Función generadora de Catalan para operadores acotados y no acotados

Prepublicaciones (2022)

A. Mahillo (UZ), P.J.M. (UZ), N. Romero (UR)

(pjmiana@unizar.es)

XVII Encuentro de la Red de Análisis Funcional y Aplicaciones,
Universidad de La Laguna, 10-12 de marzo de 2022

Instituto Universitario de Investigación
de Matemáticas
y Aplicaciones
Universidad Zaragoza

UNIVERSIDAD
DE LA RIOJA

1. Introduction

The well-known Catalan numbers $(C_n)_{n \geq 0}$ given by the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n \geq 0,$$

appear in a wide range of problems. For instance, the Catalan number C_n counts the number of ways to triangulate a regular polygon with $n+2$ sides; or, the number of ways that $2n$ people seat around a circular table are simultaneously shaking hands with another person at the table in such a way that none of the arms cross each other ([MR]).

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \ldots$$

Eugène Charles Catalan (1814 - 1894)

Triangulating Polygons

Leonhard Euler
1707 - 1783

In 1751, Euler asked the question:

In how many ways can we divide a convex polygon with n sides into triangles using non-intersecting diagonals?

Letters between L. Euler and C. Goldbach (196)
1750's Euler wrote J.A. von Segner and Segner proved in 1758

$$C_{n+1} = C_0 C_n + C_1 C_{n-1} + \dots C_1 C_{n-1} + C_0 C_n, \qquad n \geq 0,$$

We calculated C_n with $n \leq 18$ but we made a mistake $C_{13} = 742{,}900$ which invalidated all larger values.

Kotelnikow (1766), Fuss (1795), Liouville (1836, *Journal of Mathématiques Pures and Appliquées*) Lamé (1838)

The first paper of E. C. Catalan (JMPA, 1838)

III.

On sait que le $(n+1)^e$ nombre figuré de l'ordre $n+1$, a pour expression, $C_{2n,n}$: si donc, dans la table des nombres figurés, on prend ceux qui occupent la diagonale ; savoir :

$$1,\quad 2,\quad 6,\quad 20,\quad 70,\quad 252,\quad 924\ldots;$$

qu'on les divise respectivement par

$$1,\quad 2,\quad 3,\quad 4,\quad 5,\quad 6,\quad 7\ldots;$$

on obtiendra une nouvelle suite de nombres,

$$1,\quad 1,\quad 2,\quad 5,\quad 14,\quad 42,\quad 132\ldots, \qquad \text{(A)}$$

lesquels jouiront de cette propriété :

Un terme quelconque de la suite (A) *est égal à la somme des produits que l'on obtient en écrivant au-dessous d'elle-même, et dans un ordre inverse, la série des termes précédents, et en multipliant les termes correspondants des deux séries.*

Par exemple,

$$132 = 1.42 + 1.14 + 2.5 + 5.2 + 14.1 + 42.1.$$

The last paper of E. C. Catalan (Rend.Con.MatPal, 1887)

SUR LES NOMBRES DE SEGNER (*)

Par M. **Eugène Catalan**, à Liége

(Seduta del 19 dicembre 1886)

I. Introduction.

1. Divers Géomètres se sont occupés de ce problème : *De combien de manières un polygone convexe, de n côtés, peut-il être décomposé en triangles, au moyen de diagonales ?* (**)

Soit T_n le nombre des décompositions. On sait que

$$T_4 = 2, \quad T_5 = 5, \quad T_6 = 14, \quad T_7 = 42, \ \ldots .$$

Les nombres T_n, considérés par Segner (***), satisfont aux relations

$$T_{n+1} = T_2 T_n + T_3 T_{n-1} + \ldots + T_{n-1} T_3 + T_n T_2 \ (****), \qquad (1)$$

Why the term of "Catalan numbers"

The Ballot problem. Suppose A and B are candidates for office and there are $2n$ voters, n voting for A and n for B. In how many ways can the ballots be counted so that B is never ahead of A? The solution is a Catalan number C_n.

John Riordan introduced the term "Catalan number" in Math Reviews in 1948 and 1964.
Finally Riordan used "Catalan number" in *Combinatorial identies* (1968).
Martin Gardner used the term in his "Mathematical Games" column in Scientific American in 1976.

In [MR] we consider combinatorial numbers $(C_{m,k})_{m\geq 1,k\geq 0}$ given by

$$C_{m,k} := \frac{m-2k}{m}\binom{m}{k}. \tag{1}$$

$m \setminus k$	0	1	2	3	4	5	6	7	8	9	10
1	1	-1									
2	1	0	-1								
3	1	1	-1	-1							
4	1	2	0	-2	-1						
5	1	3	2	-2	-3	-1					
6	1	4	5	0	-5	-4	-1				
7	1	5	9	5	-5	-9	-5	-1			
8	1	6	14	14	0	-14	-14	-6	-1		
9	1	7	20	28	14	-14	-28	-20	-7	-1	
10	1	8	27	48	42	0	-42	-48	-27	-8	-1
...	...	...	...	...	...	...	...	...	...	...	...

(i) $C_{m+2,k} = C_{m,k} + 2C_{m,k-1} + C_{m,k-2}$,

(ii) $\displaystyle\sum_{k=0}^{n} C_{m,k} = \binom{m-1}{n}$,

(iii) $\displaystyle\sum_{k=0}^{n} (-1)^k C_{m,k} = (-1)^n C_{m-1,n}$.

(iv) $\displaystyle\sum_{k=0}^{n} C_{n,k}^2 = 2C_{n-1}$,

(v) $\displaystyle\sum_{k=0}^{n} C_{m,k}^3 = 4\binom{m-1}{n}^3 - 3\binom{m-1}{n}\sum_{j=0}^{m-1}\binom{j}{n}\binom{j}{m-n-1}$, The generating function of the Catalan sequence $c = (C_n)_{n \geq 0}$ is defined by

$$C(z) := \sum_{n=0}^{\infty} C_n z^n = \frac{1 - \sqrt{1 - 4z}}{2z}, \quad z \in D(0, \frac{1}{4}). \qquad (2)$$

This functions satisfies

$$zy^2 - y + 1 = 0.$$

The second solution is given by

$$\frac{1}{zC(z)} = \frac{1 + \sqrt{1 - 4z}}{2z}, \quad z \in D(0, \frac{1}{4}) \backslash \{0\}.$$

The main aim of this talk is to consider the quadratic equation

$$TY^2 - Y + I = 0, \qquad Y \in \mathcal{B}(X)$$

where

(1) $T \in \mathcal{B}(X)$ ([MR]).

(2) T is the infinitesimal generator of a C_0-semigroup ([MM]).

$$Y = \frac{1 \pm \sqrt{1 - 4T}}{2T}$$

The problem of quadratic equation.

The study of quadratic equations in Banach space is much complicated than in the scalar case. There are infinite symmetric square roots of $I_2 \in \mathbb{R}^{2\times 2}$ given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} a & \pm\sqrt{1-a^2} \\ \pm\sqrt{1-a^2} & -a \end{pmatrix}$$

with $a \in [-1, 1]$, i.e., solutions of $Y^2 = I_2$.

As far as we are aware, no useful necessary and sufficient conditions for the existence of solution of quadratic equations in Banach spaces are known, even in the classical case of finite-dimensional spaces and square roots.

In 1952, Newton's method was generalized to Banach space by Kantorovich.

In [McF] (1958), the author studies

$$B(x)(x) + Ax = y,$$

where B is a bilinear and A a linear operators on a Banach space X.

The iterative method

$$\begin{cases} F_0 = z, \\ F_{n+1} = (A + B(F_n))^{-1} y, \end{cases}$$

converges to the solution $F_n \to x$ under some nice conditions.

Index

2. New results about Catalan numbers

Theorem

Let A be a commutative algebra over $\mathbb{R}$ or $\mathbb{C}$, $x \in A$ and y and z solutions of the quadratic equations

$$xy^2 - y + 1 = 0, \qquad -xz^2 - z + 1 = 0.$$

Then $\frac{y+z}{2}$ is a solution of the quartic equation

$$4x^2w^4 - w^2 + 1 = 0.$$

Remark.

$$\frac{y+z}{2} = 2x\left(\frac{y-z}{2}\right)\left(\frac{y+z}{2}\right).$$

Proposition

Let $c = (C_n)_{n\geq 0}$ be the Catalan sequence. Then

$$C_e(z) := \sum_{n=0}^{\infty} C_{2n} z^{2n} = \frac{\sqrt{1+4z} - \sqrt{1-4z}}{4z},$$

$$C_o(z) := \sum_{n=0}^{\infty} C_{2n+1} z^{2n+1} = \frac{2 - \sqrt{1+4z} - \sqrt{1-4z}}{4z},$$

for $|z| \leq \frac{1}{4}$. In particular, $4z^2 C_e^4(z) - C_e(z)^2 + 1 = 0$,

$$C_o(z) = \frac{C_e(z) - 1}{2zC_e(z)}.$$

Catalan numbers have several integral representations, for example

$$C_n = \frac{1}{2\pi}\int_0^4 t^n \sqrt{\frac{4-t}{t}}dt = \frac{2^{2n+1}}{\pi}\beta(\frac{3}{2}, n+\frac{1}{2}),$$

Theorem

Given $1 \neq z \in \mathbb{C}^+$, *then*

$$\begin{aligned}\int_0^\infty \frac{\sqrt{t}}{(t+1)(t+z)}dt &= \frac{\pi}{z-1}\left(\sqrt{z}-1\right),\\ \int_0^\infty \frac{\sqrt{t}}{(t+1)(t+z)^{j+1}}dt &= \frac{\pi}{2\sqrt{z}(z-1)^j}\sum_{k=j}^{\infty} C_k \left(\frac{z-1}{4z}\right)^k,\end{aligned}$$

for $j \geq 1$ *and where the last equality holds for* $\Re(z) \geq \frac{1}{2}$.

3. The sequence of Catalan numbers

$$\lim_{z\to\frac{1}{4}} C(z) = \sum_{n=0}^{\infty} \frac{C_n}{4^n} = 2.$$

We consider the weight Banach algebra $\ell^1(\mathbb{N}^0, \frac{1}{4^n})$. This algebra is formed by sequence $a = (a_n)_{n\geq 0}$ such that

$$\|a\|_{1,\frac{1}{4^n}} := \sum_{n=0}^{\infty} \frac{|a_n|}{4^n} < \infty,$$

and the product is the usual convolution $*$ defined by

$$(a * b)_n = \sum_{j=0}^{n} a_{n-j} b_j, \qquad a, b \in \ell^1(\mathbb{N}^*, \frac{1}{4^n}).$$

The base $\{\delta_j\}_{j\geq 0}$ is defined by $(\delta_j)_n := \delta_{j,n}$ is the delta Kronecker.

This Banach algebra has identity element, δ_0, its spectrum set is $\overline{D(0, \frac{1}{4})}$ and its Gelfand transform is given by the Z-transform

$$Z(a)(z) := \sum_{n=0}^{\infty} a_n z^n, \qquad z \in \overline{D(0, \frac{1}{4})}.$$

It is straightforward to check that $Z(\delta_n)(z) = z^n$ for $n \geq 0$.

Proposition

Take $c = (C_n)_{n\geq 0}$. Then

(i) $\|c\|_{1,\frac{1}{4^n}} = 2$.

(ii) $C(z) = Z(c)(z)$ *for* $z \in D(0, \frac{1}{4})$.

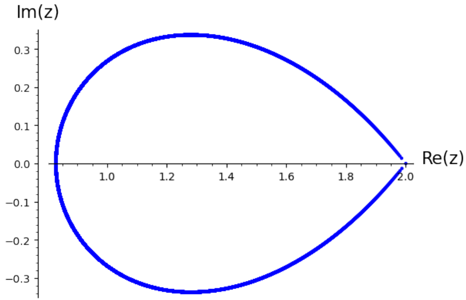
(iii) $\delta_1 * c^{*2} - c + \delta_0 = 0$.

The resolvent set $\rho(a) := \{\lambda \in \mathbb{C} \;:\; (\lambda\delta_0 - a)^{-1} \in \ell^1(\mathbb{N}^0, \frac{1}{4^n})\}$, and the spectrum of a is given by $\sigma(a) := \mathbb{C} \backslash \rho(a)$.

Proposition

The spectrum of the Catalan sequence $c = ((C_n))_{n\geq 0}$ is given by $\sigma(c) = C(\overline{D(0, \frac{1}{4})})$ and its boundary by

$$\partial(\sigma(c)) = \{2e^{-i\theta}\left(1 - \sqrt{2|\sin(\frac{\theta}{2})|e^{\frac{i(\pi-\theta)}{4}}}\right) \;:\; \theta \in (-\pi, \pi)\}.$$



Given $\lambda \in \mathbb{C}$, we consider the geometric progression $p_\lambda = (\frac{1}{\lambda^n})_{n \geq 0}$. Note that $p_\lambda \in \ell^1(\mathbb{N}^0, \frac{1}{4^n})$ if and only if $|\lambda| > \frac{1}{4}$. Moreover

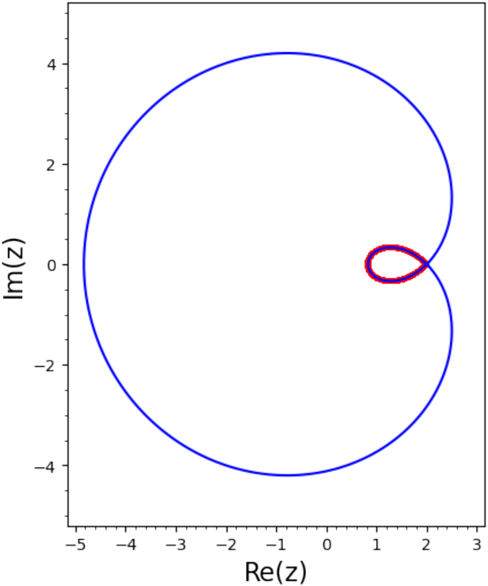
$$(\lambda - \delta_1)^{-1} = \frac{1}{\lambda} p_\lambda, \qquad |\lambda| > \frac{1}{4},$$

$$\Omega := \{\lambda \in \mathbb{C} \ : \ \left| \frac{\lambda - 1}{\lambda^2} \right| > \frac{1}{4} \}.$$

Theorem

The inverse of the Catalan sequence c is given $c^{-1} = \delta_0 - \delta_1 * c$ and

$$(\lambda - c)^{-1} = \frac{\delta_0}{\lambda} + \frac{1}{\lambda(\lambda - 1)} p_{\frac{\lambda - 1}{\lambda^2}} + \frac{1}{\lambda^2} c - \frac{1}{\lambda^2} c * p_{\frac{\lambda - 1}{\lambda^2}}, \qquad \lambda \in \Omega \backslash \{0\}.$$



The set $\partial(\Omega)$ in blue and $\partial(\sigma(c))$ in red.

4. Inverse spectral mapping theorem.

$$TY^2 - Y + I = 0, \qquad Y \in \mathcal{B}(X). \tag{3}$$

Lemma

Given $T \in \mathcal{B}(X)$ and Y a solution of (3). Then Y has left-inverse and $Y_l^{-1} = I - TY$.

Theorem

Given $T \in \mathcal{B}(X)$ and Y a solution of (3). Then TFAE

(i) $0 \in \rho(Y)$.

(ii) $T = Y^{-1} - Y^{-2}$.

(iii) *T and Y commute.*

(iv) $TY^2 = YTY$.

Corollary

Let X be a Banach space with $\dim(X) < \infty$, $T \in \mathcal{B}(X)$ and Y a solution of (3). Then Y is invertible, T and Y commute and

$$T = Y^{-1} - Y^{-2}.$$

Theorem

Given $T \in \mathcal{B}(X)$ *and* Y *a solution of (3) such that* $0 \in \rho(Y)$.

(i) *Given* $\lambda \in \mathbb{C}$ *such that* $\frac{\lambda-1}{\lambda^2} \in \rho(T)$ *then* $\lambda \in \rho(Y)$ *and*

$$(\lambda - Y)^{-1} = \frac{1}{\lambda} + \frac{1}{\lambda^3}(\frac{\lambda-1}{\lambda^2} - T)^{-1} + \frac{Y}{\lambda^2} - \frac{(\lambda-1)Y}{\lambda^4}(\frac{\lambda-1}{\lambda^2} - T)^{-1}.$$

(ii) *Given* $\lambda \in \rho(Y)$ *such that* $\frac{\lambda}{\lambda-1} \in \rho(Y)$ *then* $\frac{\lambda-1}{\lambda^2} \in \rho(T)$ *and*

$$(\frac{\lambda-1}{\lambda^2} - T)^{-1} = \frac{\lambda^4}{\lambda-1}(\frac{\lambda}{\lambda-1} - Y)^{-1}\left((\lambda - Y)^{-1} - \frac{\lambda + Y}{\lambda^2}\right).$$

5. Catalan generating functions

In this section, $T \in \mathcal{B}(X)$, such that

$$\sup_{n \geq 0} \|4^n T^n\| := M < \infty, \tag{4}$$

i.e., $4T$ is a power-bounded operator. Then $\sigma(T) \subset \overline{D(0, \frac{1}{4})}$ and we define the following bounded operator,

$$C(T) := \sum_{n \geq 0} C_n T^n. \tag{5}$$

Theorem

Given $T \in \mathcal{B}(X)$ such that $4T$ is power-bounded and $c = (C_n)_{n \geq 0}$ the Catalan sequence. Then

(i) *The operator $C(T)$ defined by (5) is well-defined, T and $C(T)$ commute, and $C(T)$ is a solution of the quadratic equation (3).*

(ii) The following integral representation holds

$$C(T)x = \frac{1}{\pi}\int_{\frac{1}{4}}^{\infty} \frac{\sqrt{\lambda - \frac{1}{4}}}{\lambda}(\lambda - T)^{-1} x d\lambda, \qquad x \in X.$$

(iii) The following integral representation holds

$$TC(T) = \frac{I}{2} - \sqrt{\frac{1}{4} - T}.$$

(iv) The spectral mapping theorem hols for $C(T)$, i.e, $\sigma(C(T)) = C(\sigma(T))$ and

$$\sigma(C(T)) \subset C(\overline{D(0, \frac{1}{4})}) \subset \sigma(c).$$

(v) Given $\lambda \in \mathbb{C}$ such that $\frac{\lambda - 1}{\lambda^2} \in \rho(T)$ then $\lambda \in \rho(Y)$ and

$$(\lambda - C(T))^{-1} =$$

$$\frac{1}{\lambda} + \frac{1}{\lambda^3}(\frac{\lambda - 1}{\lambda^2} - T)^{-1} + \frac{C(T)}{\lambda^2} - \frac{(\lambda - 1)C(T)}{\lambda^4}(\frac{\lambda - 1}{\lambda^2} - T)^{-1}.$$

In the case that $\sigma(T) \subset D(0, \frac{1}{4})$, the generating function $C(z)$ given in (2) is an holomorphic function in a neighborhood of $\sigma(T)$. Then the Dunford functional calculus, defined by the integral Cauchy-formula,

$$f(T)x = \int_{\Gamma} f(z)(z - T)^{-1} x dz, \qquad x \in X,$$

(Γ is a path around the spectrum set $\sigma(T)$) allows to defined $C(T)$, ([Y, Section VIII.7]) which, of course, coincides with the expression gives in (5).

6. Examples, applications and final comments

6.1 Matrices on $\mathbb{C}^2$ We consider $\cdot\mathbb{C}^2$ and $T = \lambda I_2$ with $0 \neq \lambda \in \mathbb{C}$. Then the solution of (3) is given by

$$Y = \begin{pmatrix} \frac{1\pm\sqrt{1-4\lambda(1+\lambda bc)}}{2\lambda} & b \\ c & \frac{1\mp\sqrt{1-4\lambda(1+\lambda bc)}}{2\lambda} \end{pmatrix},$$

for $|c| + |b| > 0$, the allowed signs are $(-, +)$ and $(+, -)$; and

$$Y = \begin{pmatrix} \frac{1\pm\sqrt{1-4\lambda}}{2\lambda} & 0 \\ 0 & \frac{1\pm\sqrt{1-4\lambda}}{2\lambda} \end{pmatrix},$$

for $c = b = 0$. In both cases, note that $\sigma(Y) = \{C(\lambda), \dfrac{1}{\lambda C(\lambda)}\}$ and $\sigma(T) = \{\lambda\}$. For $|\lambda| \leq \frac{1}{4}$.

$$C(T) = \begin{pmatrix} C(\lambda) & 0 \\ 0 & C(\lambda) \end{pmatrix}.$$

Now we consider $T = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$ with $\lambda \in \mathbb{C} \backslash \{0\}$. The solutions of (3) are given by

$$Y = \begin{pmatrix} a & \frac{a-1}{2\lambda a} \\ \frac{a-1}{2\lambda a} & a \end{pmatrix}$$

where a is a solution of the quartic equation $4\lambda^2 a^4 - a^2 + 1 = 0$. In the case that $|\lambda| \leq \frac{1}{4}$, we get that

$$C(T) = \begin{pmatrix} C_e(\lambda) & C_o(\lambda) \\ C_o(\lambda) & C_e(\lambda) \end{pmatrix}$$

where functions C_e and C_o are defined in Proposition 2.1

6. Examples, applications and final comments

6.2 Catalan operators on ℓ^p We consider the space of sequences $\ell^p(\mathbb{N}^0, \frac{1}{4^n})$ where

$$\|a\|_{p,\frac{1}{4^n}} := \left(\sum_{n=0}^{\infty} \frac{|a_n|^p}{4^{np}}\right)^{\frac{1}{p}} < \infty, \qquad \|a\|_{\infty,\frac{1}{4^n}} := \sup_{n\geq 0} \frac{|a_n|}{4^n} < \infty.$$

for $1 \leq p \leq \infty$. Note that $\ell^1(\mathbb{N}^0, \frac{1}{4^n}) \hookrightarrow \ell^p(\mathbb{N}^0, \frac{1}{4^n}) \hookrightarrow \ell^\infty(\mathbb{N}^0, \frac{1}{4^n})$.

Now we consider the convolution operator $C(f) := c * f$ for $f \in \ell^p(\mathbb{N}^0, \frac{1}{4^n})$ with $1 \leq p \leq \infty$. Since $C(f) = \sum_{n\geq 0} c_n(\delta_1)^n(f)$, then

$$\sigma(C) = C(\sigma(\delta_1)) = C(\overline{D(0, \frac{1}{4})})$$

i.e., it is independent on p and equal to $\sigma(c)$ in $\ell^1(\mathbb{N}^0, \frac{1}{4^n})$.

Now we consider $\ell^p(\mathbb{Z})$ for $1 \le p \le \infty$ and $a = \delta_1 - \delta_0$ defines

$$a * (f)(n) := f(n-1) - f(n), \qquad f \in \ell^p(\mathbb{Z})$$

for $n \in \mathbb{Z}$. Note that $\|a\| = 2$, and

$$(\lambda\delta_0 + a)^{-1} = \sum_{j\ge 0} \frac{\delta_j}{(1+\lambda)^{j+1}}, \quad 1 < |1+\lambda|,$$

see [GLM, Theorem 3.3 (4)], [B]. Now we need to consider $\dfrac{a}{8}$ and the associated Catalan generating operator $C(\frac{a}{8})$. Then

$$\begin{aligned} C(\frac{a}{8}) &= \frac{8}{\pi}\int_{\frac{1}{4}}^{\infty} \frac{\sqrt{\lambda - \frac{1}{4}}}{\lambda}(8\lambda\delta_0 + a)^{-1} x d\lambda \\ &= \frac{4}{\pi}\sum_{j\ge 0}\frac{\delta_j}{2^{j+1}}\int_0^\infty \frac{\sqrt{t}}{(t+1)(t+\frac{3}{2})^{j+1}}dt \\ &= (2\sqrt{6}-4)\delta_0 + \sum_{j=1}^{\infty}\left(\frac{\sqrt{6}}{3}\sum_{k=j}^{\infty}\frac{C_k}{12^k}\right)\delta_j. \end{aligned}$$

7. Generators of C_0-semigroups

A family of bounded operators $(T(t))_{t\geq 0}$ on a Banach space X is a C_0-semigroup if it satisfies the functional equation,

$$\begin{cases} T(t+s) = T(t)T(s), & \text{for all } t, s \geq 0; \\ T(0) = I, \end{cases}$$

and $\text{lím}_{t\to 0} T(t)x = x$ for all $x \in X$. The linear operator $(A, D(A))$ defined as,

$$Ax := \underset{h\to 0}{\text{lím}} \frac{T(h)x - x}{h}, \quad x \in D(A) := \{x \in X \,|\, Ax \text{ exists}\}$$

is the infinitesimal generator of the semigroup $(T(t))_{t\geq 0}$ with domain $D(A)$ which is closed and densely defined, see [EN].

Definition

The **Catalan kernel** is the function $c : (0, \infty) \to (0, \infty)$ defined as,

$$c(t) := \frac{1}{2\pi} \int_{\frac{1}{4}}^{\infty} e^{-\lambda t} \frac{\sqrt{4\lambda - 1}}{\lambda} \, d\lambda, \qquad t > 0.$$

Theorem

1. *For* $w_0 \leq \frac{1}{4}$, $c \in L^1(\mathbb{R}^+, e^{w_0 t})$ *and*

$$\|c\|_{L^1(\mathbb{R}^+, e^{w_0 t})} = \frac{1 - \sqrt{1 - 4w_0}}{2w_0}.$$

2. *The Laplace transform is*

$$\mathcal{L}(c)(s) = \frac{\sqrt{1 + 4s} - 1}{2s}, \quad s \geq -\frac{1}{4}.$$

Proposition

The function $\partial_t(c * c)(t)) \in L^1(\mathbb{R}^+, e^{w_0 t})$ *and*

$$\partial_t((c * c)(t)) = -c(t), \quad t > 0.$$

Definition

Let $(A, D(A))$ be the generator of the C_0-semigroup $(T(t))_{t \geq 0}$ such that $T(t) \leq Me^{w_0 t}$ with $w_0 \leq \frac{1}{4}$ for all $t > 0$. Then we define the **Catalan operator** $C(A) \in \mathcal{B}(X)$ as,

$$C(A)x := \int_0^\infty c(t)T(t)x \, dt, \quad x \in X,$$

where c is the Catalan kernel.

Theorem

Let A be the generator of the C_0-semigroup $(T(t))_{t\geq 0}$. Then,

1. *The Catalan operator verifies the quadratic Catalan equation*

$$AC(A)^2 - C(A) + I = 0.$$

2. *The following representation holds,*

$$AC(A) = \frac{1}{2} - \sqrt{\frac{1}{4} - A}.$$

3. *The spectral mapping theorem holds for $C(A)$, i.e,*

$$\sigma\left(C(A)\right) = C(\sigma(A)) \cup \{0\}.$$

Bibliography

H. Bateman, *Some simple differential difference equations and the related functions.* Bull. Amer. Math. Soc. **49** (1943) 494–512.

K.-J. Engel and R. Nagel. One-parameter semigroups for linear evolution equations. Vol. 194. Springer-Verlag, New York, 2000.

J. Gónzalez-Camus, C. Lizama and P.J. Miana: *Fundamental solutions for semidiscrete evolution equations via Banach algebras*, Adv. Difference Equ., **2021:35** (2021), 1-32.

J. E. McFarland: *An Iterative Solution of the Quadratic Equation in Banach Space*, Proc. Amer. Math. Soc., **9** (1958), 824-830.

P.J. Miana and N. Romero: *Moments of combinatorial and Catalan numbers,* J. Number Theory, **130** (2010), no. 8, 1876–1887.

K. Yosida. Functional Analysis:- Grundlehren der mathematischen Wissenchaften 123. Springer-Verlag, Berlin, 1980.

MUCHAS GRACIAS

Pedro J. Miana, IUMA-UZ

GRAFFITIS MATEMATICOS: Antonio Ledesma López

Index