *The thresholding greedy algorithm, greedy bases, and duality*, Constr.Approx. **19** (2003), no.4, 575-597.

Bidemocracy vs Quasi-greediness

F. Albiac, J.L. Ansorena, M. Berasategui, P. M. Berná, S. Lassalle, *Bidemocratic bases and their connetions with other greedy-type bases*. Submitted (2021)

Bidemocracy vs Quasi-greediness

A weight $w = (w_n)_n$ is a bounded sequence of positive numbers and its primitive weight $(s_n)_n$ is given by $s_n := \sum_{j=1}^n w_n$.

Given a weight w and $0 < q < \infty$, the weighted Lorentz seqeunce space $d_{1,q}(w)$ is the space of sequences $(a_n)_n \subset c_0$ whose non-increasing rearrangement $(a_n^*)_n$ satisfies

$$\left(\sum_{n=1}^{\infty} (a_n^*)^q s_n^{q-1} w_n\right)^{1/q} < \infty, \tag{1}$$

with the quasi-norm given the the left-hand side of (1).
When $w_n = n^{1/p-1}$ for some $1 < p < \infty$, $d_{1,q}(w)$ is the Lorentz space $\ell_{p,q}$ (up to an equivalent quasi-norm) and $s_m \approx m^{1/p}$.

Bidemocracy vs Quasi-greediness

Theorem (AABBL,2021)

Let $\mathcal{B}$ a basis for a quasi-Banach space $\mathbb{X}$ and let w be a weight whose primitive weight $(s_n)_n$ is unbounded. Assume that $\mathcal{B}$ verifies the following conditions:

- $\mathcal{B}$ is bidemocratic with $\varphi(n) \approx s_n$.
- $\mathcal{B}$ has a subsequence dominated by the unit vector basis of $d_{1,q}(w)$ for some $1 < q < \infty$.

Then, $\mathbb{X}$ has a bidemocratic basis $\mathcal{B}_1$ with $\varphi[\mathcal{B}_1, \mathbb{X}](m) \approx s_n$ that is not Markushevich (and hence, not quasi-greedy). In fact,

$$(\log(m))^{1/q'} \lesssim \mathbf{k}_m.$$

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$$(\log(m))^{1/q'} \lesssim \mathbf{k}_m.$$

Corollary

For all $1 < p < \infty$, ℓ_p has a bidemocratic basis that is not Markushevich.

Bidemocracy vs Quasi-greediness

Theorem (AABBL,2021)

Let $\mathcal{B}$ a basis for a quasi-Banach space $\mathbb{X}$ and let w be a weight whose primitive weight $(s_n)_n$ has the LRP and $(\frac{s_n}{n})_n$ is non-increasing. Assume that $\mathcal{B}$ verifies the following conditions:

- $\mathcal{B}$ is bidemocratic with $\varphi(n) \approx s_n$.
- $\mathcal{B}$ has a subsequence dominated by the unit vector basis of $d_{1,q}(w)$ for some $1 < q < \infty$.

Then, $\mathbb{X}$ has a subspace $\mathbb{Y}$ with a bidemocratic Markushevich basis $\mathcal{B}_2$ with $\varphi[\mathcal{B}_2, \mathbb{Y}](n) \approx s_n$ that is not quasi-greedy nor, in any order, a Schauder basis.

A positive sequence $(s_n)_n$ has the LRP (Lower Regularity Property) if there is $a > 0$ and $C \geq 1$ such that

$$\frac{m^a}{n^a} \leq C\frac{s_m}{s_n}.$$

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Corollary

For each $1 < p < \infty$, there is a subspace $\mathbb{Y}$ of ℓ_p with a bidemocratic Markushevich basis that is not quasi-greedy.

Bidemocracy vs Quasi-greediness

Theorem (AABBL,2021)

There is a bidemocratic Schauder basis that is not quasi-greedy.

Building bidemocratic conditional quasi-greedy bases

The DKK-method produces conditional almost-greedy bases whose fundamental function either is equivalent to $(n)_{n=1}^{\infty}$ or has both the LRP and the URP.

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If $\mathbb{X}$ is a Banach space, taking $\mathbf{k}_m := \sup_{|A|\leq m} \|P_A\|$, if $\mathcal{B}$ is quasi-greedy, then

$$\mathbf{k}_m \lesssim \log(m).$$

Thus, the DKK-method serves as a tool for constructing Banach spaces with bidemocratic conditional quasi-greedy bases whose fundamental function has both the LRP and the URP.

With bidemocracy, we develop a new method for building conditional bases that allows us to construct bidemocratic conditional quasi-greedy bases with an arbitrary fundamental function.

We write $\mathbb{X}\oplus\mathbb{Y}$ for the Cartesian product of the quasi-Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ with the quasi-norm

$$\|(f,g)\| = \max\{\|f\|,\|g\|\},\ f\in\mathbb{X}, g\in\mathbb{Y}.$$

We consider the "rotated" sequence $\mathcal{B}_x \diamond \mathcal{B}_y = (z_n)_n$ in $\mathbb{X}\oplus\mathbb{Y}$ given by

$$z_{2n-1} = \frac{1}{\sqrt{2}}(\mathbf{x}_n,\mathbf{y}_n),\quad z_{2n} = \frac{1}{\sqrt{2}}(\mathbf{x}_n,-\mathbf{y}_n),\ n\in\mathbb{N}.$$

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Proposition (AABBL,2021)

- $\mathcal{B}_x \diamond \mathcal{B}_y$ is a basis for $\mathbb{X} \oplus \mathbb{Y}$ with dual basis $\mathcal{B}_x^* \diamond \mathcal{B}_y^*$.
- If $\mathcal{B}_x$ and $\mathcal{B}_y$ are Schauder, so is $\mathcal{B}_x \diamond \mathcal{B}_y$.
- There is $C > 0$ depending only on $\mathbb{X}$ and $\mathbb{Y}$ such that

$$\varphi[\mathcal{B}_x \diamond \mathcal{B}_y](m) \leq C \max\{\varphi[\mathcal{B}_x](m), \varphi[\mathcal{B}_y](m)\}, \; \forall m \in \mathbb{N}.$$

- If $\mathcal{B}_x$ and $\mathcal{B}_y$ are bidemocratic bases with $\varphi[\mathcal{B}_x] \approx \varphi[\mathcal{B}_y]$, then, $\mathcal{B}_x \diamond \mathcal{B}_y$ is quasi-greedy if and only if $\mathcal{B}_x$ and $\mathcal{B}_y$ are quasi-greedy.
- If $\mathcal{B}_x$ and $\mathcal{B}_y$ are not equivalent, then $\mathcal{B}_x \diamond \mathcal{B}_y$ is conditional.

Theorem (AABBL,2021)

Let $(s_m)_m$ be a non-decreasing unbounded sequence of positive scalars and suppose that $(m/s_m)_m$ is unbounded and non-decreasing. Then, there is a Banach space $\mathbb{X}$ with a conditional 1-bidemocratic Schauder quasi-greedy basis whose fundamental function grows as $(s_m)_m$.

It is well known that the sequences $(\varphi(m))_m$ and $(m/\varphi(m))_m$ in Banach spaces are non-decreasing.

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