UCA Universidad de Cádiz

Órbitas de operadores que λ-conmutan con el operador diferenciación

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Hypercyclic operators

...the first examples

Hypercyclic operator

Let T be a continuous linear operator defined on a separable F-space $\mathscr{F}$, T is hypercyclic provided there exists $x \in \mathscr{F}$ such that $\{T^n x\}$ is dense in $\mathscr{F}$.

Hypercyclic operators

...the first examples

Let $\mathscr{H}(\mathbb{C})$ the space of entire functions endowed with the topology of the uniform convergence on compact subsets.

Theorem (1929-G. D. Birkhoff)

The translation operator $Tf(z) = f(z+a)$ is hypercyclic in $\mathscr{H}(\mathbb{C})$. compact subsets.

Theorem (1952. G. R. MacLane)

The differentiation operator $Df = f'$ is hypercyclic in $\mathscr{H}(\mathbb{C})$

Hypercyclic operators

...unification of classical results

Theorem (1991-Godedroy-Shapiro)

Let $L : \mathscr{H}(\mathbb{C}) \to \mathscr{H}(\mathbb{C})$ *be a continuous operator commuting with the differentiation operator D. Then L is hypercyclic if and only if L is not a multiple of the identity.*

- A characterization of the commutant of D.
- The Hypercyclicity Criterion.

The Hypercyclicity Criterion

...a sufficient condition for hypercyclicity.

Theorem (Hypercyclicity Criterion (Bès-Peris))

Let T be an operator on a F-space $\mathscr{X}$ satisfying the following conditions: there exist X_0 and Y_0 dense subsets of $\mathscr{X}$, a sequence (n_k) of non-negative integers, and (not necessarily continuous) mappings $S_{n_k} : Y_0 \longrightarrow \mathscr{X}$ so that:

- ***i)*** $T^{n_k} \longrightarrow 0$ *pointwise on* X_0.
- ***ii)*** $S_{n_k} \longrightarrow 0$ *pointwise on* Y_0.
- ***iii)*** $T^{n_k} S_{n_k} \longrightarrow Id_{Y_0}$ *pointwise on* Y_0.

Then the operator T is hypercyclic.

Extended eigenoperators

...λ-commutativity.

Definition

Let T be a continuous linear operator and $\lambda \in \mathbb{C}$. An operator $X \neq 0$ is said to be an extended λ-eigenoperator of T provided $TX = \lambda XT$, in such a case λ is called extended eigenvalue of T.

In operator theory this concept arises in order to extend Lomonosov's result.

...questions

In general

Cyclic and hypercyclic properties are usually transferred to the commutant of the operator. How does it transfer to the extended eigenoperators?

For today

Let L be a continuous operator on $\mathscr{H}(\mathbb{C})$ and assume that $DL = \lambda LD$ for some $\lambda \neq 1$. Is L hypercyclic?

...selected background

Let $C_{\lambda,b}f(z) = f(\lambda z + b)$ be the composition operator induced by an affine endomorphism of $\mathbb{C}$.

Clearly $DC_{\lambda,b} = \lambda C_{\lambda,b}D$.

Theorem (1994-Bernal-Montes.)

$C_{\lambda,b}$ is not hypercyclic unless $\lambda = 1$ and $b \neq 0$.

...selected background

Let $T_{\lambda,b}f(z) = f'(\lambda z + b)$

Again $DT_{\lambda,b} = \lambda T_{\lambda,b}D$.

Theorem (Aron-Markose(2004), Fernández-Hallack (2005), L.-Romero M.P.(2014))

Assume $\lambda \neq 1$. $T_{\lambda,b}$ is hypercyclic if and only if $|\lambda| \geq 1$.

Main result

Theorem (Bensaid-González-L.-Romero)

Let L be such that $DL = \lambda LD$ then, L is hypercyclic if and only if $|\lambda| \geq 1$ and L is not a multiple of a composition operators induced by an affine endomorphism of $\mathbb{C}$.

Main result

...first steep

Theorem

If L is an λ-extended eigenoperator of D then L factorizes as

$$L = R_\lambda \phi(D)$$

where ϕ is an entire function of exponential type and $R_\lambda f(z) = f(\lambda z)$ is the dilation operator.

...as a consequence

The proof of hypercyclicity of L splits naturally into several cases in terms of $|\lambda|$ and the zeros of ϕ.

An easy case

...$|\lambda| = 1$, *a root of the unity*

Consequence of Godefroy-Shapiro and Ansari results

If $|\lambda| = 1$, $\lambda \neq 1$ is a root of the unity and ϕ has some zero, iff $L = R_\lambda \phi(D)$ is hypercyclic.

(Some power of L commutes with D)

The case $|\lambda| < 1$.

...not hypercyclicity

A general result for operators in Banach spaces

Let A and T be two operators on a Banach space. If T is an extended λ-eigenoperator of A and $|\lambda| < 1$ then T is not hypercyclic.

....the above result is not true on *F* spaces.

For $|\lambda| > 1$, $T_{\lambda,b}D = (1/\lambda)DT_{\lambda,b}$. Hence D is an hypercyclic extended $(1/\lambda)$-eigenoperator of $T_{\lambda,b}$ with $|1/\lambda| < 1$.

The case $|\lambda| < 1$.

...not hypercyclicity

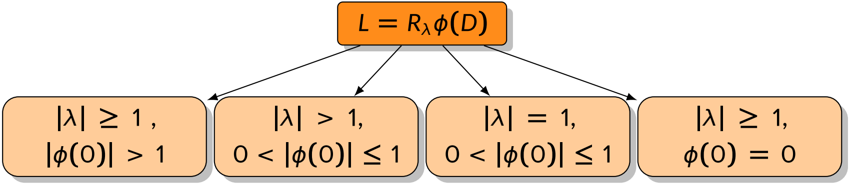
$$L = R_\lambda \phi(D)$$

$$L^n f(z) = \int \cdots \int f(\lambda^n z + \lambda^{n-1} w_1 + \cdots + w_n)\, d\mu(w_n) \cdots d\mu(w_1)$$

- μ is a complex Borel measure with compact support in $\mathbb{C}$
- If $|\lambda| < 1$ then the argument $(\lambda^n z + \lambda^{n-1} w_1 + \cdots + w_n)$ lies in a compact subset of $\mathbb{C}$.
- If $\{L^n f\}$ is dense, since D has dense range $\{D^p L^n f\}$ should be also dense.
- By selecting p such that $|\lambda|^p < \|\mu\|$ we get that $D^p L^n f \longrightarrow 0$ as $n \longrightarrow \infty$.

The case $|\lambda| \geq 1$

When $|\lambda| \geq 1$, we divide the proof of the Main Theorem in cases:



We assume that $\lambda \neq 1$.

We can assume without loss of generality that ϕ has some zero. If not, we can show that $L = R_\lambda \phi(D)$ is a multiple of a composition operator induced by an affine endomorphisms.

Case 1: $|\lambda| \geq 1$, $|\phi(0)| > 1$

$L = R_\lambda \phi(D)$

↓

$|\lambda| \geq 1$, $|\phi(0)| > 1$

- $X_0 = \operatorname{span}\{e^{(a/\lambda^n)z}\,;\, n \geq 0\}$. **is dense.** $L^n e^{(a/\lambda^k)z} = 0$ if $n > k$.
- $Y_0 = \operatorname{span}\{p_k(z) \,:\, k \geq 0\}$ **is dense** and $Lp_k = \phi(0)\lambda^k p_k$, $\forall k \geq 0$.
- Define $Sp_k = \frac{1}{\phi(0)\lambda^k} p_k$ and extend S to Y_0 by linearity. Since $|\phi(0)| > 1$, $S^n p_k \to 0$ as $n \to \infty$ for every $|\lambda| \geq 1$, hence $S^n y \to 0$ as $n \to \infty$ for all $y \in Y_0$. $LS = \mathrm{Id}_{Y_0}$.

According to the Hypercyclicity Criterion, L is hypercyclic.

Case 2: $|\lambda| > 1$, $0 < |\phi(0)| \leq 1$

Definition (Borel transform)

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of exponential type. The Borel transform Bf of f is defined as

$$Bf(z) = \sum_{n=0}^{\infty} \frac{n! a_n}{z^{n+1}}. \tag{0.1}$$

and is analytic on $|z| > c$ for some $c > 0$.

In particular, $f_n(z) = z^n/n!$, we have $Bf_n(z) = 1/z^{n+1}$ which is analytic on $|z| > 0$.

Case 2: $|\lambda| > 1$, $0 < |\phi(0)| \leq 1$

Definition (Pólya Integral Representation Formula)

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of exponential type. If $Bf(z)$ is analytic on $|z| > c$, then for any $R > c$, f can be expressed as:

$$f(z) = \frac{1}{2\pi i} \oint_{|t|=R} e^{zt} Bf(t)\, dt. \tag{0.2}$$

This is called the Pólya Integral Representation of f.

In particular, for any $R > 0$ the Pólya Integral Representation of $f_n(z) = z^n/n!$ is

$$f_n(z) = \frac{1}{2\pi i} \oint_{|t|=R} e^{zt} Bf_n(t)\, dt. \tag{0.3}$$

Case 2: $|\lambda| > 1$, $0 < |\phi(0)| \le 1$

Getting an idea about the right inverse...

$$L^n f(z) = \frac{1}{2\pi i} \oint_{|t|=R} \phi(t)\phi(\lambda t) \cdots \phi(\lambda^{n-1} t) e^{\lambda^{n-1} zt} Bf(t)\, dt. \qquad (0.4)$$

Denoting $\omega = 1/\lambda$, if for some $R > c$ using the above representation it show up many different ways to define a right inverse L. For instance:

$$S_1 f(z) = \frac{1}{2\pi i} \oint_{|t|=R} \frac{1}{\phi(\omega t)} e^{\omega zt} Bf(t)\, dt, \qquad (0.5)$$

so as we get $LS_1 f = f$..

The idea here is to define S_k on f_0. S_k on f_1, S_k on f_j,...

Case 2: $|\lambda| > 1,\ 0 < |\phi(0)| \leq 1$

Denote $P(z) = \phi(0)(1 - z/a)$. Then, there exists a sequence of positive numbers $R_k \to \infty$ such that

$$L_k f_n(z) = \frac{1}{2\pi i} \oint_{|t|=R_k} \frac{1}{P(\omega t)\cdots P(\omega^k t)} e^{\omega^k z t} B f_n(t)\, dt \to 0 \qquad (0.6)$$

uniformly on compact subsets.

Case 2: $|\lambda| > 1,\ 0 < |\phi(0)| \leq 1$

$$L = R_\lambda \phi(D)$$

$$|\lambda| > 1,\ 0 < |\phi(0)| \leq 1$$

- Set $\omega = 1/\lambda$ since $|\omega| < 1$, the subset $X_0 = \mathrm{span}\,\{e^{a\omega^n z} : n \geq 0\}$ is dense in $H(\mathbb{C})$, and $L^n x_0 = 0$ for n large enough on X_0.
- Defining

$$S_k f_m = L_k f_m - \sum_{j=0}^{m-1} a_{m-j}^{(k)} S_k f_j \tag{0.7}$$

we get $(R_\lambda \phi(D))^k S_k f_m = f_m$ by construction, and $S_k f_m \to 0$. on compact subsets as $k \to \infty$ for $n = 0, \ldots, n_0$.

Case 3: $|\lambda| = 1$ irrational rotation.

- In this case the right inverse is defined on the exponentials as follows ($\omega = \lambda^{-1}$):

$$S^n e^{bz} = \frac{1}{\phi(\omega b)\cdots\phi(\omega^n b)} e^{\omega^n bz}. \tag{0.8}$$

- The problem is reduced to find a complex number b such that $\limsup \phi(\omega b)\cdots\phi(\omega^n b) = \infty$.
- If ϕ has a finite numbers of zeros, the complex number b can be obtained by standard arguments.

Case 3: $|\lambda| = 1$, $0 < |\phi(0)| \leq 1$

Set $f_n(z) = \phi(\omega z) \cdots \phi(\omega^n z)$ We proved that:

1. There exists an open subset G such that $\mathscr{F} = \{f_n\}$ is normal at no point of G.
2. Apply Montel's Theorem and deduce that

$$\bigcup_{f_n \in \mathscr{F}} f_n(G)$$

   is dense in $\mathbb{C}$.

3. There exists $z_0 \in G$ such that $\{f_n(z_0)\}_{n \geq 1}$ is dense in $\mathbb{C}$.
4. In particular, there exists a subsequence $\{n_k\}_k$ such that $\lim_{k \to \infty} f_{n_k}(z_0) = \infty$.

Theorem

Suppose that ω is an irrational rotation and $0 < |\phi(0)| \leq 1$. Then $L = R_\lambda \phi(D)$ is hypercyclic.

Case 4: $|\lambda| \geq 1$, $\phi(0) = 0$

$T = R_\lambda \phi(D)$

$|\lambda| \geq 1, \phi(0) = 0$

Let $\phi(z) = z^m \psi(z)$ with $\psi(0) \neq 0$ and set $A_\lambda = R_\lambda \psi(D)$. So, $T = A_\lambda D^m$.

- $T^n p(z) = 0$ for $n > \deg(p)$
- In view of $TV^m p_k = A_\lambda p_k = \psi(0)\lambda^k p_k$, we define

$$S_k p_n = \frac{V^{mk} p_n}{\lambda^m \lambda^{2m} \cdots \lambda^{(k-1)m} (\psi(0)\lambda^n)^k}, \tag{0.9}$$

  and extend S_k to $Y_0 = \mathrm{span}\,\{p_k(z) \; : \; k \geq 0\}$ by linearity.
- Since $|\lambda| \geq 1$, $|S_k(p_n)(z)| \leq \frac{|V^{mk} p_n(z)|}{|\psi(0)|^k} \to 0$ uniformly on compact sets. $T^k S_k p_n = p_n$

According to the Hypercyclicity Criterion, T is hypercyclic.

Thank you very much for your attention