# Linear spaces of strongly norm-attaining Lipschitz mappings

**Óscar Roldán**

A joint work with
**Vladimir Kadets**

XVII Encuentro de la Red de Análisis Funcional y Aplicaciones.
Universidad de La Laguna, Tenerife.
10-12 de marzo de 2022.

MICIU grant FPU17/02023.
Project MTM2017-83262-C2-1-P/MCIN/AEI/10.13039/501100011033 (FEDER)

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#StopThisWar
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# About the talk

The contents of this talk are from:

V. Kadets, Ó. Roldán, Closed linear spaces consisting of strongly norm attaining Lipschitz functionals. Preprint (2022). ArXiv/2202.06855.

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- $\text{NA}(X, Y)$ is **NOT ALWAYS** dense in $\mathcal{L}(X, Y)$ (**Lindenstrauss, 1963**).

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**OBJECTIVE:** Study similar questions adapted to Lipschitz mappings.

## Lipschitz mappings

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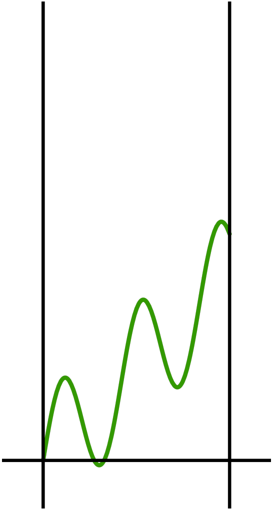
# Lipschitz mappings

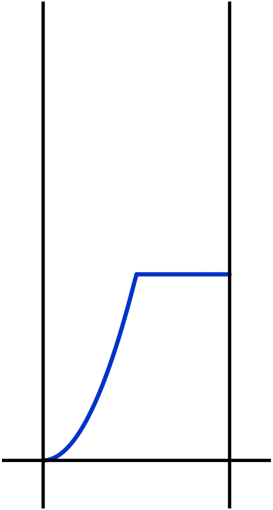
Figure: Non vertical(∗)

(∗): Picture from: https://unsplash.com/photos/y0N4XwM70yA

# Lipschitz mappings

These **ARE** Lipschitz mappings:





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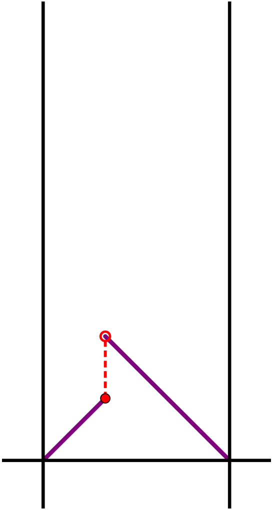

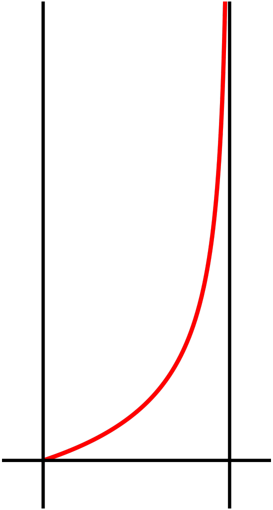

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There is a natural way to define norm-attainment. $f \in \text{Lip}_0(M)$ **attains its norm strongly** if there exist $x \neq y \in M$ such that

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- This norm-attainment is called **strong** because there are other weaker, much less restrictive, norm-attainments considered that are also natural.

# The strong norm-attainment is restrictive

**Lemma 2.2 (Kadets-Martín-Soloviova, 2016)**

If $f \in \mathrm{Lip}_0(M)$ attains its norm on a pair $(x, y) \in M \times M$, $x \neq y$, and if $z \in M \backslash \{x, y\}$ is such an element that $\rho(x, y) = \rho(x, z) + \rho(z, y)$, then $f$ strongly attains its norm on the pairs $(x, z)$ and $(z, y)$, and

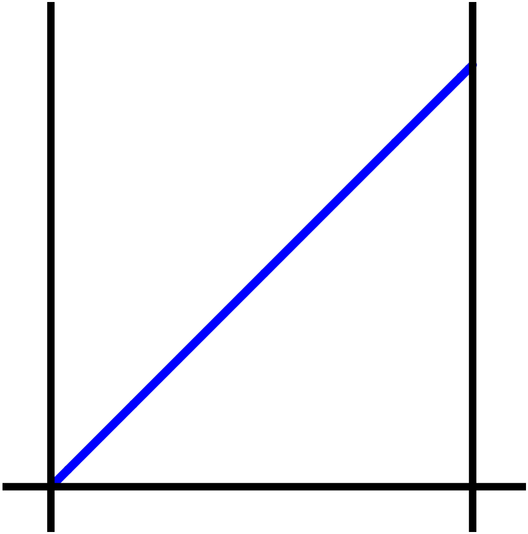
$$f(z) = \frac{\rho(z, y)f(x) + \rho(x, z)f(y)}{\rho(x, y)}.$$

In particular, if $M$ is a convex subset of a Banach space, then $f$ is affine on the closed segment $[x, y]$, that is, $f(\theta x + (1-\theta)y) = \theta f(x) + (1-\theta)f(y)$ for every $\theta \in [0, 1]$.

In other words: if $f$ attains its norm strongly at $(x, y)$, it must also attain its norm strongly at any pair of distinct points in between them!

Actually, if the maximum possible slope of $f$ is attained at the pair $(x, y)$, $f$ must be affine in between those points!

# The strong norm-attainment is restrictive



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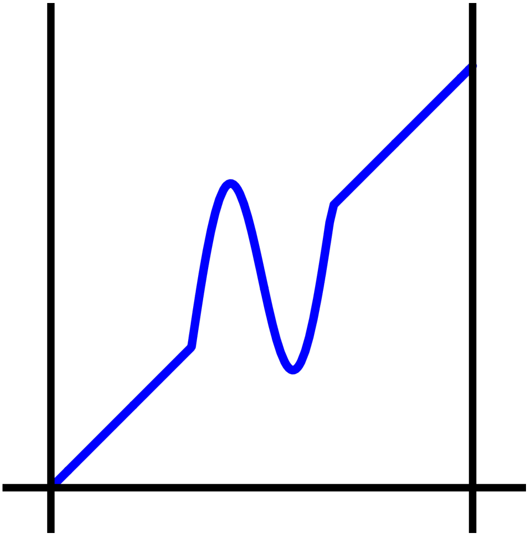

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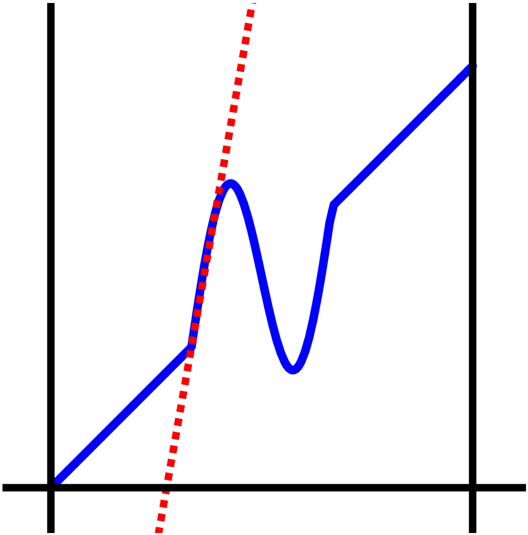

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Figure: Space (unrelated to our topic).

Picture from: https://pixabay.com/es/photos/estrellas-cielo-noche-1845140/

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Recall the title of the talk:

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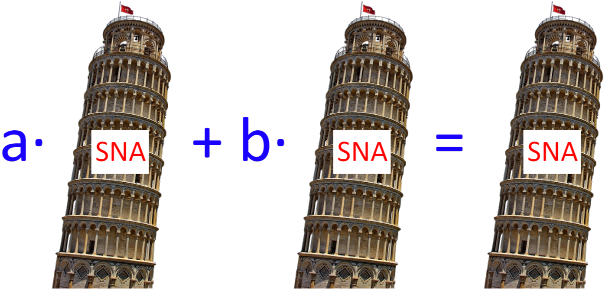


Figure: This is more like what we mean.

---

Pisa tower picture from:
https://www.kindpng.com/picc/m/109-1098941_leaning-tower-of-pisa-building-places-of-interest.png

# Index

## Remarks

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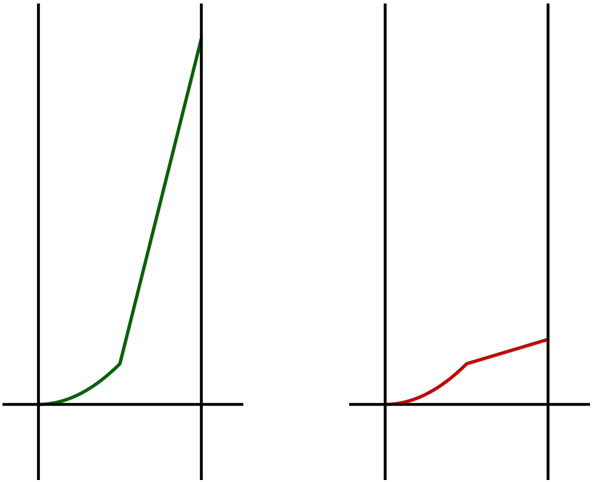

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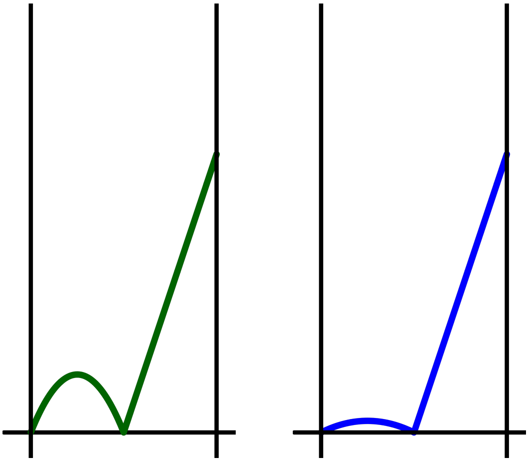

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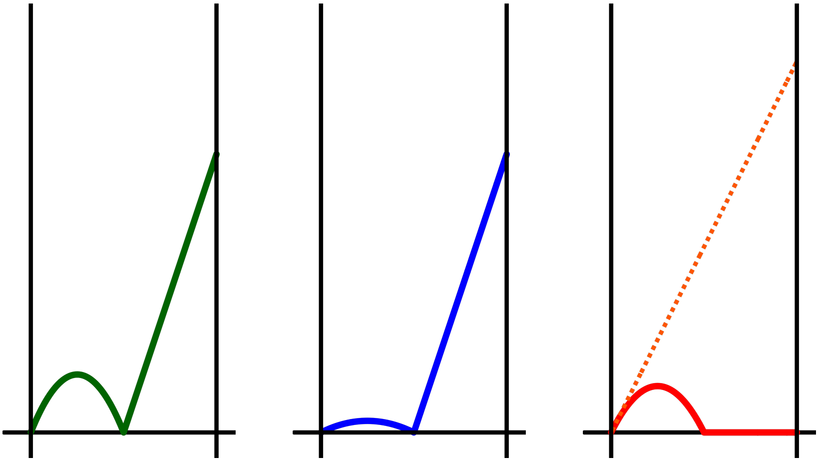

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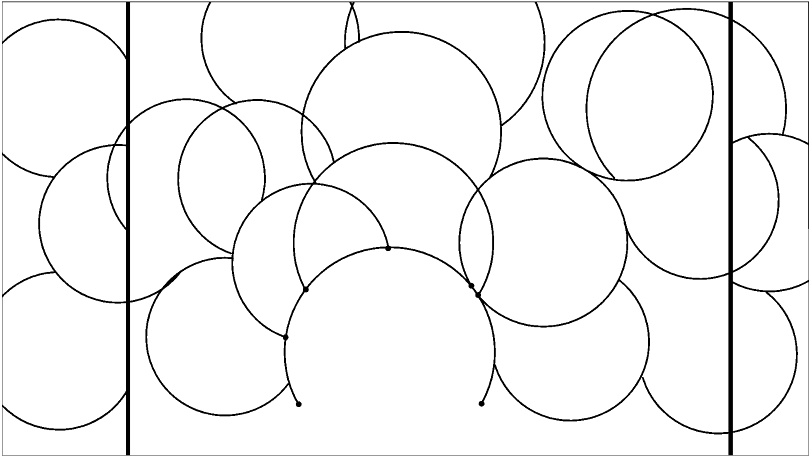

Picture made by and posted with the consent of Andrés Quilis.

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The **Lipschitz-free** space associated to $M$ is

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1) $\mathcal{F}(M)^*$ is isometrically isomorphic to $\operatorname{Lip}_0(M)$. This allows us to "linearize" Lipschitz mappings.

2) Link to classical functionals theory: $\operatorname{SNA}(M) \subset \operatorname{NA}(\mathcal{F}(M), \mathbb{R})$ in a natural way.

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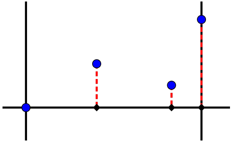

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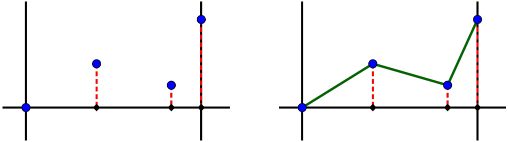

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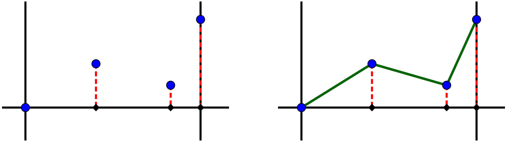

**McShane's extension theorem:** If $f \in \mathrm{Lip}_0(M)$ and $M$ is a metric subspace of $M'$, there is a mapping $F \in \mathrm{Lip}_0(M')$ such that $F = f$ on $M$ and $\|F\| = \|f\|$.

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Let $M \subset M'$ be pointed metric spaces and $n \in \mathbb{N}$. If $\ell_1^n$ embeds isometrically into $\mathrm{SNA}(M)$, then it also embeds isometrically into $\mathrm{SNA}(M')$.

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**Question:** is it possible to get $\ell_1^n$ spaces, $n > 1$, in $\text{SNA}(M)$, where $M$ is finite?

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**Corollary**

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**Remark:** Contrast with classical norm-attainment theory!

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If $|M| = 2n$, then $\mathcal{F}(M)$ contains a 1-complemented subspace isometric to $\ell_1^n$.

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If $n \in \mathbb{N}$, then $\ell_1^n$ is isometric to a subspace of $\ell_\infty^{2^{n-1}}$.

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(4) By Lemma 1, $\mathrm{SNA}(M)$ contains $\ell_1^n$ isometrically. ■.

**Corollary**

If $M$ is infinite, $\mathrm{SNA}(M)$ contains all the $\ell_1^n$ isometrically.

# One step further

**Theorem**

If $|M| \geq 2^n$, then SNA$(M)$ contains $\ell_1^n$ isometrically.

**Sketch of proof:**

(1) Let $K \subset M$ with $|K| = 2^n$. By Lemma 2, $\mathcal{F}(K)$ contains a 1-complemented subspace isometric to $\ell_1^{2^{n-1}}$.

(2) By Lemma 3, $\mathrm{Lip}_0(K) = \mathrm{SNA}(K)$ contains $\ell_\infty^{2^{n-1}}$ isometrically.

(3) By Lemma 4, SNA$(K)$ contains $\ell_1^n$ isometrically.

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**Corollary**

If $M$ is infinite, SNA$(M)$ contains all the $\ell_1^n$ isometrically.

Compare this to the classical theory again, where there are Banach spaces $X$ such that NA$(X)$ does not contain 2-dimensional linear subspaces.

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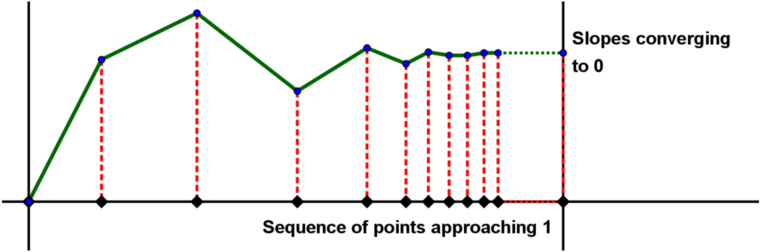

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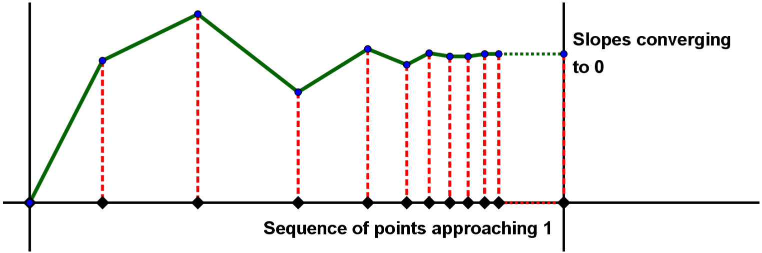


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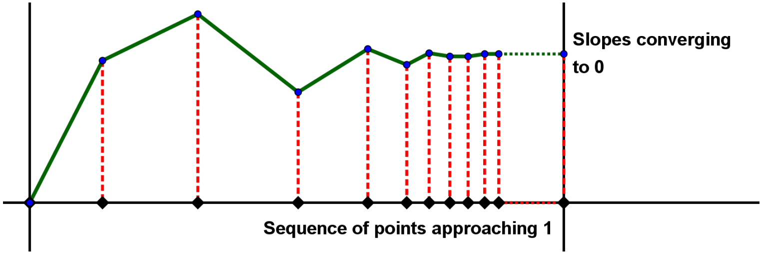


Actually, **ANY** Banach space can be subspace of SNA($M$) for an appropriate $M$.

**Theorem**

If $Y$ is a Banach space, then it embeds isometrically in SNA($B_{Y^*}$).

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So if $Y^*$ is not separable, $M$ **CANNOT** be separable either. Hence, some $\mathrm{SNA}(M)$ with $M$ infinite don't contain $\ell_1$, despite containing all the $\ell_1^n$ spaces.

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**Theorem**

Let $M$ be a $\sigma$-precompact pointed metric space, then all Banach subspaces in SNA($M$) are separable and isomorphic to polyhedral spaces.

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So all the subspaces of SNA([0, 1]) are separable and isomorphically polyhedral, and the same happens on all the $\mathbb{R}^n$ spaces, for instance.

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If $M$ is a metric space containing $[0, 1]$ isometrically, then SNA($M$) contains $c_0$ isometrically.

**Sketch of proof:** The identity on $[0, 1]$ can be extended to $M$ as a retraction $F$ with Lipschitz constant 1. By using $T : \text{Lip}_0([0, 1]) \to \text{Lip}_0(M)$ defined as $T(f) := f \circ F$, we can deduce from here that any subspace of SNA($[0, 1]$) yields a subspace of SNA($M$). ■.

# Index

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The small ball property is a weaker property than $\sigma$-precompactness.

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Let $M$ be a pointed metric space with the small ball property. Is it true that all subspaces of SNA($M$) are separable and isomorphically polyhedral?

# Sample of references

For the interested reader.

Figure: Interested reader

Manipulated pictures. Originals: Pisa tower, Book, Sunglasses.

# Sample of references

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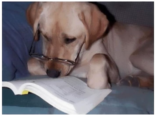

# Thank you for your attention!

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