Frequent Recurrence via Invariant Measures

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What are we studying?

We will discuss some properties about Linear Dynamics so:

$T : X \to X$ is an operator; X is a (separable, infinite-dimens.) Banach space
(or a continuous map); (or a Polish space)

(X, T) is a linear dynamical system and given $x \in X$ we study its T-orbit:
(or a Polish dynamical system)

$$\mathrm{Orb}(x, T) := \{T^n x : n \in \mathbb{N}_0\}.$$

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$$\mathrm{Orb}(x, T) := \{ T^n x : n \in \mathbb{N}_0 \}.$$

We focus on the frequency in which it visit some sets, i.e. in the "bigness" of

$$N(x, U) := \{ n \in \mathbb{N}_0 : T^n x \in U \},$$

the return set from x to $U \subset X$; where U is a neighbourhood or $U \in O(X)$ ...

$$O(X) := \{ U \text{ non-empty open subset of } X \}.$$

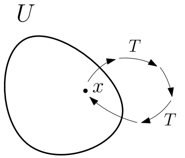
Topological Dynamics: Recurrence and Hypercyclicity Notions

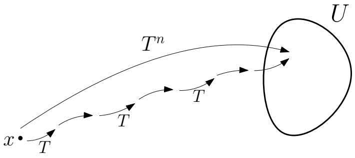
Definition

A vector $x \in X$ is said to be:

(i) recurrent/hypercyclic if $N(x, U)$ is infinite ...

$\forall U$ neighbourhood of x / $\forall U$ non-empty open subset of X.





$x \in \mathrm{Rec}(T) \Leftrightarrow \lim_{k\to\infty} T^{n_k}x = x$ / $x \in \mathrm{HC}(T) \Leftrightarrow \overline{\mathrm{Orb}(x, T)} = X$

Topological Dynamics: Recurrence and Hypercyclicity Notions

Definition

A vector $x \in X$ is said to be:

(i) recurrent/hypercyclic if $N(x, U)$ is infinite ...

(ii) frequently recurrent/hypercyclic if $\underline{\text{dens}}(N(x, U)) > 0$ ...

$\forall U$ neighbourhood of x / $\forall U$ non-empty open subset of X.

Definition

Given $A \subset \mathbb{N}_0$ its lower density is

$$\underline{\text{dens}}(A) := \liminf_{N \to \infty} \frac{\#A \cap [0, N]}{N+1}.$$

FRec(T) (Bonilla et al., 2020) / FHC(T) (Bayart and Grivaux, 2006)

Topological Dynamics: Recurrence and Hypercyclicity Notions

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(i) recurrent/hypercyclic if $N(x, U)$ is infinite ...

(ii) frequently recurrent/hypercyclic if $\underline{\text{dens}}(N(x, U)) > 0$ ...

(iii) $\mathcal{U}$-frequently recurrent/hypercyclic if $\overline{\text{dens}}(N(x, U)) > 0$ ...

$\forall U$ neighbourhood of x / $\forall U$ non-empty open subset of X.

Definition

Given $A \subset \mathbb{N}_0$ its upper density is

$$\overline{\text{dens}}(A) := \limsup_{N\to\infty} \frac{\#A \cap [0, N]}{N+1}.$$

UFRec(T) (Bonilla et al., 2020) / UFHC(T) (Shkarin, 2009).

Topological Dynamics: Recurrence and Hypercyclicity Notions

Definition

A vector $x \in X$ is said to be:

(i) recurrent/hypercyclic if $N(x, U)$ is infinite ...

(ii) frequently recurrent/hypercyclic if $\underline{\text{dens}}(N(x, U)) > 0$ ...

(iii) $\mathcal{U}$-frequently recurrent/hypercyclic if $\overline{\text{dens}}(N(x, U)) > 0$ ...

(iv) reiteratively recurrent/hypercyclic if $\overline{\text{Bd}}(N(x, U)) > 0$ ...

$\forall U$ neighbourhood of x / $\forall U$ non-empty open subset of X.

Definition

Given $A \subset \mathbb{N}_0$ its upper Banach density is

$$\overline{\text{Bd}}(A) := \limsup_{N \to \infty} \left(\max_{m \geq 0} \frac{\# A \cap [m, m+N]}{N+1} \right).$$

RRec(T) (Bonilla et al., 2020) / RHC(T) (Bés et al., 2016).

Topological Dynamics: Recurrence and Hypercyclicity Notions

Definition

A vector $x \in X$ is said to be:

(i) recurrent/hypercyclic if $N(x, U)$ is infinite ...

(ii) frequently recurrent/hypercyclic if $\underline{\text{dens}}(N(x, U)) > 0$ ...

(iii) $\mathcal{U}$-frequently recurrent/hypercyclic if $\overline{\text{dens}}(N(x, U)) > 0$ ...

(iv) reiteratively recurrent/hypercyclic if $\overline{\text{Bd}}(N(x, U)) > 0$ ...

$\forall U$ neighbourhood of x / $\forall U$ non-empty open subset of X.

- T is (freq., $\mathcal{U}$-freq., reiter.) hypercyclic if there is such a vector
- T is (freq., $\mathcal{U}$-freq., reiter.) recurrent if the set of such vectors is dense

$$0 \leq \underline{\text{dens}}(A) \leq \overline{\text{dens}}(A) \leq \overline{\text{Bd}}(A) \leq 1 \quad \text{for each } A \subset \mathbb{N}_0$$

$$\text{FHC}(T) \subset \text{UFHC}(T) \subset \text{RHC}(T) \subset \text{HC}(T)$$

$$\text{FRec}(T) \subset \text{UFRec}(T) \subset \text{RRec}(T) \subset \text{Rec}(T)$$

Topological vs Measurable Dynamics I

Topological Dynamics:

$$\begin{array}{ccc}
\text{Frequent Recurrence} & \Longleftarrow & \text{Frequent Hypercyclicity} \\
\Downarrow & & \Downarrow \\
\mathcal{U}\text{-frequent Recurrence} & \Longleftarrow & \mathcal{U}\text{-frequent Hypercyclicity} \\
\Downarrow & & \Downarrow \\
\text{Reiterative Recurrence} & \Longleftarrow & \text{Reiterative Hypercyclicity} \\
\Downarrow & & \Downarrow \\
\text{Recurrence} & \Longleftarrow & \text{Hypercyclicity}
\end{array}$$

Measurable Dynamics (**Ergodic Theory**):

If μ is a Borel prob. measure with full support (i.e. $\mu(U) > 0$ for $U \in O(X)$) we can study the system $(X, \mathscr{B}(X), \mu, T)$ with the properties:

Topological vs Measurable Dynamics I

Topological Dynamics:

Frequent Recurrence	$\Longleftarrow$	Frequent Hypercyclicity
$\Downarrow$		$\Downarrow$
$\mathcal{U}$-frequent Recurrence	$\Longleftarrow$	$\mathcal{U}$-frequent Hypercyclicity
$\Downarrow$		$\Downarrow$
Reiterative Recurrence	$\Longleftarrow$	Reiterative Hypercyclicity
$\Downarrow$		$\Downarrow$
Recurrence	$\Longleftarrow$	Hypercyclicity

Measurable Dynamics (**Ergodic Theory**):

If μ is a Borel prob. measure with full support (i.e. $\mu(U) > 0$ for $U \in O(X)$) we can study the system $(X, \mathscr{B}(X), \mu, T)$ with the properties:

(a) invariance: for each $A \in \mathscr{B}(X)$ the equality $\mu(T^{-1}(A)) = \mu(A)$ holds.

Poincaré Recurrence Theorem $\Rightarrow$ T is recurrent.

(b) ergodicity: μ is T-invariant and if $T^{-1}(A) = A$ then $\mu(A) \in \{0, 1\}$.

Pointwise Ergodic Theorem $\Rightarrow$ T is frequently hypercyclic.

Topological vs Measurable Dynamics II

If μ is a Borel probability measure with full support ($\mu(U) > 0$ for $U \in O(X)$) then from the system $(X, \mathscr{B}(X), \mu, T)$ we get:

Invariance	$\Longleftarrow$	Ergodicity
???		$\Downarrow$
Frequent Recurrence	$\Longleftarrow$	Frequent Hypercyclicity
$\Downarrow$		$\Downarrow$
$\mathcal{U}$-frequent Recurrence	$\Longleftarrow$	$\mathcal{U}$-frequent Hypercyclicity
$\Downarrow$		$\Downarrow$
Reiterative Recurrence	$\Longleftarrow$	Reiterative Hypercyclicity
$\Downarrow$		$\Downarrow$
Recurrence	$\Longleftarrow$	Hypercyclicity
$\Uparrow$		
Invariance		

Question

Does Invariance implies Frequent Recurrence?

Topological vs Measurable Dynamics II

If μ is a Borel probability measure with full support ($\mu(U) > 0$ for $U \in O(X)$) then from the system $(X, \mathscr{B}(X), \mu, T)$ we get:

Invariance	$\Longleftarrow$	Ergodicity
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Frequent Recurrence	$\Longleftarrow$	Frequent Hypercyclicity
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$\Downarrow$		$\Downarrow$
Reiterative Recurrence	$\Longleftarrow$	Reiterative Hypercyclicity
$\Downarrow$		$\Downarrow$
Recurrence	$\Longleftarrow$	Hypercyclicity
$\Uparrow$		
Invariance		

Lemma 1 (S. Grivaux and A. L-M, 2022)

Let $T : (X, \mathscr{B}(X), \mu) \to (X, \mathscr{B}(X), \mu)$ be invariant. Then $\mu(\mathrm{FRec}(T)) = 1$ and

$$\mathrm{supp}(\mu) \subset \overline{\mathrm{FRec}(T)}.$$

Invariance $\Rightarrow$ Frequent Recurrence

STEP 1: For each $B \in \mathscr{B}(X)$ with $\mu(B) > 0$ apply the ...

Ergodic Decomposition Theorem

Let $T : (X, \mathscr{B}(X), \mu) \to (X, \mathscr{B}(X), \mu)$ be invariant. There is an **abstract probability measure space** $(\mathcal{M}, \tau)$ formed by measures such that for any $A \in \mathscr{B}(X)$ we have

$$\mu(A) = \int_{\mathcal{M}} \nu(A) \; d\tau(\nu)$$

and for τ-a.e. measure $\nu \in \mathcal{M}$, $T : (X, \mathscr{B}(X), \nu) \to (X, \mathscr{B}(X), \nu)$ is ergodic.

... obtaining an ergodic measure ν on X with (**AT LEAST**) $\nu(B) > 0$.

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... obtaining an ergodic measure ν on X with (**AT LEAST**) $\nu(B) > 0$.

STEP 2: Apply to ν the Pointwise Ergodic Theorem as in the **VERY WELL-KNOWN** Ergodicity $\Rightarrow$ Frequent Hypercyclicity **CASE**:

$\exists$ **FREQUENTLY DENSE** orbits **AROUND** $B \Rightarrow \mathrm{FRec}(T) \cap B \neq \varnothing$

By the arbitrariness of $B \Rightarrow \mu(\mathrm{FRec}(T)) = 1$.

Topological vs Measurable Dynamics III

Invariance	$\Longleftarrow$	Ergodicity
$\Downarrow$	$\nwarrow^{(*)}$	$\Downarrow$
Frequent Recurrence	$\Longleftarrow$	Frequent Hypercyclicity
$\Downarrow$		$\Downarrow$
$\mathcal{U}$-frequent Recurrence	$\Longleftarrow$	$\mathcal{U}$-frequent Hypercyclicity
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Recurrence	$\Longleftarrow$	Hypercyclicity

S. Grivaux and É. Matheron (2014): $(*) \equiv$ under some “natural” assumptions

$$\text{Frequent Hypercylcicity} \xrightarrow{(*)} \text{Invariance}$$

Topological vs Measurable Dynamics III

Invariance	$\Longleftarrow$	Ergodicity
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Frequent Hypercylcicity $\xrightarrow{(*)}$ Invariance

S. Grivaux and A. L-M (2022): $(*) \equiv$ under some “natural” assumptions

Reiterative Recurrence $\xrightarrow{(*)}$ Invariance

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Topological Assumptions and Examples

Let (X, T) be a Polish dynamical system, i.e.:

(X, τ_X) is separable and completely metrizable; $T : X \to X$ is τ_X-continuous

"Natural" Topological Assumptions

Let τ be a Hausdorff topology in X. Enumerate the properties:

(I) $T : (X, \tau) \to (X, \tau)$ is τ-continuous;

(II) $\tau \subset \tau_X$, i.e. τ is coarser than τ_X;

(III*) every $x \in X$ has a τ_X-neighbourhood basis of τ-compact sets.

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- If (X, τ_X) is compact, take $\tau = \tau_X$. **Compact Dynamical Systems**, ...

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- If (X, τ_X) is locally compact, take $\tau = \tau_X$. **Diff. Manifolds**, $(\mathbb{R}^n, T)$, ...

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- If (X, τ_X) is compact, take $\tau = \tau_X$. **Compact Dynamical Systems**, ...
- If (X, τ_X) is locally compact, take $\tau = \tau_X$. **Diff. Manifolds**, $(\mathbb{R}^n, T)$, ...
- Given an adjoint operator $T : X \to X$ on a dual Banach space $(X, \|\cdot\|)$ take $\tau = w^*$ (weak-star topology) and $\tau_X = \tau_{\|\cdot\|}$ (norm topology)

Invariant Measures from Reiterative Recurrence

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Main Theorem (S. Grivaux and A. L-M, 2022)

Let (X, T) be a Polish dynamical system and τ fulfilling (I), (II) and (III*)

$\Longrightarrow$ for each $x_0 \in \mathrm{RRec}(T)$ (reiteratively recurrent point) one can find an invariant probability measure μ_{x_0} on X such that

$$x_0 \in \mathrm{supp}(\mu_{x_0}).$$

We divide the proof in 2 facts:

Measures' "Constructing Machine"

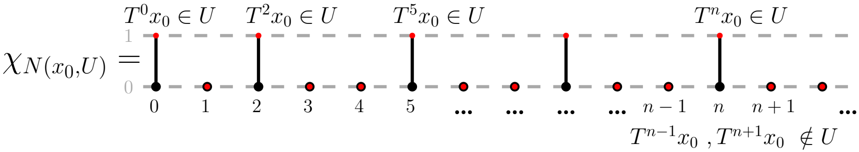
Fact 1

$\overline{\mathrm{Bd}}(N(x_0, U)) > 0$, U τ-compact $\Rightarrow \exists \mu$ inv. prob. measure with $\mu(U) > 0$.

Technical Lemma (S. Grivaux and A. L-M, 2022)

Let (X, T) be a Polish dynamical system and τ fulfilling (I), (II) and (III*) $\Longrightarrow$ for each $x_0 \in X$ and each Banach limit $\mathfrak{m} : \ell^\infty(\mathbb{N}_0) \to \mathbb{R}$ one can find a (non-negative) invariant finite Borel regular measure μ on X such that

$$\mu(K) \geq \mathfrak{m}(\chi_{N(x_0,K)}) \qquad \text{for every } \tau\text{-compact } K \subset X.$$



$\chi_{N(x_0,U)} \in \ell^\infty(\mathbb{N}_0)$ ... choose a Banach limit $\mathfrak{m}$ such that $\mathfrak{m}(\chi_{N(x_0,U)}) > 0$

$$\mu(U) \geq \mathfrak{m}(\chi_{N(x_0,U)}) = \overline{\mathrm{Bd}}(N(x_0, U)) > 0$$

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$$\mu(K) \geq \mathfrak{m}(\chi_{N(x_0,K)}) \quad \text{for every } \tau\text{-compact } K \subset X.$$

Fact 2

For $x_0 \in \mathrm{RRec}(T) \Rightarrow \exists \mu_{x_0}$ inv. probability measure such that $x_0 \in \mathrm{supp}(\mu_{x_0})$.

(III*) $\Rightarrow \exists$ a τ_X-neighbourhood basis $(U_n)_{n\in\mathbb{N}}$ formed by τ-compact sets

$x_0 \in \mathrm{RRec}(T) \Rightarrow \overline{\mathrm{Bd}}(N(x_0, U_n)) > 0 \overset{\text{Fact 1}}{\Longrightarrow} \exists \mu_n$ with $\mu_n(U_n) > 0$ $\forall n \in \mathbb{N}$ ...

$$\mu_{x_0} := \sum_{n\in\mathbb{N}} \frac{\mu_n}{2^n} \quad \text{is invariant and } x_0 \in \mathrm{supp}(\mu_{x_0}).$$

Reiterative Recurrence $\Rightarrow$ Full Support Invariant Measure

Main Theorem (S. Grivaux and A. L-M, 2022)

Let (X, T) be a Polish dynamical system and τ fulfilling (I), (II) and (III*)

$\Longrightarrow$ for each $x_0 \in \mathrm{RRec}(T)$ (reiteratively recurrent point) one can find an invariant probability measure μ_{x_0} on X such that

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Reiterative Recurrence ⇒ Full Support Invariant Measure

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Let (X, T) be a Polish dynamical system and τ fulfilling (I), (II) and (III*)

$\Longrightarrow$ for each $x_0 \in \mathrm{RRec}(T)$ (reiteratively recurrent point) one can find an invariant probability measure μ_{x_0} on X such that

$$x_0 \in \mathrm{supp}(\mu_{x_0}).$$

If $\mathrm{RRec}(T)$ is dense there is an invariant prob. measure μ with full support.

There is $\{x_n : n \in \mathbb{N}\} \subset \mathrm{RRec}(T)$ dense ...

Fact 2

For $x_n \in \mathrm{RRec}(T) \Rightarrow \exists \mu_{x_n}$ inv. probability measure such that $x_n \in \mathrm{supp}(\mu_{x_n})$.

...

$$\mu := \sum_{n\in\mathbb{N}} \frac{\mu_{x_n}}{2^n}$$

is an invariant probability measure with full support.

Índex

Adjoint Operators and Reflexive Spaces

Theorem (S. Grivaux and A. L-M, 2022)

For an adjoint operator $T : X \to X$ on a separable dual Banach space X we have $\mathrm{FRec}(T) = \mathrm{RRec}(T)$. Moreover, the following statements are equivalent:

(i) T is frequently recurrent;

(ii) T is $\mathcal{U}$-frequently recurrent;

(iii) T is reiteratively recurrent;

(iv) T admits an invariant probability measure with full support.

(I) T is adjoint $\Longrightarrow T : (X, w^*) \to (X, w^*)$ is continuous;

(II) $w^* \subset \tau_{\|\cdot\|}$;

(III*) Alaoglu-Bourbaki $\Longrightarrow$ closed balls are w^*-compact and $\|\cdot\|$-neighbour.

Given $x_0 \in \mathrm{RRec}(T)$ there is μ_{x_0} invariant with $x_0 \in \mathrm{supp}(\mu_{x_0}) \overset{\text{Lemma 1}}{\subset} \overline{\mathrm{FRec}(T)}$

- True whenever X is reflexive ... L^p and ℓ^p-spaces $(1 < p < \infty)$

Product Dynamical Systems

When $T : X \to X$ has a property ...

Usual Question

Does $T \oplus T : X \oplus X \longrightarrow X \oplus X$ has that property?

$$\begin{aligned} T \oplus T : X \oplus X &\longrightarrow X \oplus X \\ (x, y) &\longmapsto (Tx, Ty) \end{aligned}$$

Examples

There are negative and **positive** answers; and open problems:

- T hypercyclic $\not\Longrightarrow$ $T \oplus T$ hypercyclic;
- T **reiteratively hypercyclic** $\Longrightarrow$ $T \oplus T$ **reit.hypercyclic**;
- T $\mathcal{U}$-**frequently hypercyclic** $\Longrightarrow$ $T \oplus T$ $\mathcal{U}$-**freq. hypercyclic**;
- T frequently hypercyclic $\overset{???}{\Longrightarrow}$ $T \oplus T$ freq. hypercyclic;
- T recurrent $\overset{???}{\Longrightarrow}$ $T \oplus T$ recurrent.

Product Dynamical Systems

When $T : X \to X$ has a property ...

Usual Question

Does $T \oplus T : X \oplus X \longrightarrow X \oplus X$ has that property?

$$T_n = \underbrace{T \oplus \cdots \oplus T}_{n} : \underbrace{X \oplus \cdots \oplus X}_{n} \longrightarrow \underbrace{X \oplus \cdots \oplus X}_{n}$$
$$(x_1, \ldots, x_n) \longmapsto (Tx_1, \ldots, Tx_n)$$

Corollary (S. Grivaux and A. L-M, 2022)

For an adjoint operator $T : X \to X$ on a separable dual Banach space X, then the following statements are equivalent:

(i) for every $n \in \mathbb{N}$, T_n is frequently recurrent;
(ii) for every $n \in \mathbb{N}$, T_n is $\mathcal{U}$-frequently recurrent;
(iii) for every $n \in \mathbb{N}$, T_n is reiteratively recurrent;
(iv) T is reiteratively recurrent.

Product Dynamical Systems

When $T : X \to X$ has a property ...

Usual Question

Does $T \oplus T : X \oplus X \longrightarrow X \oplus X$ has that property?

$$\begin{aligned} T = T_1 \oplus \cdots \oplus T_N : X_1 \oplus \cdots \oplus X_N &\longrightarrow X_1 \oplus \cdots \oplus X_N \\ (x_1, ..., x_n) &\longmapsto (T_1 x_1, ..., T_N x_N) \end{aligned}$$

Theorem (S. Grivaux and A. L-M, 2022)

Fix $N \in \mathbb{N}$ and for each $1 \leq i \leq N$ let $T_i : X_i \to X_i$ be an adjoint operator on a separable dual Banach space X_i. For the direct sum operator

$$T = T_1 \oplus \cdots \oplus T_N \text{ on the direct sum space } X = X_1 \oplus \cdots \oplus X_N,$$

we have the equality $\overline{\mathrm{FRec}(T)} = \prod_{i=1}^{N} \overline{\mathrm{RRec}(T_i)}$.

Proof for Product Dynamical Systems

$$\begin{array}{rcl} T = T_1 \oplus \cdots \oplus T_N : X_1 \oplus \cdots \oplus X_N & \longrightarrow & X_1 \oplus \cdots \oplus X_N \\ (x_1, ..., x_n) & \longmapsto & (T_1 x_1, ..., T_N x_N) \end{array}$$

We clearly have the inclusion

$$\mathrm{FRec}(T) \subset \prod_{i=1}^{N} \mathrm{RRec}(T_i).$$

For $\boldsymbol{x}_0 = (x_1, ..., x_N) \in X$ with $x_i \in \mathrm{RRec}(T_i)$, $\exists \mu_{x_i}$ T_i-invariant, $x_i \in \mathrm{supp}(\mu_{x_i})$. Since

$$\mathscr{B}(X, \tau_X) = \prod_{i=1}^{N} \mathscr{B}(X_i, \tau_{X_i}),$$

the product measure $\mu_{\boldsymbol{x}_0} := \prod_{i=1}^{N} \mu_{x_i}$ on $X_1 \oplus \cdots \oplus X_N$ is T-invariant, so

$$\boldsymbol{x}_0 = (x_1, ..., x_N) \in \mathrm{supp}(\mu_{\boldsymbol{x}_0}) \overset{\text{Lemma 1}}{\subset} \overline{\mathrm{FRec}(T)}.$$

Inverse Dynamical Systems

When $T : X \to X$ has a property ... and $T^{-1} : X \to X$ exists ...

Usual Question

Does $T^{-1} : X \to X$ has that property?

Examples

There are **positive** and negative answers:

- T **hypercyclic** $\iff$ T^{-1} **hypercyclic**;
- T **reiteratively hypercyclic** $\iff$ T^{-1} **reit.hypercyclic**;
- T $\mathcal{U}$-frequently hypercyclic $\not\Longrightarrow$ T^{-1} $\mathcal{U}$-freq. hypercyclic;
- T frequently hypercyclic $\not\Longrightarrow$ T^{-1} freq. hypercyclic;
- T **recurrent** $\iff$ T^{-1} **recurrent**.

Inverse Dynamical Systems

When $T : X \to X$ has a property ... and $T^{-1} : X \to X$ exists ...

Usual Question

Does $T^{-1} : X \to X$ has that property?

$$T \text{ is adjoint} \iff T^{-1} \text{ is adjoint}$$

Theorem (S. Grivaux and A. L-M, 2022)

For an invertible adjoint operator $T : X \to X$ on a sep. dual Banach space X,

$$\overline{\mathrm{RRec}(T)} = \overline{\mathrm{FRec}(T)} = \overline{\mathrm{FRec}(T^{-1})} = \overline{\mathrm{RRec}(T^{-1})}.$$

T is reiteratively (and then $\mathcal{U}$-frequently and frequently) recurrent iff so is T^{-1}.

Note that $(X, \mathscr{B}(X), T, \mu)$ invariant $\implies (X, \mathscr{B}(X), T^{-1}, \mu)$ invariant since

$$\mu([T^{-1}]^{-1}(A)) = \mu(T(A)) \overset{\mu \text{ is } T\text{-invariant}}{=} \mu(T^{-1}(T(A))) = \mu(A)$$

Conclusion

In closing ...

- PART 1: Invariant Measure $\Longrightarrow$ there are **Frequently Recurrent Points**;
- PART 2$^{(*)}$: Reiterative Recurrent Points $\Longrightarrow$ there are Invariant Measures;
- PART 3$^{(*)}$: Reiterative Recurrence $\Longleftrightarrow$ **Frequent Recurrence**;
- FINALLY: good prop. for **PRODUCT** and **INVERSE** dynamical systems.

$(*) \equiv$ under some "natural" assumptions ...

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Thank you for your attention