Multiplicative convex functions: a redefinition

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A function $f : \mathbb{R} \to \mathbb{R}$ is *convex* if

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for every number x, y and $0 \leq \lambda \leq 1$.

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Nicolescu: A function is *multiplicative convex* if

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for every $x, y > 0$ and $0 \leq \lambda \leq 1$.

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Our definition

A function $f : (0, \infty) \to [0, \infty)$ is *multiplicative convex* (*mc*$-$function for short) if

$$f(x^{\mu} y^{1/\mu}) \leq f(x)^{\mu} f(y)^{1/\mu}$$

for every number $x, y \geq 0$ and $\mu > 0$.

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If $f(1) = 1$, we will say that f is *1-multiplicative convex* ($mc1-$function for short).

Let us focus first on $mc1-$functions, and let us see what we can say of them...

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Theorem 1

An $mc1-$function is either increasing, decreasing or decreasing on $(0, 1)$ and increasing $[1, \infty)$ (decreasing-increasing type, for short).

Let us focus first on $mc1-$functions, and let us see what we can say of them...

Proposition 1

Let f be an $mc1-$function and $q \in \mathbb{Q}^+$. Then

$$f(x^q) = f(x)^q.$$

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Theorem 3

Let $f : (0,\infty) \to (0,\infty)$. Then, f is a $mc1-$function if and only if f is of the form

$$f(x) = \begin{cases} b^{\log_a(x)} & \text{if } 0 < x < 1, \\ b'^{\log_{a'}(x)} & \text{if } x \geq 1, \end{cases}$$

where

1. $0 < a < 1$ and $a' > 1$,
2. if $b < 1$, then $\log_b(b') \leq \log_a(a')$,
3. if $b > 1$, then $\log_b(b') \geq \log_a(a')$.

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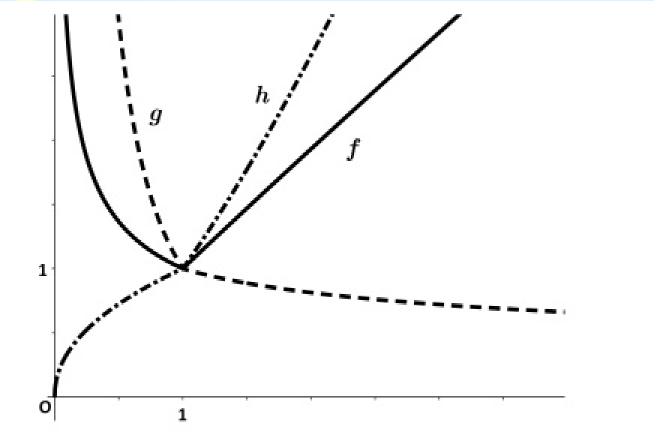

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3. If f is an *mc*−function and $K \geq 1$, Kf is an *mc*−function.
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Theorem 4

The set of *mc*−functions is a closed algebraic truncated cone.

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$\mathfrak{A}$(V) stands for the set of all algebraic combinations of elements of the set V

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Is every *mc*−function continuous?

The function

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$\mathcal{MC} \setminus C(0, \infty)$ contains a closed algebraic truncated cone with a set of cardinality $\mathfrak{c}$ of algebraically independent elements.

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$\mathcal{MC} \setminus C(0, \infty)$ contains a closed algebraic truncated cone with a set of cardinality $\mathfrak{c}$ of algebraically independent elements.

$$f(x) = \begin{cases} \alpha f(x) & \text{if } 0 < x < 1, \\ \beta f(x) & \text{if } x \geq 1, \end{cases}$$

where f is a $mc1-$function and $1 < \alpha < \beta \leq \alpha^2$.

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Theorem 5

If $\mathfrak{X}$ is a monotonous sequence, then there exists an $mc-$function which is discontinuous on $\mathfrak{X}$ and continuous on $(0, \infty) \setminus \mathfrak{X}$.

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Theorem 5

If $\mathfrak{X}$ is a monotonous sequence, then there exists an $mc-$function which is discontinuous on $\mathfrak{X}$ and continuous on $(0, \infty) \setminus \mathfrak{X}$.

Theorem 6

There exists a truncated cone consisting of $mc-$functions discontinuous over an infinite set and containing an uncountable set of algebraically independent elements.

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  Answer: $\mathfrak{c}$
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  Answer: The following Theorem

Theorem 7

An *mc*−function is either increasing, decreasing or decreasing-increasing.

An open question:

$\mathcal{MC} \cap C(0,\infty)$ is a truncated cone.
Is $\mathcal{MC} \setminus C(0,\infty)$ a truncated cone?

THANK YOU VERY MUCH FOR YOUR ATTENTION!!