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On the Permanence of the Null Character of Maxwell Fields

Bartolomé Coll[1] and Joan Josep Ferrando[1,2,3]

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A critical review of known results about the permanence conditions for the null character of the solutions to the (vacuum) Maxwell equations, is presented. Concomitants of the electromagnetic field and the metric tensor are constructed, which give the principal directions of the field in covariant form. The known permanence conditions are generalized in order to include *all* the (local) null fields; the above concomitants allow these conditions to be explicitly formulated in terms of the electromagnetic field.

1. INTRODUCTION

One of the general problems, still open, concerning the Maxwell equations is that of the conditions under which the permanence of the null electromagnetic field may be insured: the (vacuum) Maxwell equations do not guarantee that an electromagnetic field which is null at an instant will remain null near it. The only known result about such *permanence conditions* is the Mariot–Lichnerowicz theorem,[4] which states that *among the Maxwell fields admitting an integrable principal direction, the null fields are permanent.*

In this paper we solve mainly two problems: (i) to obtain covariantly the principal directions of an electromagnetic field; this will allow us, in particular, to write the differential equations that a Maxwell field satisfies

[1] Département de Mécanique Relativiste, Université Paris VI, Tour 66. 4, Place Jussieu. 75252 Paris Cedex 05, France.

[2] Permanent address: Departament de Física Teòrica, Facultat de Físiques, Burjassot, València, Spain.

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[4] See the discussion of Section 2.

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when it admits an integrable principal direction: the above theorem will then become a permanence statement for the null initial data of this differential system. (ii) To obtain a generalization of the Mariot–Lichnerowicz result in order to include the null fields having no integrable principal directions.

The literature on this subject being rather confusing, we devoted Section 2 to explaining the nature of the permanence problem for the null Maxwell fields, the role played by the Mariot–Lichnerowicz theorem and the possible generalizations of it that one expects to obtain.

In Section 3 we solve in a covariant way the algebraic problem of finding the principal directions of an arbitrary electromagnetic field. To do this, we introduce a pair of algebraic concomitants of the electromagnetic field and the metric which project arbitrary time-like directions onto the principal directions. This section is self-consistent, and its interest exceeds the use of it that we make here.

Finally, in Section 4 we generalize the Mariot–Lichnerowicz theorem in order to include *all* the null Maxwell fields, and use the results of the above section to formulate explicitly the differential system in the electromagnetic field variables with respect to which the null fields are permanent. Under our permanence conditions, the property of admitting an integrable principal direction becomes also permanent.

The results without proof of this paper were communicated to the Spanish relativistic meeting E.R.E. 86 [1].

2. SURVEY OF THE PROBLEM

(a) A field F is said to be a *Maxwell field* if it is a solution of the vacuum Maxwell equations[5]

$$\delta F = 0, \qquad \delta * F = 0 \qquad \{M\}$$

From the evolution point of view, these equations split, with respect to a timelike direction u, into an *evolution system*[6]

$$\perp(u)\, \delta F = 0, \qquad \perp(u)\, \delta * F = 0 \qquad \{E\}$$

and a *constraint system*

$$i(u)\, \delta F = 0, \qquad i(u)\, \delta * F = 0 \qquad \{C\}$$

[5] δ and $*$ are, respectively, the divergence and dual operators.

[6] $\perp(u)$ is the spatial projector with respect to u, and $i(u)$ is the u projection (interior product).

The Maxwell system $\{M\} = \{E\} \cup \{C\}$ is *in involution*: if F is a solution of $\{E\}$ in a domain and verifies $\{C\}$ on an instant,[7] then F is a solution of $\{C\}$ in the domain. One says then that the constraints $\{C\}$ are *permanent* for $\{E\}$.

All the considerations in this paper are of local character. This is so because spacetimes do not admit, in general, global instants, because the Maxwell equations are hyperbolic (and hyperbolicity is intrinsically local) and because their solutions are not, generically, globally continuous (and consequently global evolution problems are not well-posed). Thus, we restrict ourselves to domains where the electromagnetic fields are sufficiently differentiable and, there, to neighborhoods included in the influence domain of a given instant.

(b) An electromagnetic field F is said to be *singular* or *null* if its invariant scalars[8] $\phi \equiv (F, F)$ and $\psi \equiv (F, *F)$ are zero

$$\phi^2 + \psi^2 = 0 \qquad \{N\}$$

otherwise it is said to be *regular*. It is well-known that the physical significance of the null electromagnetic fields represents *pure radiation*; here we are interested in the rather paradoxal fact that the system $\{N\}$ is not permanent for the system $\{E\}$ or, in other words, that:

Proposition 1. A Maxwell field that is null on an instant is not necessarily null in its neighbor.

This statement contradicts an old result by Mariot [2]. For this reason, and because it is the starting point of the present work, we have presented it as a proposition. The proof by a qualitative analysis of the system $\{M\} \cup \{N\}$ is rather difficult; we prove it here by explicit construction of such a null-regular Maxwell field. Let e and h be, respectively, the electric and magnetic fields characterizing F for an inertial observer in Minkowski spacetime; the system $\{E\}$, $\{C\}$, and $\{N\}$ become, respectively

$$\partial e/\partial t = -\text{rot}\, h, \qquad \partial h/\partial t = \text{rot}\, e$$

$$\text{div}\, e = 0, \qquad \text{div}\, h = 0$$

and

$$e^2 - h^2 = 0, \qquad e \cdot h = 0$$

[7] In a domain Ω of the spacetime (V, g), an *instant* is a specelike hypersurface of Ω.

[8] $(\ ,\ )$ denotes the scalar product with the spacetime metric g.

Taking the time derivative of the last equation and using the first, we obtain

$$\begin{aligned} h \cdot \mathrm{rot}\, e + e \cdot \mathrm{rot}\, h &= 0 \\ e \cdot \mathrm{rot}\, e - h \cdot \mathrm{rot}\, e &= 0 \end{aligned} \tag{1}$$

But $\{M\}$ is in involution, and $\{E\}$ is well-posed (in the Cauchy sense), so that to every pair (e_0, h_0) verifying $\{C\}$ *on an instant* corresponds, in the neighborhood of it, to a unique Maxwell field. Let the instant be given by $t = t_0$ and consider the following fields at t_0

$$e_0(x^i) = (x, y, 0), \qquad h_0(x^i) = (0, 0, r)$$

where $x^i \equiv \{x, y, z\}$ (Cartesian coordinates) and $r^2 \equiv x^2 + y^2$. It is easy to see that these fields verify the systems $\{N\}$ and $\{C\}$, and so they generate in the neighborhood of t_0 a Maxwell field. Nevertheless, as may be seen by substitution, our data are not solutions of (1) at t_0 and, consequently, the corresponding Maxwell field, null at t_0, is regular in its neighborhood, as stated in Proposition 1.

It may be noted that the verification of the first-order system (1) by the initial data (e_0, h_0) does not imply the null character of the field out of t_0: taking the time derivative of (1) and using $\{E\}$, one obtains a second-order system at t_0 which, in general, will not be verified by the solutions to the system $\{C\} \cup \{N\} \cup \{(1)\}$. The prolongation of this procedure to any finite order of derivation will always give the same negative result.

(c) Proposition 1 leads naturally to the following:

Problem. To find the conditions under which the null Maxwell fields are permanent.

The only known answer to this problem is the Mariot–Lichnerowicz theorem. Remember that the *principal directions* of an electromagnetic field are given by the null eigen-directions and that an electromagnetic field is said to be of *integrable type* if it has an integrable principal direction, say $\ell : \ell \wedge d\ell = 0$. One has then:

Theorem (Mariot–Lichnerowicz). If a Maxwell field of integrable type is null on an instant, it is null in the neighborhood.

This statement, and an easy proof of it based on the eigen-direction equations, is due to Lichnerowicz [3]. Before him, Mariot [4] obtained it in an adapted Cartan's moving frame, but his proof, based partially in a previous erroneous result [2], is incomplete, and his statement, for the same reason, contains some inaccuracies. In spite of these errors, Mariot seems to have been the first to ask for the permanence of the null fields; he discovered also another fundamental property of them [5].

(d) According to its definition, an electromagnetic field F is of integrable type if there exists a function θ such that

$$i(d\theta)F \wedge d\theta = 0, \qquad i(d\theta) *F \wedge d\theta = 0 \tag{2}$$

Thus, it is for the Maxwell fields verifying (2) that the Mariot–Lichnerowicz theorem insures the permanence of the null character of the fields. But, while the Maxwell system $\{M\}$ is a differential system in F, system (2) is a mixed system, explicitly algebraic in F and implicitly differential, depending on a function whose null gradient itself depends on F. The analysis of such a system is not easy to make. It would be desirable to substitute system (2) by an equivalent one, in which the dependence on F of $d\theta$ is explicit. In the following section we introduce the essential algebraic elements for this task.

The Mariot–Lichnerowicz theorem does not address the permanence of Maxwell fields of a nonintegrable type. Because of the great variety and physical interest of such fields, it would be convenient to have a permanence theorem for them. We obtain it in the last section.

(e) Here we indicate three aspects of the permanence problem we think important. The first concerns the *non*involutive character of the system $\{E\} \cup \{N\}$: it is *not* possible to find *initial conditions* which ensure, in general, the permanence of the null character of Maxwell fields. In other words: there do not exist common relations to all null Maxwell fields (and, eventually, to some other regular ones) such that their verification on *one* instant ensure, for a null field at that instant, the permanence of its null character in the neighbor of the instant. Nevertheless, let us note that this does not deny the possibility of finding initial conditions ensuring the permanence of some, *particular*, null fields. Thus, for example, the initial conditions given, in Minkowski's inertial frames, by $\{\nabla e_0 = 0, \nabla h_0 = 0\}$, although without physical interest, show that "the Maxwell fields which are null and constant at *one* instant remain null and constant everywhere."

The conditions which select *all* the permanent null Maxwell fields are to be imposed in *all* domains in question (as is already the case for the Mariot–Lichnerowicz conditions). In other words, the system of the Maxwell equations has to be necessarily completed, in all given domains, with a convenient differential system in such a way that the *common* solutions to both systems be either everywhere null or everywhere regular in the domain. A permanence theorem for all the null Maxwell fields is thus *also* a permanence theorem (everywhere regularity) for some class of regular fields. Then, it is clear that the interest of a permanence theorem for the null fields is intimately related to the interest presented by the regular fields for which the theorem also insures permanence. For example, from the physical point of view, the one attached to the Maxwell fields verifying the conditions

$$d(F, F) = 0, \qquad d(*F, F) = 0 \tag{3}$$

will not be considered a "good" permanence theorem (that is to say, the one which selects the null fields among the fields having constant their two invariant scalars). The physical interest of such fields is too small. This is the second aspect of the permanence problem we desire to explain.

Finally, the third aspect we consider important is the *independence* of the permanence theorems for null fields from the Robinson theorem on null fields [6]. Generalizing a result by Mariot [5], this last theorem states that: *a null direction is geodesic and shear-free if, and only if, it is a principal direction of a null Maxwell field.* We know that when the spacetime has nonvanishing Weyl tensor, it admits at most two geodesic and shear-free null directions, whereas when its Weyl tensor vanishes, the distortion $\mathscr{D}$ of a null geodesic congruence ℓ satisfies the transport law $i(\ell)\nabla\mathscr{D} = -\mathrm{tr}\,\mathscr{L}\cdot\mathscr{D}$ where $\mathscr{L} \equiv \mathscr{L}(\ell)\,g$ is the Lie derivative of the metric g. Thus, in both cases (though by very different methods), the geodesic and shear-free character of a null congruence may be discerned from initial data. Nevertheless, the null Maxwell fields associated to such a congruence by the Robinson theorem *cannot be selected by initial conditions* from among the electromagnetic fields which, *at the given instant*, are null fields and admit that congruence as principal direction. These last fields may become regular in the neighborhood of the initial instant; we see in the last section that in such a case their principal directions diverge necessarily from the geodesic and shear-free congruence initially considered.

3. PRINCIPAL DIRECTIONS OF THE ELECTROMAGNETIC FIELD

(a) To every 2-form F is associated the energy tensor $T \equiv 1/2\{F^2 + (*F)^2\}$.[9] Conversely [7], a symmetric tensor T, is the energy tensor of some 2-form iff it verifies the algebraic Rainich conditions $\mathrm{tr}\, T = 0$, $T^2 = \chi^2 g$; the corresponding 2-forms are then related by a duality

[9] $F^2 = F \times F$, where $\times$ denotes the *cross* product (contraction of the tensorial product over the adjacent indices).

rotation, and are null fields iff $\chi = 0$. Let us remember also that the characteristic polynomials of F and $*F$ are of the form

$$P(\lambda) = (\lambda^2 - \alpha^2)(\lambda^2 + \beta^2), \qquad P^*(\lambda) = (\lambda^2 + \alpha^2)(\lambda^2 - \beta^2) \tag{4}$$

thus, denoting by $\ell_\pm$ their principal directions, we have

$$i(\ell_\pm)F = \pm\alpha\ell_\pm, \qquad i(\ell_\pm) *F = \mp\beta\ell_\pm \tag{5}$$

Let us introduce

$$\begin{aligned} P_\pm(\lambda) &\equiv (\lambda + \alpha)^{-1} P(\lambda) = (\lambda \pm \alpha)(\lambda^2 + \beta^2) \\ P^*_\pm(\lambda) &\equiv (\lambda + \beta)^{-1} P^*(\lambda) = (\lambda \pm \beta)(\lambda^2 + \alpha^2) \end{aligned} \tag{6}$$

from the Cayley–Hamilton theorem it follows that $P_\pm(F)$ (resp., $P^*_\pm(*F)$) is an eigentensor of F (resp., of $*F$) with eigenvalue $\pm\alpha$ (resp., $\pm\beta$) and commutes with F (resp., with $*F$).

Starting from the known identities

$$\begin{aligned} F^2 - (*F)^2 &= (\alpha^2 - \beta^2)\, g \\ F \times *F = *F \times F &= -\alpha\beta g \end{aligned} \tag{7}$$

it can be shown that the above eigentensors are of the form

$$\begin{aligned} P_+(F) &= \alpha\mathscr{F}, \qquad & P_-(F) &= -\alpha\, {}'\mathscr{F} \\ P^*_-(*F) &= -\beta\mathscr{F}, \qquad & P^*_+(*F) &= \beta\, {}'\mathscr{F} \end{aligned} \tag{8}$$

where $\mathscr{F}$ is the tensor defined by

$$\mathscr{F} \equiv \alpha F - \beta{*F} + T + \chi g \tag{9}$$

and ${}'\mathscr{F}$ denotes its transpose.

Definition. We call principal concomitants of a 2-form F their eigentensors $\mathscr{F}$ and ${}'\mathscr{F}$.

(b) Let us analyze some properties of these concomitants. The expression (9) shows that $\mathscr{F}$ never reduces to a 2-form and that it becomes symmetric iff $\mathscr{F} = T$, that is to say, iff F is null. In such a case, we know that $T \times F = F \times T = 0$; we obtain now the analog of this relation for the regular case. When F is nonnull, there is, at last, one nonzero eigenvalue, say α; then, as $P^\pm(F)$ are eigentensors of F commuting with it, from the first two relations (8) it follows that

$$\mathscr{F} \times F = F \times \mathscr{F} = \alpha\mathscr{F} \tag{10}$$

and, taking into account the second identity in (7)

$$\mathscr{F} \times *F = *F \times \mathscr{F} = -\beta\mathscr{F} \tag{11}$$

From (10), (11) and their transposed relations, it follows that Im $\mathscr{F}$ and Im $'\mathscr{F}$ are the principal directions of F. Therefore, we may write

$$\mathscr{F} = \ell_+ \times \ell_- \tag{12}$$

It is then easy to see that:

Proposition 2. The principal concomitants of a 2-form F are, up to a scalar factor, the only eigentensors of F and $*F$ that commute with them.

An endomorphism $\mathscr{G}$ is said to be a generator of a pair of directions, say $\{\ell_\pm\}$, if Im $\mathscr{G} = \{\ell_+\}$ and Im $'\mathscr{G} = \{\ell_-\}$. The above result may be stated as follows:

Proposition 2′. An endomorphism $\mathscr{G}$ is the generator of the principal directions of a 2-form F iff it coincides, up to a scalar factor, with a principal concomitant of F.

Now, from (12), it is clear that Ker $\mathscr{F}$ and Ker $'\mathscr{F}$ have no timelike directions; we have thus the following rule to obtain the principal directions:

Theorem 1. The principal directions $\ell_\pm$ of a 2-form F are given by

$$\ell_+ = \mathscr{F}(x), \qquad \ell_- = '\mathscr{F}(x) \tag{13}$$

where x is an arbitrary timelike direction and $\mathscr{F}$ is the principal concomitant of F given by (9).[10]

This theorem is the "covariant solution" to the problem of finding the principal directions of an electromagnetic field.

(c) The Rainich algebraic relations are the necessary and sufficient conditions for a symmetric tensor T to be the energy tensor of an electromagnetic field; we give here the corresponding relations for $\mathscr{F}$. From (12) it follows that $\mathscr{F} \times '\mathscr{F} = '\mathscr{F} \times \mathscr{F} = 0$; conversely, if a tensor $\mathscr{F}$ verifies these relations, Im $\mathscr{F}$ and Im $'\mathscr{F}$ are null directions and so $\mathscr{F}$ may be written as (12). Consequently, every 2-form F having these null directions as principal directions admits $\lambda\mathscr{F}$ as principal concomitants. But, the trace-

[10] $\mathscr{F}(x) \equiv i(x)\,'\mathscr{F}$; in local coordinates $\ell_-^{\alpha} = \mathscr{F}^{\alpha}{}_{\beta} x^{\beta}$.

free symmetric part of $\mathscr{F}$ is an energy tensor verifying the algebraic Raynich conditions and, from (9), $\operatorname{tr}\mathscr{F}=4\chi\geqslant 0$. We thus have:

Proposition 3. The necessary and sufficient conditions for a 2-tensor $\mathscr{F}$ to be the principal concomitant of a 2-form are

$$\mathscr{F}\times{}'\mathscr{F}={}'\mathscr{F}\times\mathscr{F}=0, \qquad \operatorname{tr}\mathscr{F}\geqslant 0 \tag{14}$$

Then $\mathscr{F}$ determines F up to a duality rotation. It is then easy to show that:

Proposition 4. Suppose $\mathscr{F}$ verifies the relations (14). If $\operatorname{tr}\mathscr{F}=0$, put

$$F_0\equiv|z|^{-1}\{i^2(x)\mathscr{F}\}^{-1/2}\, i(x)\mathscr{F}\wedge z$$

where x and y are time-like vectors submitted to the condition that $z\equiv\mathscr{F}(x,y)x-\mathscr{F}(x,x)y$ be nonzero; if $\operatorname{tr}\mathscr{F}>0$, put

$$F_0\equiv(2\operatorname{tr}\mathscr{F})^{-1/2}(\mathscr{F}-{}'\mathscr{F})$$

Then, the 2-forms F for which $\mathscr{F}$ is a principal concomitant are given by

$$F=\cos\mu F_0+\sin\mu{*F_0}, \qquad \forall\mu\in R$$

(d) For the sake of completeness we indicate here the covariant characterization of the two-invariant 2-planes, say π and $\pi^{\perp}$, of a 2-form F.

When F is regular, $P=\chi^{-1}T$ defines a $(2+2)$-almost-product structure. Let π be the time-like 2-plane and G the volume element 2-form on π; then $*G$ is the volume element 2-form on the (space-like) orthogonal 2-plane $\pi^{\perp}$, and F may be written $F=\alpha G+\beta{*G}$. Conversely, the volume element G is given by $G=(2\chi)^{-1}(\alpha F-\beta{*F})$, where $2\chi=\alpha^2+\beta^2$. On the other hand, let v and h be, respectively, the induced metrics on π and $\pi^{\perp}$; one has $v+h=g$ and $v-h=\chi^{-1}T$; so that

$$v=(2\chi)^{-1}(T+\chi g), \qquad h=(2\chi)^{-1}(T-\chi g) \tag{15}$$

Now, let x be a timelike arbitrary vector, not contained in π $({*G(x)}\neq 0)$; one has

$$v(x)\in\pi, \qquad G(x)\in\pi \tag{16}$$

$$h(x)\in\pi^{\perp}, \qquad {*G(x)}\in\pi^{\perp} \tag{17}$$

x being timelike and $G^2\neq 0$, $G(x)$ is neither zero nor null, and $v(x)=G[G(x)]$ is not colinear to $G(x)$; for similar reasons, $*G(x)$ being nonzero, the vectors $h(x)$ and $*G(x)$ are independent. Thus, the vectors given by

(16) (resp. by (17)) generate the 2-plane π (resp., $\pi^{\perp}$). Taking into account (15), we have

Proposition 5. The invariant 2-planes, π and $\pi^{\perp}$, of a regular 2-form F are given, respectively, by

$$\pi \equiv \{i(x)\,\mathscr{F}_{\lambda,\mu}\}, \qquad \pi^{\perp} \equiv \{i(x)\,\mathscr{F}_{\lambda,\mu}{}^{\perp}\}$$

where x is any timelike vector such that $T(x) \neq \chi x$, and the biparametric concomitants of F, $\mathscr{F}_{\lambda,\mu}$, and $\mathscr{F}_{\lambda,\mu}{}^{\perp}$, $\lambda, \mu \in R$, are given by

$$\mathscr{F}_{\lambda,\mu} \equiv \lambda(T+\chi g) + \mu(\alpha F - \beta {*F}), \qquad \mathscr{F}_{\lambda,\mu}{}^{\perp} \equiv \lambda(T-\chi g) + \mu(\beta F + \alpha {*F})$$

Note that π (resp., $\pi^{\perp}$) is an eigenspace of F, with eigenvalue zero, when F is spacelike (resp., timelike); that is to say, when $\alpha = 0$ (resp., $\beta = 0$).

When F is singular, both invariant 2-planes are null and contain the principal direction of F; they are defined by $\pi \equiv \{p/p \wedge F = 0\}$ and $\pi^{\perp} \equiv \{q/q \wedge {*F} = 0\}$. Let x be an arbitrary time-like vector; one has $x \notin \pi \cup \pi^{\perp}$ and then $i(x)F \neq 0$, $i(x) {*F} \neq 0$. Also, from $F \wedge F = F \wedge {*F} = 0$, one has $i(x)F \in \pi$ and $i(x) {*F} \in \pi^{\perp}$. The principal concomitant $\mathscr{F}$ reduces now to T, so that we have

Proposition 6. The invariant 2-planes, π and $\pi^{\perp}$, of a null 2-form F are given by

$$\pi \equiv \{i(x)(\lambda T + \mu F), \lambda, \mu \in R\}, \qquad \pi^{\perp} \equiv \{i(x)(\lambda T + \mu {*F}), \lambda, \mu \in R\}$$

where x is an arbitrary timelike vector.

(e) For the null 2-forms, $\mathscr{F}$ and $'\mathscr{F}$ reduce to their energy tensor; in the opposite end, for the *completely* regular 2-forms ($\alpha \neq 0$, $\beta \neq 0$), $\mathscr{F}$ and $'\mathscr{F}$ coincide with the Frobenius covariants of the associated matrix. But, as algebraic regular functions on the space of *all* the 2-forms, our principal concomitants $\mathscr{F}$ and $'\mathscr{F}$ seem not to have been considered up to now.

Let us note that this method of "covariant resolution" of the problem of eigen-directions of a 2-form, may be extended to arbitrary tensors. Its extension, in particular, to symmetric tensors has been recently made [8].

4. PERMANENCE OF NULL FIELDS

(a) The principal concomitants of a 2-form F allow one to formulate, in terms of F, the differential equations imposed on their principal directions by any particular problem.

Theorem 1 states that, if x is an arbitrary timelike 1-form, $\ell = \mathscr{F}(x)$ is a principal direction of F. Thus, the integrability of ℓ, $\ell \wedge d\ell = 0$, may be written $\mathscr{F}(x) \wedge d\mathscr{F}(x) = 0$. Considered as an endomorphism on the space of the 1-forms, $\mathscr{F}$ is a Λ'-valued vector field; when exterior algebra operators act on $\mathscr{F}$, it must be understood they act on the "1-form part" of it.[11] Thus, the above expression for the integrability of ℓ being valid for all x, it is easy to show that:

Proposition 7. An electromagnetic field F is of integrable type if, and only if, its principal concomitant $\mathscr{F}$ verifies

$$\mathscr{F} \wedge d\mathscr{F} = 0 \tag{18}$$

Taking into account this result, the Mariot–Lichnerowicz theorem may be stated as a property of a differential system for F:

Theorem (Mariot–Lichnerowicz). For the electromagnetic fields F verifying the differential system

$$dF = d{*}F = \mathscr{F} \wedge d\mathscr{F} = 0 \tag{19}$$

the null fields are permanent.

(b) A real null tetrad $\{\ell, n, p, q\}$, with $\ell \cdot n = -p^2 = -q^2 = 1$ as nonzero scalar products, defines an almost-product structure given by the two 2-planes $\pi(\ell, n)$ and $\pi(p, q)$; let P, $P^2 = g$ be its structure tensor. A simple algebraic calculation shows that:

Lemma 1. Let ℓ be a principal direction for a symmetric tensor T verifying the algebraic Rainich conditions. With respect to the real null tetrad $\{\ell, n, p, q\}$, T may be written

$$T = k^2 \ell \otimes \ell + \chi P + r\ell \,\tilde{\otimes}\, p + s\ell \,\tilde{\otimes}\, q \tag{20}$$

where P is the structure tensor of the tetrad $r^2 + s^2 = k^2\chi$ and $\tilde{}$ denotes symmetrization.

From (20), with $P = 2n \,\tilde{\otimes}\, \ell - g$ and remembering that $i(l)\,{}^t\nabla l = 0$, we have

$$\mathrm{tr}(T \times \nabla\ell) = \chi\delta\ell + \{2i(n) + ri(p) + si(q)\}\, i(\ell)\,\nabla\ell$$

and thus:

[11] That is to say, on the first index of their local coordinate components.

Lemma 2. If the null direction ℓ of Lemma 1 is geodesic, $i(\ell)\nabla\ell = \gamma\ell$, one has

$$\mathrm{tr}(T \times \mathscr{L}) = 2\chi(2\gamma + \delta\ell) \tag{21}$$

where $\mathscr{L} \equiv \mathscr{L}(\ell)\, g$ is the Lie derivative of g along ℓ.

Now, taking divergences in the eigenvectors equation for T, $i(\ell)T = \chi\ell$, we have, from (21).

Lemma 3. The variation of χ along the geodesic principal direction ℓ of T is given by

$$\mathscr{L}(\ell)\chi = 2\chi(\delta\ell + \gamma) - i(\ell)\,\delta T \tag{22}$$

When $\delta T = 0$, (22) becomes a first-order homogeneous propagation system for χ; we have thus

Proposition 8. Let T be a conservative tensor verifying the algebraic Rainich conditions and having a geodesic principal direction. If T^2 vanishes on an instant, then it vanishes in the neighborhood.

Due to the one-to-one correspondence, up to duality rotation, between T and F [7], it follows:

Theorem 2. Let F be a Maxwell field admitting a geodesic principal direction. If F is null on an instant, then it is null in the neighborhood.

The principal direction of a null Maxwell field being geodesic [5], this theorem allows, in fact, the selection of *all* the permanent null Maxwell fields in the neighborhood of the given instant. Note that the proof involves only one-half of the Maxwell equations, those imposing the conservation of T; thus, the theorem remains true for the so-called pre-Maxwellian fields [9]. Furthermore, let us note that (22) is a *transport* law; consequently, the electromagnetic fields of Theorem 2 are such that, when they are null at a point, they remain null on the integral curve of ℓ containing the point.

The geodesic condition for ℓ, expressed in parameter-independent form, $\ell \wedge i(\ell)\nabla\ell = 0$, may be written, according to our Theorem 1, as $\mathscr{F}(x) \wedge i[\mathscr{F}(x)]\,\nabla\mathscr{F}(x) = 0$; this relation being valid for all timelike x, we obtain

Proposition 9. An electromagnetic field F has a geodesic principal direction iff its principal concomitant F satisfies

$$\mathscr{H} \equiv \mathscr{F} \wedge i(\mathscr{F})\,\nabla\mathscr{F} = 0 \tag{23}$$

When F is a null field, (23) is satisfied if T is conserved. When F is regular it may be shown that (23) is equivalent to $\operatorname{tr} \mathscr{H} = 0$, tr being the contraction of the first covariant index with the first contravariant one.[12] We have thus

Proposition 10. The necessary and sufficient condition for a Maxwell field F to have a geodesic principal direction is

$$\operatorname{tr} \mathscr{H} = 0$$

where $\mathscr{H}$ is the differential concomitant of F given by (23).

Theorem 2 then may be stated in the following form:

Theorem 2′. For the electromagnetic fields satisfying the differential system

$$dF = d{*F} = \operatorname{tr}[\mathscr{F} \wedge i(\mathscr{F})\nabla\mathscr{F}] = 0 \tag{24}$$

the null character is permanent.

Let us note that, as has already been indicated in Section 2(e), this result is also a permanence statement for regularity: if F verifies (24) in a domain and is regular on an instant, then F is regular in the domain. Theorem 2′ is the wanted generalization of the Mariot–Lichnerowicz theorem.

(c) Let $w \equiv *(v \wedge dv)$ be the *rotation* of v and denote by $D \equiv i(v)\nabla$ the directional derivative; we have the identities

$$[*, D] = 0, \qquad [d, D]v = {}'\nabla v \times dv + dv \times \nabla v \tag{25}$$

and, for every 2-form A and every 2-tensor K

$$*(A \times K + {}'K \times A) = \operatorname{tr} K \cdot {*A} - (*A \times {}'K + K \times *A) \tag{26}$$

Applying (25) and (26) to w, we have

$$\begin{aligned} Dw &= *D(v \wedge dv) = *\{Dv \wedge dv + v \wedge Ddv\} \\ &= *d(v \wedge Dv) + i(v) * \{{}'\nabla v \times dv + dv \times \nabla v\} \\ &= *d(v \wedge Dv) + i(v)\{-{*dv} \times {}'\nabla v - \nabla v \times *dv + \operatorname{tr} \nabla v \cdot *dv\} \\ &= *d(v \wedge Dv) + *(v \wedge dv) \times {}'\nabla v - i(Dv) *dv + \delta v \cdot *(v \wedge dv) \end{aligned}$$

[12] $\mathscr{H}$ has local components of the form $\mathscr{H}_{\alpha\beta}{}^{\lambda\mu\sigma}$. The antisymmetry in $\alpha\beta$ follows from the exterior product form; meanwhile, the symmetry in $\lambda\mu\sigma$ is a consequence of the identity $\mathscr{F} \wedge \mathscr{F} = 0$, obtained from (12).

that is

$$Dw = i(w)\,'\nabla v + \delta v \cdot w + \mathscr{K}(v, \nabla v) \tag{27}$$

where

$$\mathscr{K}(v, \nabla v) = *\{d(v \wedge Dv) + Dv \wedge dv\}$$

When $\mathscr{K}(v, \nabla v)$ depends on w and vanishes with it, (27) becomes a propagation system for w; thus, for geodesic fields ($v \wedge Dv = 0$) where $\mathscr{K} = \lambda w$, we have:

Proposition 11. Let v be a geodesic field in a domain. If v is not tangent to an instant and is integrable on it, it is integrable in the domain.

From this proposition and Theorem 2, we have:

Theorem 3. If an electromagnetic field F is a solution of (24) in a domain and is of integrable type on an instant, then it is of integrable type in the domain.

This theorem reduces to *initial* conditions a part of the conditions required in *all* the domain by the Mariot–Lichnerowicz theorem (that part that must be imposed to a geodesic principal direction in order to be integrable). It is thus a nonempty refinement of their statement. In fact, the Mariot–Lichnerowicz theorem ensures the permanence of the null fields *among* the ones of integrable type, whereas Theorem 3 ensures the permanence of the null fields of integrable type *among* the null fields.

REFERENCES

1. Coll, B. and Ferrando, J. J. (1986). In *Actas de los E.R.E.* (Publicacions de la Universitat de València, València, Spain).
2. Mariot, L. (1954). *C.R.A.S. Paris*, **239**, 1189–1190.
3. Lichnerowicz, A. (1960). *Ann. Mat. Pura Ed App.*, **50**, 1–96.
4. Mariot, L. (1955). *C.R.A.S. Paris*, **241**, 175–176.
5. Mariot, L. (1954). *C.R.A.S. Paris*, **238**, 2055–2056.
6. Robinson, I. (1961). *Jour. Math. Phys.*, **2**, 290–291.
7. Raynich, G. Y. (1925). *Trans. Am. Math. Soc.*, **27**, 106–136.
8. Bona, C., Coll, B., and Morales, J. A. (1986). In *Actas de los E.R.E.* (Publicacions de la Universitat de València, València, Spain).
9. Debever, R. (1976). *Bull. Acad. R. Belgi. Class Sci.*, **62**, 662–677.