Nonlocal operators are chaotic

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1 Introduction

2 Chaos of Nonlocal Operators

3 Chaos of operators associated to Numerical Schemes

Objectives

Analize the dynamics of certain nonlocal operators:

- The fractional difference operator in the sense of Riemann-Liouville: Δ^{α} and the Nabla difference operator ∇_a^{α} for $0 < \alpha \leq 1$.
- Nonlocal difference operators which arise in the study of time-stepping schemes for fractional operators.

Linear dynamical systems

Definition

Given X a Banach space and an operator $T : X \to X$:

- $T : X \to X$ is **hypercyclic**, if there exists a vector $x \in X$ such that $Orb(x, T) = \{T^n x : n \in \mathbb{N}\}$ is dense in X.
- T is said to have **sensitive dependence on initial conditions** if there exists some $\delta > 0$ such that, for every $x \in X$ and $\varepsilon > 0$, there exists some $y \in X$ with $d(x, y) < \varepsilon$ such that, for some $n \geq 0$, $d(T^n x, T^n y) > \delta$.

Theorem (Banks, Brooks, Cairns, Davis and Stacey)

Let T be a hypercyclic operator. Then T has sensitive dependence on initial conditions.

Definition

An operator $T : X \to X$ is called **chaotic in the sense of Devaney** if it is hypercyclic, and $Per(T)$ is dense in X, where
$Per(T) := \{\text{periodic points of } T\} = \{x \in X \ ; \ T^n x = x \text{ for some } n\}$.

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Example: Multiples of the backward shift (Rolewicz, 1969)

If $|\lambda| > 1$, the operator $\lambda B : \ell^p \to \ell^p$, $1 \leq p < \infty$,
$(x_1, x_2, \dots) \mapsto (\lambda x_2, \lambda x_3, \dots)$ is Devaney chaotic.

Definition

Given two operators T and S defined on Banach spaces X and Y, respectively, we say T is **quasi-conjugate** to S if there exists a continuous map $\Phi : Y \to X$ with dense range such that $T \circ \Phi = \Phi \circ S$.

Usual notions of linear dynamics are preserved under quasiconjugacy: hypercyclicity and Devaney chaos.

Fractional calculus

- Studies differential operators of an arbitrary real order not only integer order.
- In contrast to ordinary derivative operators, fractional operators are non-local and incorporate memory effects into modelling.
- They capture the memory and the heredity of the process. It is an effective tool for revealing phenomena in nature because nature has memory.
- Applications in science, engineering, and mathematics: viscoelasticity, electrical circuits, chemistry, neurology, diffusion, control theory, statistics,....

Some history

- **Kutter (1956):** Mentioned by the first time differences of fractional order.
- **Diaz and Osler (1974):** A fractional difference operator as an infinite series.
- **Gray and Zhang (1988):** A fractional calculus for the discrete nabla (backward) difference operator.
- **Miller and Ross (1989):** A fractional sum via the solution of a linear difference equation.
- **Atici and Eloe (2007):** The Riemann-Liouville like fractional difference using the fractional sum of Miller and Ross.
- **Anastassiou (2010):** The Caputo like fractional difference using the fractional sum from Miller and Ross.

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1 Introduction

2 Chaos of Nonlocal Operators

3 Chaos of operators associated to Numerical Schemes

Given $a \in \mathbb{N}$, we denote $\mathbb{N}_a := \{a, a+1, a+2, ...\}$. and $s(\mathbb{N}_a)$ the vectorial space consisting of all complex-valued sequences $f : \mathbb{N}_a \to \mathbb{C}$.

Forward Euler operator

$\Delta_a : s(\mathbb{N}_a) \to s(\mathbb{N}_a)$ is defined by

$$\Delta_a f(n) := f(n+1) - f(n), \quad n \in \mathbb{N}_a.$$

For $m \in \mathbb{N}_2$, we define recursively $\Delta_a^m : s(\mathbb{N}_a) \to s(\mathbb{N}_a)$ by $\Delta_a^m := \Delta_a^{m-1} \circ \Delta_a$, and is called the m-th order forward difference operator.

We denote $\Delta \equiv \Delta_0$ and $\Delta_a^0 \equiv I_a$, with $I_a : s(\mathbb{N}_a) \to s(\mathbb{N}_a)$ the identity operator. For instance, for any $f \in s(\mathbb{N}_0)$, we have

$$\Delta^m f(n) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} f(n+j), \quad n \in \mathbb{N}_0.$$

The sequence k^α

For any $\alpha \in \mathbb{R} \setminus \{0\}$, we set

$$k^\alpha(n) = \begin{cases} \dfrac{\alpha(\alpha+1)...(\alpha+n-1)}{n!} & n \in \mathbb{N}_0, \\ 0 & \textit{otherwise.} \end{cases}$$

In case $\alpha = 0$ we define $k^0(n) = 1$ if $n = 0$ and 0 otherwise. Note that if $\alpha \in \mathbb{R} \setminus \{-1, -2, ..\}$, we have $k^\alpha(n) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)}$, $n \in \mathbb{N}_0$ where Γ is the Euler gamma function.

Definition (Atici and Eloe (2009))

Let $\alpha > 0$. For any given positive real number *a*, the α **-th fractional sum** of a function *f* is

$$\nabla_a^{-\alpha} f(n) := \sum_{j=a}^{n} k^{\alpha}(n-j) f(j).$$

The α-th fractional sum

For each $\alpha > 0$ and a sequence $f \in s(\mathbb{N}_0)$, we define the α-th fractional sum

$$\Delta^{-\alpha} f(n) := (k^{\alpha} * f)(n) = \sum_{j=0}^{n} k^{\alpha}(n-j) f(j), \quad n \in \mathbb{N}_0.$$

The fractional difference operator in the Riemann-Liouville sense

The **fractional difference operator** $\Delta^\alpha : s(\mathbb{N}_0) \to s(\mathbb{N}_0)$ of order $\alpha > 0$ (in the sense of Riemann-Liouville) is defined by

$$\Delta^\alpha f(n) := \Delta^m \circ \Delta^{-(m-\alpha)} f(n), \quad n \in \mathbb{N}_0,$$

where $m - 1 < \alpha < m,\ m := \lceil \alpha \rceil$.

$0 < \alpha < 1$

For every $n \in \mathbb{N}_0$,

$$\Delta^\alpha u(n) = \Delta(k^{1-\alpha} * u)(n) = \sum_{j=0}^{n+1} k^{1-\alpha}(n+1-j)u(j) - \sum_{j=0}^{n} k^{1-\alpha}(n-j)u(j).$$

Nabla fractional difference operator

The **nabla fractional difference operator** $\nabla^{\alpha} : s(\mathbb{N}_a) \to s(\mathbb{N}_a)$ of order $\alpha > 0$ is defined by

$$\nabla_a^{\alpha} f(t) = \Delta_a^m \circ \nabla_a^{-(m-\alpha)} f(t), \quad t \in \mathbb{N}_a,$$

where $m - 1 < \alpha < m$, $m = \lceil \alpha \rceil$.

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where $m - 1 < \alpha < m$, $m = \lceil \alpha \rceil$.

Transference principle

Let $\alpha > 0$ and $a \in \mathbb{R}$ be given. Then we have

$$\tau_a \circ \nabla_a^\alpha = \Delta^\alpha \circ \tau_a,$$

where $\tau_a : s(\mathbb{N}_a) \to s(\mathbb{N}_0)$ by $\tau_a g(n) := g(a+n), \quad n \in \mathbb{N}_0$.

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Proof:

First, we compute $\tau_a \circ \nabla_a^{-\alpha}$:

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Proof:

$$\tau_a \circ \nabla_a^\alpha f(n) \quad = \quad (\tau_a \circ (\Delta_a^m \circ \nabla_a^{-(m-\alpha)})f)(n) = (\Delta_a^m \circ \nabla_a^{-(m-\alpha)} f)(n+a)$$

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&= \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} (\nabla_a^{-(m-\alpha)} f)(n+a+j)
\end{aligned}$$

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&\overset{(1)}{=} \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} (\Delta^{-(m-\alpha)} \circ \tau_a f)(n+j)
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□

Complex analysis

Toeplitz operators

The **Hardy space** is defined as

$$H^2(\mathbb{D}) = \{f \in H(\mathbb{D}) \; ; \; \|f\| := \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^2 d\theta \right)^{1/2} < \infty \}.$$

Let $P : L^2(\mathbb{T}) \to H^2(\mathbb{D})$ be the projection. Any $g \in L^\infty(\mathbb{T})$ defines a multiplication operator M_g on $L^2(\mathbb{T})$.
The **Toeplitz operator** with symbol g is defined as

$$T_g = P \circ M_g.$$

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A bounded operator on H^2 is Toeplitz if and only if its matrix representation in the basis $\{z^n \; ; \; n \geq 0\}$ has constant diagonals.

In what follows we denote by $\widehat{\mathbb{D}} = \mathbb{C} \setminus \overline{\mathbb{D}}$.

Theorem (Baranov and Lishanskii (2016) and L.M.P (2020))

Let $\Phi(z) = \frac{\gamma}{z} + \varphi(z)$ with $\gamma \in \mathbb{C}$ and $\varphi \in A(\overline{\mathbb{D}}) = H^\infty(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ satisfying

(i) the function Φ is univalent (injective) in $\overline{\mathbb{D}} \setminus \{0\}$;

(ii) $\mathbb{D} \cap (\mathbb{C} \setminus \Phi(\mathbb{D})) \neq \emptyset$ and $\widehat{\mathbb{D}} \cap (\mathbb{C} \setminus \Phi(\mathbb{D})) \neq \emptyset$.

Then the Toeplitz operator $T_\Phi : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$ is Devaney chaotic.

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Theorem (Ganigi and Uralegaddi, 1989)

Let M_n denote the class of functions of the form $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ which are regular in $0 < |z| < 1$ and satisfy

$$\Re\left(\frac{D^{n+1}f(z)}{D^n f(z)} - 2\right) < -\frac{n}{n+1}, \quad \text{for} \quad |z| < 1,$$

where $D^n f(z) = \frac{1}{z}(z^{n+1}\frac{f(z)}{n!})^{(n)}$, $n \in \mathbb{N}_0$. Then $M_{n+1} \subset M_n$ for all $n \in \mathbb{N}_0$ and all functions in M_n are univalent.

Lemma (Matrix representation of Δ^α)

The representation of Δ^α in the canonical basis $\{e_l(j)\}_{j,l\in\mathbb{N}_0}$ of $\ell^2(\mathbb{N}_0)$ is a **Toeplitz matrix**. For $0 < \alpha < 1$ we have

$$\Delta^\alpha e_l(n) = \begin{cases} -\alpha \dfrac{k^{1-\alpha}(n-l)}{n-l+1} & \text{if } n \geq l \\ 1 & \text{if } n = l-1 \\ 0 & \text{if } n < l-1. \end{cases}$$

The symbol of Δ^α as a Toeplitz operator is $\Phi(z) = \frac{(1-z)^\alpha}{z}$.

Proof:
In general, if $f \in \ell^2(\mathbb{N}_0)$:

$$\Delta^\alpha f(n) = \Delta(k^{1-\alpha} * f)(n) = \sum_{j=0}^{n+1} k^{1-\alpha}(n+1-j)f(j) - \sum_{j=0}^{n} k^{1-\alpha}(n-j)f(j).$$

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Let $l \in \mathbb{N}_0$:

- Case $l \leq n$:

$$\begin{aligned} \Delta^\alpha e_l(n) &= \sum_{j=0}^{n+1} k^{1-\alpha}(n+1-j)e_l(j) - \sum_{j=0}^{n} k^{1-\alpha}(n-j)e_l(j) \\ &= k^{1-\alpha}(n+1-l) - k^{1-\alpha}(n-l) \\ &= k^{1-\alpha}(n-l)\left(\frac{1-\alpha+n-l}{n-l+1} - 1\right) \\ &= -\alpha \frac{k^{1-\alpha}(n-l)}{n-l+1}. \end{aligned}$$

Proof:

- Case $l = n + 1$:

$$\begin{aligned}\Delta^{\alpha} e_l(n) &= \sum_{j=0}^{n+1} k^{1-\alpha}(n+1-j)e_l(j) - \sum_{j=0}^{n} k^{1-\alpha}(n-j)e_l(j) \\ &= k^{1-\alpha}(0) = 1.\end{aligned}$$

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- Case $l > n + 1$:

$$\begin{aligned} \Delta^{\alpha} e_l(n) &= \sum_{j=0}^{n+1} k^{1-\alpha}(n+1-j)e_l(j) - \sum_{j=0}^{n} k^{1-\alpha}(n-j)e_l(j) \\ &= 0. \end{aligned}$$

Proof:
To sum up:

$$\begin{bmatrix} -\alpha & 1 & 0 & 0 & \dots \\ -\alpha\frac{k^{1-\alpha}(1)}{2} & -\alpha & 1 & 0 & \dots \\ -\alpha\frac{k^{1-\alpha}(2)}{3} & -\alpha\frac{k^{1-\alpha}(1)}{2} & -\alpha & 1 & \dots \\ -\alpha\frac{k^{1-\alpha}(3)}{4} & -\alpha\frac{k^{1-\alpha}(2)}{3} & -\alpha\frac{k^{1-\alpha}(1)}{2} & -\alpha & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Consequently the symbol is

$$\begin{aligned} \Phi(z) &= \frac{1}{z} - \alpha - \alpha\frac{k^{1-\alpha}(1)}{2}z - \alpha\frac{k^{1-\alpha}(2)}{3} - \dots \\ &= \frac{1}{z}\left(1 - \alpha\sum_{j=0}^{\infty}\frac{k^{1-\alpha}(j)}{j+1}z^{j+1}\right) \end{aligned}$$

Proof:

$$\begin{aligned}\Phi(z) &= \frac{1}{z}\left(1-\alpha\int\sum_{j=0}^{\infty}k^{1-\alpha}(j)z^j\,dz\right)\\ &= \frac{1}{z}\left(1-\alpha\int\frac{1}{(1-z)^{1-\alpha}}\,dz\right)\\ &= \frac{1}{z}\left(1+(1-z)^{\alpha}+C\right)\\ &= \frac{(1-z)^{\alpha}}{z}.\end{aligned}$$

□

Theorem

For any $0 < \alpha \leq 1$, the operator Δ^{α} defines a Devaney chaotic Toeplitz operator on $\ell^2(\mathbb{N}_0)$ with symbol $\Phi(z) = \frac{(1-z)^{\alpha}}{z}$.

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Proof:

1 For $0 < \alpha < 1$ and $u \in \ell^2(\mathbb{N}_0)$:

$$\Delta^\alpha u = \Delta(k^{1-\alpha} * u) = \Delta k^{1-\alpha} * u + \tau_1 u,$$

where τ_1 denotes the translation operator. Using Young's convolution inequality:

$$\|\Delta^\alpha u\|_2 \leq \|\Delta k^{1-\alpha} * u\|_2 + \|u\|_2 \leq \|\Delta k^{1-\alpha}\|_1 \|u\|_2 + \|u\|_2,$$

where $\Delta k^{1-\alpha}(n) \sim \frac{C}{n^{\alpha+1}}$.

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where $D^n f(z) = \frac{1}{z}(z^{n+1}\frac{f(z)}{n!})^{(n)}$, $n \in \mathbb{N}_0$. Then $M_{n+1} \subset M_n$ for all $n \in \mathbb{N}_0$ and all functions in M_n are univalent.

Theorem (Ganigi and Uralegaddi, 1989)

Let M_n denote the class of functions of the form $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ which are regular in $0 < |z| < 1$ and satisfy

$$\Re\left(\frac{D^{n+1}f(z)}{D^n f(z)} - 2\right) < -\frac{n}{n+1}, \quad \text{for } |z| < 1,$$

where $D^n f(z) = \frac{1}{z}(z^{n+1}\frac{f(z)}{n!})^{(n)}$, $n \in \mathbb{N}_0$. Then $M_{n+1} \subset M_n$ for all $n \in \mathbb{N}_0$ and all functions in M_n are univalent.

2 Using the criterion by Ganigi and Uralegaddi for univalence: given $z = a + ib$ with $z \in \mathbb{D}$:

$$\begin{aligned} \Re\left(\frac{D^1\Phi(z)}{\Phi(z)} - 2\right) &= \Re\left(-1 - \frac{\alpha z}{1-z}\right) \\ &= \frac{(1-a)(-1+a(1-\alpha)) - 2b^2}{(1-a)^2 + b^2} < 0. \end{aligned} \tag{2}$$

Theorem (Baranov and Lishanskii (2016) and L.M.P (2020))

Let $\Phi(z) = \frac{\gamma}{z} + \varphi(z)$ with $\gamma \in \mathbb{C}$ and $\varphi \in A(\overline{\mathbb{D}}) = H^\infty(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ satisfying

(i) the function Φ is univalent (injective) in $\overline{\mathbb{D}} \backslash \{0\}$;

(ii) $\mathbb{D} \cap (\mathbb{C} \backslash \Phi(\mathbb{D})) \neq \emptyset$ and $\widehat{\mathbb{D}} \cap (\mathbb{C} \backslash \Phi(\mathbb{D})) \neq \emptyset$.

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3 Show that $[-2^\alpha, 0] \subset \mathbb{C} \backslash \Phi(\mathbb{D})$. Taking the parametrization $z = e^{it}$,

$$\begin{aligned} \Phi(e^{it}) &= \frac{(1-e^{it})^\alpha}{e^{it}} \\ &= 2^\alpha \sin(t/2)^\alpha e^{i(-\alpha\pi/2 + (\alpha/2-1)t)}. \end{aligned} \tag{3}$$

Taking $t = 0$ and $t = \pi$ in the parametrization, we see that $\{-2^\alpha, 0\} \subset \mathbb{C} \backslash \Phi(\mathbb{D})$.

□

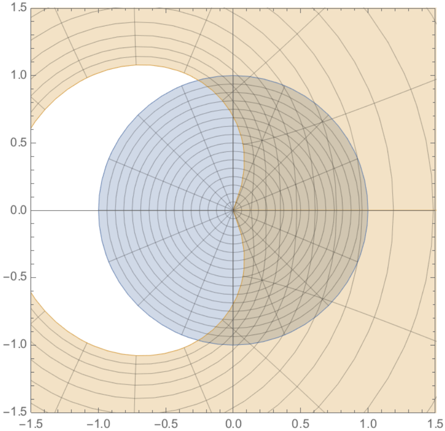

Figure: In orange $\Phi(\mathbb{D})$, $\alpha = \frac{3}{4}$.

Corollary

For any $0 < \alpha \leq 1$ and $a > 0$, the nabla fractional difference operator ∇_a^α is chaotic in $\ell^2(\mathbb{N}_a)$.

Proof:

1. Transference principle:

$$\tau_a \circ \nabla_a^\alpha = \Delta^\alpha \circ \tau_a.$$

2. Devaney chaos is preserved under quasi-conjugacy.

□

1 Introduction

2 Chaos of Nonlocal Operators

3 Chaos of operators associated to Numerical Schemes

Dynamics of operators associated to numerical schemes

We consider the fractional evolution equation for $0 < \alpha < 1$

$$\partial_t^\alpha u(t) = Au(t) + f(t), \quad t > 0,$$

with initial conditions $u(0) = 0$ and ∂_t^α denotes the Riemann-Liouville fractional derivative:

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t (t-s)^{n-\alpha-1} u(s)ds, \quad n-1 < \alpha < n, n \in \mathbb{N}.$$

We study chaos for relevant nonlocal difference operators arising in the study of time-stepping schemes for fractional operators.

Time-stepping schemes for fractional operators

They are defined by a convolution operator $\partial_b^\alpha : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$:

$$\partial_b^\alpha u(n) := (b * u)(n) = \sum_{j=0}^{n} b(n-j)u(j), \quad n \in \mathbb{N}_0, b \in \ell^1(\mathbb{N}_0).$$

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Each scheme is uniquely determined by the generating series, called the Gelfand transform of b,

$$\delta(\xi) := \sum_{n=0}^{\infty} b(n)\xi^n, \quad \xi \in \mathbb{T},$$

where $\delta(\xi)$ represents the symbol of the scheme.

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where $\delta(\xi)$ represents the symbol of the scheme.
The adjoint operator of ∂_b^α in $\ell^2(\mathbb{N}_0)$, i.e., $\langle (\partial_b^\alpha)^* u, v\rangle = \langle u, \partial_b^\alpha v\rangle$:

$$(\partial_b^\alpha)^* u(n) = F_b u(n) = \sum_{j=0}^{\infty} b(j) B^j u(n), \quad n \in \mathbb{N}_0.$$

Theorem

Let $b \in \ell^1(\mathbb{N}_0)$ be given and $F_b : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$ given by

$$F_b u(n) = \sum_{j=0}^{\infty} b(j) B^j u(n), \quad n \in \mathbb{N}_0,$$

where B denotes the backward shift operator. Then F_b defines a bounded operator on $\ell^2(\mathbb{N}_0)$ and the following assertions are equivalent

(i) F_b is chaotic;

(ii) $\delta(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$.

Corollary

The forward Euler operator, $\Delta u(n) = u(n+1) - u(n)$, is chaotic.

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The forward Euler operator, $\Delta u(n) = u(n+1) - u(n)$, is chaotic.

Proof:

$\Delta = B - I \Rightarrow \delta(z) = z - 1 \Rightarrow \delta(0) = -1 \in \mathbb{T}$. □

The Weil fractional difference operator

For the fractional backward Euler scheme:
The sequence kernel $b_\tau(n) = \tau^{-\alpha} k^{-\alpha}(n)$ defines the scheme and we can consider the nonlocal operator:

$$\partial_k^\alpha u(n) = \sum_{j=0}^{n} \tau^{-\alpha} k^{-\alpha}(n-j) u(j),$$

where $\tau > 0$ denotes the step size of the scheme.
The symbol is:

$$\delta(\xi) = \tau^{-\alpha}(1-\xi)^\alpha.$$

It is remarkable that $(\partial_k^\alpha)^* = W_\tau^\alpha$ corresponds to the Weil fractional difference operator or order $\alpha > 0$.

Theorem

For any $\alpha > 0$, the Weil difference operator W_τ^α is chaotic on $\ell^2(\mathbb{N}_0)$ if and only if $0 < \tau < 2$.

Proof:
Note that $w \in \delta_\tau(\mathbb{D}) \cap \mathbb{T}$ if and only if

$$w = \tau^{-\alpha}(1 - z)^\alpha, \qquad \text{where } |w| = 1, \ |z| < 1.$$

Then, $|1 - \tau w^{1/\alpha}| = |z|$ shows that the complex number $\tau w^{1/\alpha}$ must belong to the disk of center 1 and radius 1.

Consequently, $0 < \tau < 2$ iff $\delta_\tau(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$. □

Theorem

Let $0 < \alpha < 1$. The operator F_b, which is the dual of the operator that defines the fractional second order backward difference scheme with step size τ, is chaotic on $\ell^2(\mathbb{N}_0)$ if and only if $0 < \tau < 4$.

The symbol for the fractional second order backward difference scheme is:

$$\delta(\xi) = \tau^{-\alpha}(\frac{3}{2} - 2\xi + \frac{1}{2}\xi^2)^{\alpha}.$$

Theorem

Let $0 < \alpha < 1$. The operator F_b, which is the dual of the operator that defines the fractional Crank-Nicholson scheme with step size τ, is chaotic on $\ell^2(\mathbb{N}_0)$ if and only if $0 < \tau < \dfrac{2}{(1-\alpha)^{1/\alpha}}$.

The symbol for the Crank-Nicholson scheme is:

$$\delta(\xi) = \tau^{-\alpha}\frac{(1-\xi)^{\alpha}}{1 - \frac{\alpha}{2} + \frac{\alpha}{2}\xi}.$$

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