A note on Hahn–Banach extensions: uniqueness and renormings

A joint work with Antonio José Guirao and Vicente Montesinos

Christian Cobollo

I Workshop de la Red de Análisis Funcional y Aplicaciones

$$(X, \|\cdot\|)$$

Motivation

Renorming

$$(X, \|\cdot\|)$$

$$\rightsquigarrow$$

Properties

$$(X, \|\cdot\|)$$

$$\rightsquigarrow$$

Properties

$\swarrow$

Topological
(Isomorphic)

Motivation

Renorming

$$(X, \|\cdot\|)$$

$$\rightsquigarrow$$

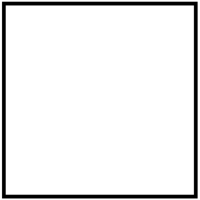
Properties

Topological (Isomorphic) | Geometrical (Isometric)

Motivation
Renorming

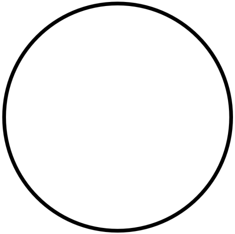
$$X = \mathbb{R}^2$$

$\|\cdot\|_\infty$



$S_{\|\cdot\|_\infty}$

$\|\cdot\|_2$



$S_{\|\cdot\|_2}$

$$(X, \|\cdot\|)$$

$$\rightsquigarrow$$

Properties

Topological (Isomorphic)

Geometrical (Isometric)

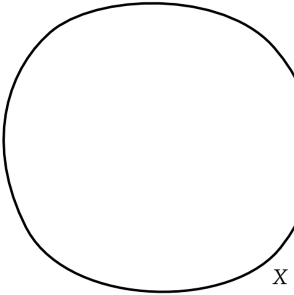
Motivation

Oja–Viil–Werner

E. Oja, T. Viil, and D. Werner, *Totally smooth renormings* (2019)

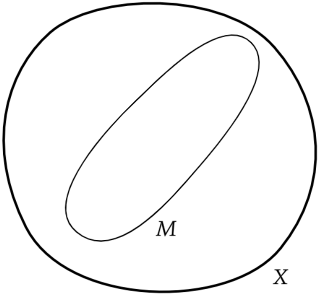
Motivation

Unique Extension Properties



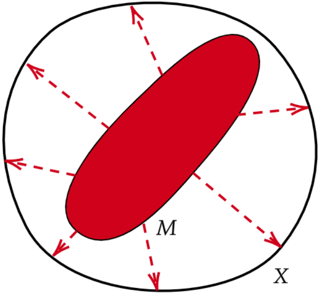
Motivation

Unique Extension Properties



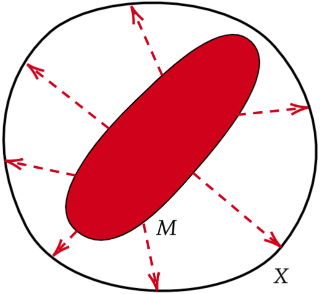
Motivation

Unique Extension Properties



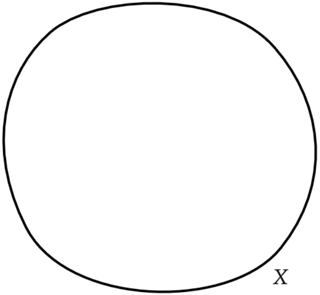
Definition (Phelps, 1960)

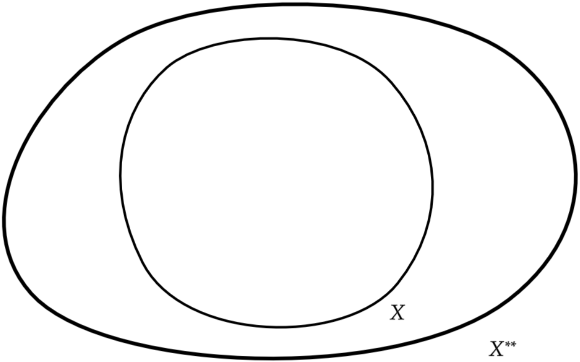
$M \hookrightarrow X$ has **property U in X** if: every $f^* \in M^*$ has unique norm-preserving extension to X.



Motivation

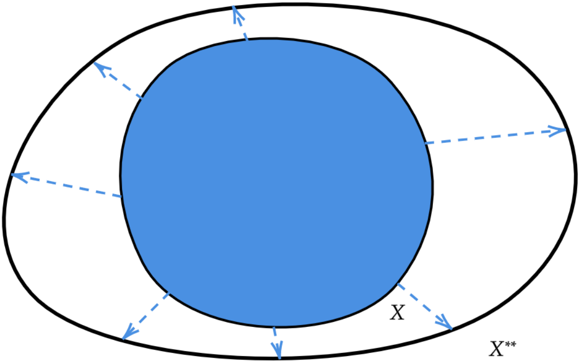
Unique Extension Properties




X
X**

Motivation

Unique Extension Properties

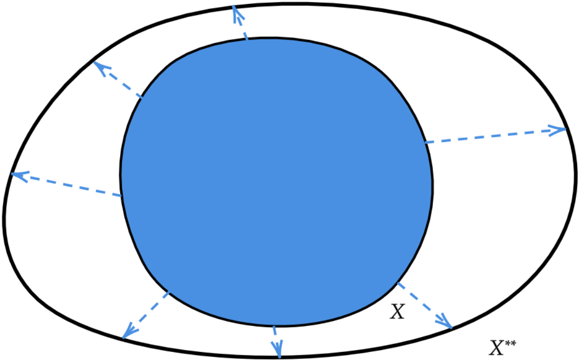


Motivation

Unique Extension Properties

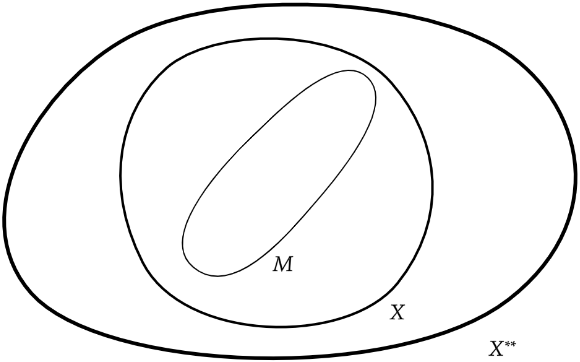
Definition (Sullivan, 1977)

X is **HBS** if: X has property U in X^{**}.



Motivation

Unique Extension Properties

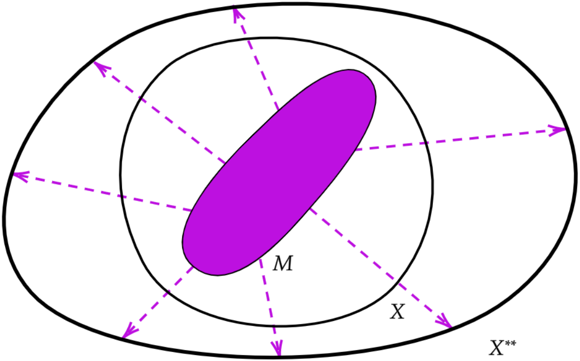


Motivation

Unique Extension Properties

Definition

X is **TS** if: every $M \hookrightarrow X$ has property U in X^{**}.



Motivation

The Problem

Question

HBS + **?** $\implies$ renormable TS.

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Theorem (Sullivan, 1977)

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Theorem (C.C., A. J. Guirao, V. Montesinos)

HBS $\implies$ renormable TS...

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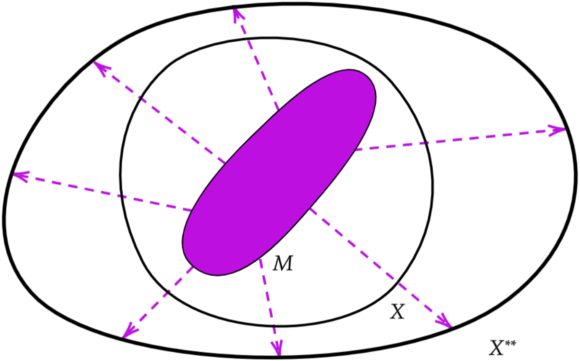
Theorem (C.C., A. J. Guirao, V. Montesinos)

HBS $\implies$ renormable TS... and even more.

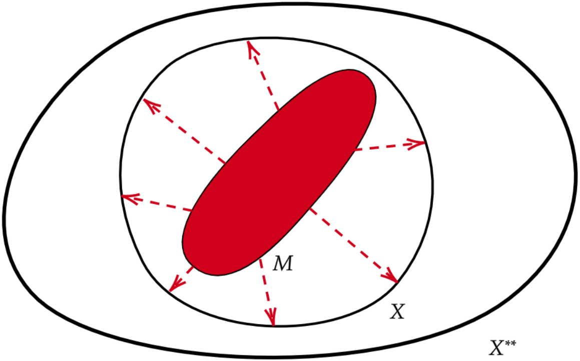
On unique extensions
TS decomposition

TS



On unique extensions
TS decomposition

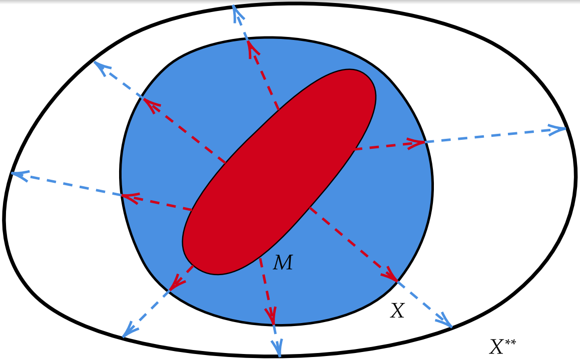
TS = $\forall M \hookrightarrow X$ has U in X



On unique extensions

TS decomposition

TS = $\forall\ M \hookrightarrow X$ has U in X + HBS



TS = $\forall\, M \hookrightarrow X$ has U in X + HBS

Theorem (Taylor–Foguel, 1958)

$\forall\, M \hookrightarrow X$ has U in $X \iff (X^*, \|\cdot\|^*)$ is rotund (X has R*).

On unique extensions

TS decomposition

TS = R* + HBS

Theorem (Taylor–Foguel, 1958)

$\forall\, M \hookrightarrow X$ has U in $X \iff (X^*, \|\cdot\|^*)$ is rotund (X has R*).

Proposition (Godefroy, 1981)

$(X, \|\cdot\|)$ is HBS $\iff$ $(X^*, \|\cdot\|^*)$ has w^*-w-KK

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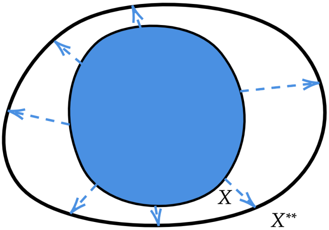

$$\iff \quad w^* = w \quad in \quad S_{\|\cdot\|^*}$$

Proposition (Godefroy, 1981)

$(X, \|\cdot\|)$ is HBS $\iff$ $(X^*, \|\cdot\|^*)$ has w^*-w-KK

Definition

Let $\tau_1 \subset \tau_2 \subset \|\cdot\|$ and $A \subset X$ a cone. We say $\|\cdot\|$ is **τ_1-τ_2-KK w.r.t. A** if $\tau_1 = \tau_2$ when restricted to $A \cap S_X$.

Proposition (Godefroy, 1981)

$$(X, \|\cdot\|) \text{ is HBS } \iff (X^*, \|\cdot\|^*) \text{ has } w^*\text{-}w\text{-KK}$$

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Proposition (C.C., A. J. Guirao, V. Montesinos)

Let $\|\cdot\|$ be τ_1-τ_2-KK w.r.t. A. If $\|\cdot\|$ is τ_2-lsc and $\overline{A \cap B_X}^{\|\cdot\|} = B_X$, then $\|\cdot\|$ is τ_1-lsc.

On unique extensions

HBS and topologies

Proposition (Godefroy, 1981)

$(X, \|\cdot\|)$ is HBS $\iff$ $(X^*, \|\cdot\|^*)$ has w^*-w-KK

Theorem (C.C., A. J. Guirao, V. Montesinos)

X admits HBS $\iff$ X^* admits w^*-w-KK

Troyanski (1985) : X admits LUR $\leftrightarrows$ X admits R + X admits w-$\|\cdot\|$-KK

A landmark in renorming theory

Raja's Theorem

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A landmark in renorming theory

Raja's Theorem

Troyanski (1985) : X admits LUR $\leftrightarrows$ X admits R + X admits w-$\|\cdot\|$-KK

Raja (2002) : X^* admits dual LUR $\leftrightarrows$ X^* admits dual R + X^* admits w^*-w-KK

Theorem (M. Raja, 2002)

X^* has a LUR norm $\iff$ X^* has w^*-w-KK norm.

TS decomposition + Godefroy + Raja + w^*-w-KK norms are dual

More than a TS renorming

Putting the pieces together

TS decomposition + Godefroy + Raja + w^*-w-KK norms are dual

Theorem (C.C., A. J. Guirao, V. Montesinos)

Let $(X, \|\cdot\|)$ be a Banach space. TFAE:

(i) X admits HBS norm.

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$$\text{TS} \;=\; \text{R}^* + \text{HBS}$$

More than a TS renorming

A key example

Proposition

Let K be a Ciesielski–Pol compact set. Then:
1) $K^{(3)} = \emptyset$.
2) There is no bounded linear one-to-one $T : C(K) \to c_0(\Gamma)$.

More than a TS renorming

A key example

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More than a TS renorming

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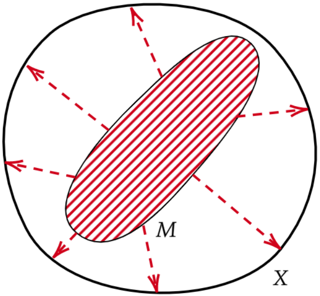
1) $\implies C(K)^*$ has dual LUR renorming.
2) $\implies C(K)$ is not WCG.

About a Sullivan extension

Properties wU and wHBS



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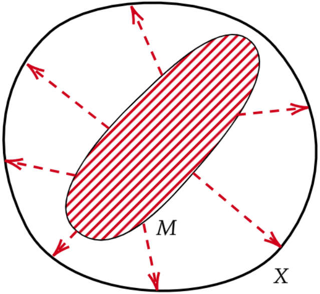

About a Sullivan extension

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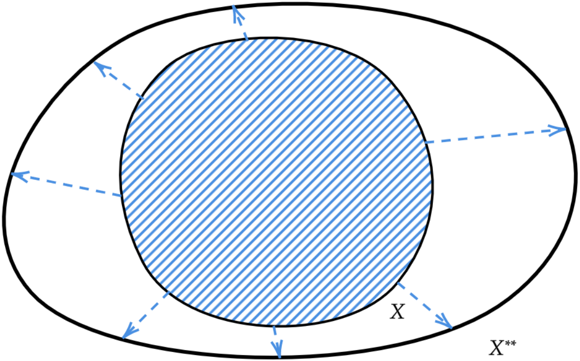

About a Sullivan extension

Properties wU and wHBS

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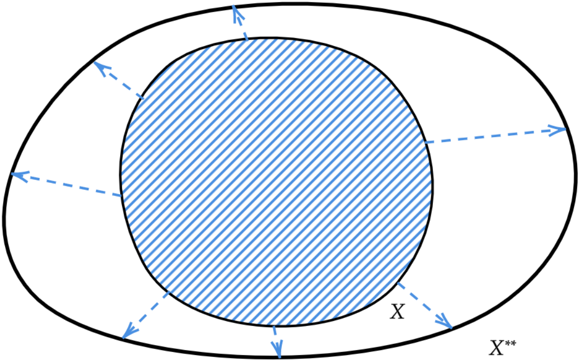
X has **wHBS** if: X has property wU in X^{**}.



About a Sullivan extension

On wHBS

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TS = R* + **HBS**

About a Sullivan extension

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TS = R* + **HBS**

VS = G + **wHBS**

Theorem (C.C., A. J. Guirao, V. Montesinos)

Let $(X, \|\cdot\|)$ be a Banach space. TFAE:

(i) X admits HBS norm.
(ii) X^* admits w^*-w-KK norm.
(iii) X^* admits dual LUR norm.
(iv) X admits TS norm.

Theorem?

Let $(X, \|\cdot\|)$ be a Banach space. TFAE:

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(iii) X^* admits **dual R norm.**
(iv) X admits VS norm.

About a Sullivan extension

On wHBS

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NO. Example: $C([0, \mu])$ with uncountable μ.

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HBS$>>>$wHBS

Very Smoothness

Gâteaux $<$ Very Smooth $<$ Fréchet

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Gâteaux: $\partial\|\cdot\| : X \to X^*$ is $\|\cdot\|$-w^*-continuous.

Fréchet: $\partial\|\cdot\| : X \to X^*$ is $\|\cdot\|$-$\|\cdot\|$-continuous.

Very Smoothness

Gâteaux $<$ Very Smooth $<$ Fréchet

Gâteaux: $\partial\|\cdot\| : X \to X^*$ is $\|\cdot\|$-w^*-continuous.

Very Smooth: $\partial\|\cdot\| : X \to X^*$ is $\|\cdot\|$-w continuous (Diestel, 1975).

Fréchet: $\partial\|\cdot\| : X \to X^*$ is $\|\cdot\|$-$\|\cdot\|$-continuous.

Very Smoothness

$(X^*, \|\cdot\|^*)$		$(X, \|\cdot\|)$
R	$\Longrightarrow$	Gâteaux diff.
??	$\Longrightarrow$	Very Smoothness
LUR	$\Longrightarrow$	Fréchet diff.

Very Smoothness

$(X^*, \|\cdot\|^*)$		$(X, \|\cdot\|)$
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$(X^*, \|\cdot\|^*)$		$(X, \|\cdot\|)$
R	$\Longrightarrow$	Gâteaux diff.
VR	$\Longrightarrow$	Very Smoothness
LUR	$\Longrightarrow$	Fréchet diff.

Definition

$(X, \|\cdot\|)$ VR $\iff$ $(X^*, \|\cdot\|^*)$ Gâteaux in NA$(X)\setminus\{0\}$

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Theorem (Moltó, Orihuela, Troyanski, Valdivia, 2009)

$(X, \|\cdot\|)$ Fréchet and $(X^*, \|\cdot\|^*)$ Gâteaux. Then X admits LUR norm.

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$(X, \|\cdot\|)$ Fréchet and VR. Then X admits LUR norm.

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Example: $c_0(\Gamma)$ with uncountable Γ.

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Open Problem

$(X, \|\cdot\|)$ Fréchet $\implies$ X admits LUR norm?

Fréchet and Morris norm

Question (Orihuela)

F+R $\implies$ F+VR?

Fréchet and Morris norm

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F+R $\implies$ F+VR?

Definition (Guirao, Montesinos, Zizler, 2014)

$(X, \|\cdot\|)$ is Morris if is R but no element of S_X is extreme point of $S_{X^{**}}$.

Fréchet and Morris norm

Question (Orihuela)

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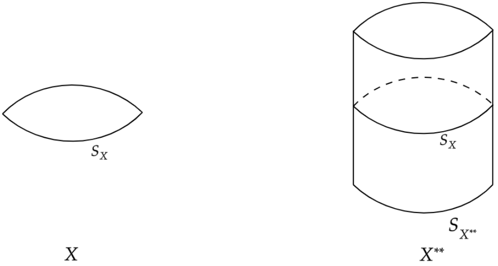

Fréchet and Morris norm

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Sketch (c_0 case):

Fréchet and Morris norm

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Define in $c_0^* = \ell^1$ the norm $|\cdot| := \|\cdot\|_1 + \|\cdot\|_2$ $\quad$ (w^*-w-KK + R).

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Sketch (c_0 case):
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Take $|\cdot|_*$ in c_0 (F).

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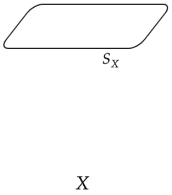
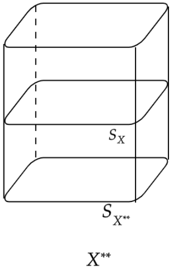


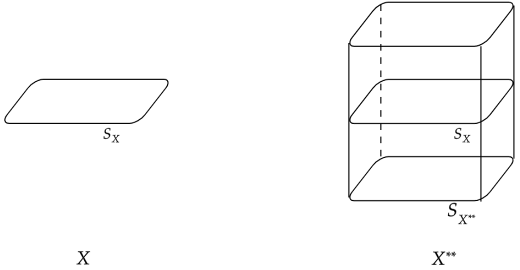


Fréchet and Morris norm

Theorem (C.C., A. J. Guirao, V. Montesinos)

Every $c_0(\Gamma)$ with infinite Γ admits an equivalent F+M norm.

Sketch (c_0 case):
Define in $c_0^* = \ell^1$ the norm $|\cdot| := \|\cdot\|_1 + \|\cdot\|_2$ (w^*-w-KK + R).
Take $|\cdot|_*$ in c_0 (F).
Build an appropiate $T : c_0 \to \ell^2$. Define $|||x||| := |x|_* + \|Tx\|_2$.



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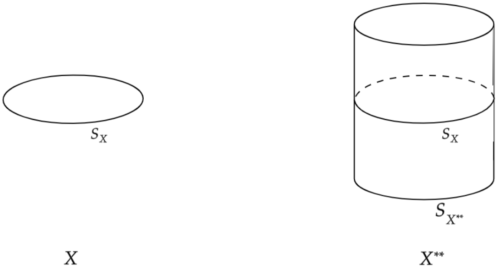

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but VR $\implies$ all extreme points are preserved!

So, $(c_0(\Gamma), |||\cdot|||)$ is F+R but no F+VR.

Open Problem

$(X, \|\cdot\|)$ has F+R norm $\implies$ admits LUR (or F+VR)?

Some References

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The End

Thanks For Your Attention!