Non-commutative L^p-spaces and some orthogonality related problems

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Contents

Introduction

- $\mathbb{F} = \mathbb{C}$.
- Banach algebra: Banach space with product
  $B(\mathcal{H}) =$ continuous operators on a Hilbert space $\mathcal{H}$
- C^*-algebra: Banach algebra with involution $(*)$
  Example: $\mathbb{C}$, $z^* = \overline{z}$
  Example: $C(K)$, $(f^*)(x) = \overline{f(x)}$.
  Example: $B(\mathcal{H})$, $T^* =$ adjoint operator of T
- If $\mathcal{A}$ is a C^*-algebra, there is a Hilbert space $\mathcal{H}$ such that
  $\mathcal{A} \subset B(\mathcal{H})$ *as a* C^**-algebra*.

A von Neumann algebra is a C^*-algebra $\mathcal{M} \subset B(\mathcal{H})$ that is WOT-closed and has unit.

Examples: $L^\infty(\mu)$, $\mathbb{M}_n(\mathbb{C})$, $B(\mathcal{H})$ ...

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- Self-adjoint elements: $\mathcal{M}_{sa} = \{x \in \mathcal{M} : x = x^*\}$ ($\mathbb{C}_{sa} = \mathbb{R}$)
- Positive elements: $\mathcal{M}_+ = \{x^2 : x \in \mathcal{M}_{sa}\}$ ($\mathbb{C}_+ = \mathbb{R}_0^+$)

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- Continuous functional calculus:
  - If $f \in C_0(\mathbb{C})$, $\exists f(x)$ for $x \in \mathcal{M}$ if $x^*x = xx^*$.
  - If $f \in C_0(\mathbb{R})$, $\exists f(x)$ for $x \in \mathcal{M}_{sa}$.
  - If $f \in C_0(\mathbb{R}_0^+)$, $\exists f(x)$ for $x \in \mathcal{M}_+$.

  (and it works fine: the map $f \mapsto f(x)$ is a $*$-homomorphism)
  Example: if $x \in \mathcal{M}_+$, $\exists x^{1/n} \in \mathcal{M}_+$. If $x \in \mathcal{M}$, there are $x_{Re}^+, x_{Re}^-, x_{Im}^+, x_{Im}^- \in \mathcal{M}_+$ such that $x = x_{Re}^+ - x_{Re}^- + ix_{Im}^+ - ix_{Im}^-$.

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  Example: if $x \in \mathcal{M}_+$, $\exists x^{1/n} \in \mathcal{M}_+$. If $x \in \mathcal{M}$, there are $x_{Re}^+, x_{Re}^-, x_{Im}^+, x_{Im}^- \in \mathcal{M}_+$ such that $x = x_{Re}^+ - x_{Re}^- + ix_{Im}^+ - ix_{Im}^-$.
- Projections: $\mathrm{Proj}(\mathcal{M}) = \{e \in \mathcal{M} : e^2 = e = e^*\}$.
- Every self-adjoint element is limit of self-adjoint linear combinations of projections.

Definition

Let $\mathcal{M}$ be a von Neumann algebra. A *trace* on $\mathcal{M}$ is a map $\tau : \mathcal{M}_+ \to [0, \infty]$ satisfying:

- $\tau(x + y) = \tau(x) + \tau(y)$ for all $x, y \in \mathcal{M}_+$.
- $\tau(\lambda x) = \lambda \tau(x)$ for all $x \in \mathcal{M}_+$ and $\lambda \geq 0$.
- $\tau(xx^*) = \tau(x^*x)$ for all $x \in \mathcal{M}$.

1. τ is *normal* is $\sup_\alpha \tau(x_\alpha) = \tau(\sup_\alpha x_\alpha)$ for any bounded increasing net (x_α) in $\mathcal{M}_+$.
2. τ is *semifinite* if for any non-zero $x \in \mathcal{M}_+$ there is a non-zero $y \in \mathcal{M}_+$ such that $y \leq x$ and $\tau(y) < \infty$.
3. τ is *faithful* if $\tau(x) = 0$ implies $x = 0$.

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This means that we can integrate operators of $\mathcal{M}$.

Example

- Let $\mathcal{M} = L^\infty(\mathbb{R})$.
  $L^\infty(\mathbb{R})_+ = \{f \in L^\infty(\mathbb{R}) : f(x) > 0 \text{ almost everywhere}\}$.

$$\tau(f) = \int_{\mathbb{R}} f(x)dx, \quad (f \in L^\infty(\mathbb{R})_+)$$

- $S = \{\text{bounded functions with compact support}\}$,

$$S = \operatorname{lin}\left\{f \in L^\infty(\mathbb{R})_+ : \int_{\mathbb{R}} \operatorname{supp} f(x)dx < \infty\right\}.$$

- If $0 < p < \infty$,

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p \, dx\right)^{1/p}, \quad (f \in S).$$

- $L^p(\mathbb{R})$ is the completion of $(S, \|\cdot\|_p)$.

Tracial L^p-spaces

- Let $\mathcal{M}$ be a semifinite von Neumann algebra with normal semifinite faithful trace τ.
- Let $S(\mathcal{M},\tau) = \text{lin}\{x \in \mathcal{M}_+ : \tau(\text{supp}(x)) < \infty\}$.
- If $0 < p < \infty$ we define

$$\|x\|_p = \big(\tau(|x|^p)\big)^{1/p}, \quad (x \in S(\mathcal{M},\tau)).$$

- $L^p(\mathcal{M},\tau)$ is the completion of $(S(\mathcal{M},\tau), \|\cdot\|_p)$.
- We set $L^\infty(\mathcal{M},\tau) = (\mathcal{M}, \|\cdot\|)$ and $L^0(\mathcal{M},\tau) =$ measurable closed densely defined operators affiliated to $\mathcal{M}$.
- $L^p(\mathcal{M},\tau) = \{x \in L^0(\mathcal{M},\tau) : \big(\tau(|x|^p)\big)^{1/p} < \infty\}$.

What if $\mathcal{M}$ is not semifinite?

A posible definition:

- $L^\infty(\mathcal{M}) = \mathcal{M}$.
- $L^1(\mathcal{M})^* = L^\infty(\mathcal{M}) = \mathcal{M}$, so $L^1(\mathcal{M}) = \mathcal{M}_*$.
- We can define $L^p(\mathcal{M})$ $(1 < p < \infty)$ by interpolation.

Problems:

- We can't define $L^p(\mathcal{M})$ for $p < 1$.
- We lost the multiplicative structure.

Definition

Let $\mathcal{M}$ be a von Neumann algebra. A *weight* on $\mathcal{M}$ is a map $\varphi : \mathcal{M}_+ \to [0, \infty]$ satisfying:

- $\varphi(x + y) = \varphi(x) + \varphi(y)$ for all $x, y \in \mathcal{M}_+$.
- $\varphi(\lambda x) = \lambda\varphi(x)$ for all $x \in \mathcal{M}_+$ and $\lambda \geq 0$.

1. φ is *normal* is $\sup_\alpha \varphi(x_\alpha) = \varphi(\sup_\alpha x_\alpha)$ for any bounded increasing net (x_α) in $\mathcal{M}_+$.
2. φ is *semifinite* if for any non-zero $x \in \mathcal{M}_+$ there is a non-zero $y \in \mathcal{M}_+$ such that $y \leq x$ and $\varphi(y) < \infty$.
3. φ is *faithful* if $\varphi(x) = 0$ implies $x = 0$.

Every von Neumann algebra admits a normal semifinite faithful weight.

Haagerup's construction

- Let $\mathcal{M}$ be a von Neumann algebra.
- Let $\mathcal{R}$ be the crossed product of $\mathcal{M}$ by the modular automorphism group $\{\sigma_t\}$ associated with a normal semifinite faithful weight φ on $\mathcal{M}$.

Haagerup's construction

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- Let $\mathcal{R}$ be the crossed product of $\mathcal{M}$ by the modular automorphism group $\{\sigma_t\}$ associated with a normal semifinite faithful weight φ on $\mathcal{M}$.
- $\mathcal{R}$ is a semifinite von Neumann algebra and, for each $s \in \mathbb{R}$, there is a linear map $\theta_s : \mathcal{R} \to \mathcal{R}$ such that

$$\tau \circ \theta_s = e^{-s}\tau \quad (s \in \mathbb{R}).$$

- $L^p(\mathcal{M}) = \{x \in L^0(\mathcal{R}, \tau) : \theta_s(x) = e^{-s/p}x \ (s \in \mathbb{R})\} \ (p < \infty)$.
- $L^\infty(\mathcal{M}) = \{x \in L^0(\mathcal{R}, \tau) : \theta_s(x) = x \ (s \in \mathbb{R})\} = \mathcal{M}$.
- $L^1(\mathcal{M}) = \mathcal{M}_* \subset \mathcal{M}^*$, so we can define

$$\mathrm{Tr}(x) = x(1) \quad (x \in L^1(\mathcal{M})).$$

- For $0 < p < \infty$, we define $\|\cdot\|_p \colon L^p(\mathcal{M}) \to \mathbb{R}$ by

$$\|x\|_p = \mathrm{Tr}(|x|^p)^{1/p} \quad (x \in L^p(\mathcal{M})).$$

Summary

Tracial L^p-spaces

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- $\mathcal{M}$ must be semifinite
- τ is a n.s.f. trace (integral)
- $\mathcal{S}(\mathcal{M},\tau) \subset \mathcal{M}$
- $L^p(\mathcal{M},\tau) = \overline{S(\mathcal{M},\tau)}^{\|\cdot\|_p}$

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Tracial L^p-spaces

- $\mathcal{M}$ must be semifinite
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Haagerup L^p-spaces

- Any $\mathcal{M}$
- $\mathcal{R}$ is semifinite
- $L^p(\mathcal{M}) \subset L^0(\mathcal{R},\tau)$
- $L^p(\mathcal{M}) = \dots$

Let $\mathcal{M}$ be a von Neumann algebra and let $0 < p \leq \infty$. Let L^p be the non-commutative L^p-space associated with $\mathcal{M}$ (tracial or Haagerup).

Properties

- L^p is a Banach space if $1 \leq p \leq \infty$.
- L^p is a quasi-Banach space if $0 < p < 1$.
- Hölder's inequality: if $0 < p, q, r \leq \infty$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then

$$x \in L^p, \ y \in L^q \implies xy \in L^r$$

  and $\|xy\|_r \leq \|x\|_p \|y\|_q$.
- $\left(L^p\right)^* = L^{p^*}$ if $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p^*} = 1$.
- $\left(L^1\right)^* = L^\infty = \mathcal{M}$.
- L^p is a Banach (or quasi-Banach) $\mathcal{M}$-bimodule.

Problem 1: Orthogonally additive polynomials

Definition

Let X and Y be linear spaces. A map $P: X \to Y$ is said to be an *m-homogeneous polynomial* if there exists an m-linear map $\varphi : X^m \to Y$ such that

$$P(x) = \varphi(x, \ldots, x) \quad (x \in X).$$

Example

Let X be a linear space that has an additional structure that allow us to multiply its elements (algebra, function space, etc).
If $X_{(m)}$ is a linear space containing the set $\{x^m : x \in X\}$ and $\Phi : X_{(m)} \to Y$ is a linear map, then we can define an m-homogeneous polynomial $P : X \to Y$ as follows:

$$P(x) = \Phi(x^m) \quad (x \in X).$$

Question

If P is a polynomial on X, then $P(x) = \Phi(x^m)$ $(x \in X)$ for some linear map Φ?

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Answer: no.

Example

If $P(x) = \Phi(x^m)$ $(x \in X)$, then P satisfies that

$$x, y \in X,\ xy = yx = 0 \implies P(x+y) = P(x) + P(y).$$

Let $P : \mathbb{M}_2 \to \mathbb{C}$, $P(A) = a_{11}a_{22}$ $(A = (a_{ij}) \in \mathbb{M}_2)$.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \implies P(A+B) \neq P(A) + P(B)$$

Let X and Y be linear spaces.

- We say that $x, y \in X$ are orthogonal if $xy = yx = 0$. In that case, we write $x \perp y$.
- A map $P\colon X \to Y$ is said to be *orthogonally additive* on a subset $\mathcal{S} \subset X$ if

$$x, y \in \mathcal{S},\ x \perp y \implies P(x+y) = P(x) + P(y).$$

- A map $P\colon X \to Y$ is said to be orthogonally additive if it is orthogonally additive on X.

Question

If P is a polynomial on X, and P is orthogonally additive on a certain subset $\mathcal{S} \subset X$, then $P(x) = \Phi(x^m)$ $(x \in X)$ for some linear map Φ?

Theorem

Let $\mathcal{M}$ be a von Neumann algebra with a normal semifinite faithful trace τ, let X be a topological linear space, and let $P\colon L^p(\mathcal{M},\tau) \to X$ be a continuous m-homogeneous polynomial with $0 < p < \infty$. If P is orthogonally additive on $S(\mathcal{M},\tau)_+$, then there exists a unique continuous linear map $\Phi\colon L^{p/m}(\mathcal{M},\tau) \to X$ such that

$$P(x) = \Phi(x^m) \quad (x \in L^p(\mathcal{M},\tau)).$$

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$$P(x) = \Phi(x^m) \quad (x \in L^p(\mathcal{M},\tau)).$$

Lemma

Let $\mathcal{M}$ be a von Neumann algebra, let X be a topological linear space, and let $P\colon \mathcal{M} \to X$ be a continuous m-homogeneous polynomial. If P is orthogonally additive on $\mathcal{M}_+$, then there exists a unique continuous linear map $\Phi\colon \mathcal{M} \to X$ such that

$$P(a) = \Phi(a^m) \quad (a \in \mathcal{M}).$$

Proof of the theorem:

- Let $e \in \mathrm{Proj}(M)$ with $\tau(e) < \infty$ and let $\mathcal{M}_e = e\mathcal{M}e$.
- $\mathcal{M}_e \subset S(\mathcal{M}, \tau)$.
- $P \mid_{\mathcal{M}_e}$ is continuous.
- There exists a unique continuous linear map $\Phi_e : \mathcal{M}_e \to X$ such that $P(x) = \Phi_e(x^m)$ $(x \in \mathcal{M}_e)$.
- For each $x \in S(\mathcal{M}, \tau)$, define $\Phi(x) = \Phi_e(x)$, where $e \in \mathrm{Proj}(\mathcal{M})$ is such that $\tau(e) < \infty$ and $x \in \mathcal{M}_e$.
- Φ is linear.
- Φ is continuous with respect to the norm $\|\cdot\|_{p/m}$.
- Φ extends to a continuous linear map from $L^{p/m}(\mathcal{M}, \tau)$ to the completion of X.
- $\Phi(L^{p/m}(\mathcal{M}, \tau)) \subset X$.

What we don't know

Does this hold for Haagerup L^p-spaces?

It makes sense, but:

- Tracial L^p: $S \subset L^p \cap L^{p/m}$, and S is dense in both.
- Haagerup L^p: $L^p \cap L^{p/m} = \{0\}$ if $m \neq 1$.

Problem 2: Reflexivity and hyperreflexivity

Definition

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces, and let $\mathcal{A}$ be a closed linear subspace of $B(\mathcal{X}, \mathcal{Y})$.

- $\mathcal{A}$ is called *reflexive* if

$$\mathcal{A} = \left\{ T \in B(\mathcal{X}, \mathcal{Y}) : T(x) \in \overline{\{S(x) : S \in \mathcal{A}\}} \ \forall x \in \mathcal{X} \right\}.$$

- $\mathcal{A}$ is called *hyperreflexive* if there exists C such that

$$\operatorname{dist}(T, \mathcal{A}) \leq C \sup_{\|x\| \leq 1} \inf \left\{ \| T(x) - S(x) \| : S \in \mathcal{A} \right\}$$

  for all $T \in B(\mathcal{X}, \mathcal{Y})$, and the optimal constant is called the *hyperreflexivity constant* of $\mathcal{A}$.

Example

- Each von Neumann algebra is reflexive.
- The algebra

$$\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\} \subset \mathbb{M}_2(\mathbb{C})$$

  is not reflexive.
- Each injective von Neumann algebra is hyperreflexive with hyperreflexivity constant less or equal than 4.
- We don't know if each von Neumann algebra is hyperreflexive.

Definition

An operator $T \in B(L^p, L^q)$ is a *right $\mathcal{M}$-module homomorphism* if

$$T(xa) = T(x)a \quad \forall x \in L^p, \ \forall a \in \mathcal{M}.$$

$\mathrm{Hom}_{\mathcal{M}}(L^p, L^q)$ is the space of right $\mathcal{M}$-module homomorphisms from L^p to L^q.

- For $T \in B(L^p, L^q)$ and $a \in \mathcal{M}$, define $aT, \ Ta \colon L^p \to L^q$ by

$$(aT)(x) = T(xa), \quad (Ta)(x) = T(x)a \quad (x \in L^p).$$

- Define $\mathrm{ad}(T) \colon \mathcal{M} \to B(L^p, L^q)$ by

$$\mathrm{ad}(T)(a) = aT - Ta \quad (a \in \mathcal{M}).$$

- This way,

$$T \in \mathrm{Hom}_{\mathcal{M}}(L^p, L^q) \iff ad(T) = 0.$$

Lemma

Let $T \in B(L^p, L^q)$.

1. *If*

$$e \in \mathrm{Proj}(\mathcal{M}) \Longrightarrow eT(1-e) = 0,$$

   then $T \in \mathrm{Hom}_{\mathcal{M}}(L^p, L^q)$.

2. *If* $p, q \geq 1$, *then*

$$\|\mathrm{ad}(T)\| \leq 8 \sup_{\|x\| \leq 1} \inf\big\{\|T(x) - \Phi(x)\| : \Phi \in \mathrm{Hom}_{\mathcal{M}}(L^p, L^q)\big\}$$

Definition

$\mathrm{Hom}_{\mathcal{M}}(L^p, L^q)$ is reflexive if

$$\mathrm{Hom}_{\mathcal{M}}(L^p, L^q) = \left\{ T \in B(L^p, L^q) : T(x) \in \overline{\{\Phi(x) : \Phi \in \mathrm{Hom}_{\mathcal{M}}(L^p, L^q)\}} \; \forall x \in L^p \right\}.$$

$\mathrm{Hom}_{\mathcal{M}}(L^p, L^q)$ is hyperreflexive if

$$\mathrm{dist}(T, \mathrm{Hom}_{\mathcal{M}}(L^p, L^q)) \leq C \sup_{\|x\| \leq 1} \inf \left\{ \| T(x) - \Phi(x) \| : \Phi \in \mathrm{Hom}_{\mathcal{M}}(L^p, L^q) \right\}.$$

Corollary

The space $\mathrm{Hom}_{\mathcal{M}}(L^p, L^q)$ *is reflexive.*

Proof:

Let $T \in B(L^p, L^q)$ such that

$$T(x) \in \overline{\{\Phi(x) : \Phi \in \mathrm{Hom}_{\mathcal{M}}(L^p, L^q)\}}, \quad (x \in L^p).$$

Let $e \in \mathrm{Proj}(\mathcal{M}), x \in L^p$. Let $(\Phi_n)_{n\in\mathbb{N}} \subset \mathrm{Hom}_{\mathcal{M}}(L^p, L^q)$ such that $\lim_n \Phi_n(xe) = T(xe)$. Then

$$(eT(1-e))(x) = T(xe)(1-e) = \lim_{n\to\infty} \Phi_n(xe)(1-e) = 0.$$

Theorem (1)

If $p = \infty$ or $q = 1$, then $\mathrm{Hom}_{\mathcal{M}}(L^p, L^q)$ is hyperreflexive and the hyperreflexivity constant is less or equal than 8.

Idea of the proof:
Let $T \in B(L^p, L^q)$.

- If $p = \infty$, $L^p = \mathcal{M}$. Take $y = T(1)$.
- If $q = 1$ and $p \neq \infty$, define $\Phi \in \left(L^p\right)^*$ by $\Phi(x) = \mathrm{Tr}(T(x))$ $(x \in L^p)$, and take $y \in L^{p^*}$ such that $\Phi(x) = \mathrm{Tr}(yx)$ $(x \in L^p)$.

$$\|T - L_y\| \leq \|\mathrm{ad}(T)\| \implies \mathrm{dist}(T, \mathrm{Hom}_{\mathcal{M}}(L^p, L^q)) \leq \|\mathrm{ad}(T)\|.$$

Theorem (2)

If $\mathcal{M}$ is injective and $p, q \geq 1$, then $\mathrm{Hom}_{\mathcal{M}}(L^p, L^q)$ is hyperreflexive and the hyperreflexivity constant is less or equal than 8.

Idea of the proof:
If $p = \infty$ or $q = 1$, we apply the previous theorem.
If $p \neq \infty$ and $q \neq 1$, then $\left(L^p\right)^* = L^{p^*}$ and $L^q = \left(L^{q^*}\right)^*$.
Define $\Phi : L^p \to L^q = \left(L^{q^*}\right)^*$ by

$$\langle y, \Phi(x)\rangle = \int_G \langle y, T(xu^*)u\rangle \, d\mu(u) \quad (x \in L^p,\, y \in L^{q^*}).$$

$\Phi \in \mathrm{Hom}_{\mathcal{M}}(L^p, L^q)$ and $\|T - \Phi\| \leq \|\mathrm{ad}(T)\|$, so
$\mathrm{dist}(T, \mathrm{Hom}_{\mathcal{M}}(L^p, L^q)) \leq \|\mathrm{ad}(T)\|$.

Theorem (3)

If $1 \leq q < p$, then $\mathrm{Hom}_{\mathcal{M}}(L^p, L^q)$ is hyperreflexive and the hyperreflexivity constant is less or equal than a constant $C_{p,q}$ that does not deppend on $\mathcal{M}$.

Idea of the proof:

Assume towards a contradiction that, for each $n \in \mathbb{N}$, there is a von Neumann algebra $\mathcal{M}_n$ and an operator $T_n \in B(L^p(\mathcal{M}_n), L^q(\mathcal{M}_n))$ such that

$$\mathrm{dist}(T_n, \mathrm{Hom}_{\mathcal{M}_n}(L^p(\mathcal{M}_n), L^q(\mathcal{M}_n))) > n \left\| \mathrm{ad}(T_n) \right\|.$$

What we don't know

What if $p \leq q$, $p \neq \infty$, $q \neq 1$ and $\mathcal{M}$ is not injective?

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