On the electromagnetic field and the Teukolsky–Press relations in arbitrary space-times

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The relations on the electromagnetic field obtained by Teukolsky and Press for type D vacuum space-times are considered; these are four second-order equations in two complex components of the field with respect to a principal null tetrad. A rigorous geometric interpretation of these relations is given, showing the essential role played by the Maxwellian character of the basic null tetrad. It appears that, generically, the Teukolsky–Press relations are incomplete. Once completed, their generalizations to the general Maxwell equations (with source term) with respect to non-necessarily Maxwellian tetrads on arbitrary space-times are given.

I. INTRODUCTION

(a) Let ϕ_0, ϕ_1, ϕ_2 be the components of the electromagnetic field F with respect to a principal null tetrad in a type D vacuum space-time. Here we call *Teukolsky–Press relations*[1] the following set of four second-order partial differential equations in the two components ϕ_0, ϕ_2:

$$T_0\!:\ \tau_0\phi_0 = 0, \quad T_2\!:\ \tilde{\tau}_0\phi_2 = 0,$$
$$\underline{T}_0\!:\ \underline{\tau}_0\phi_0 + \underline{\tau}_2\phi_2 = 0, \quad \underline{T}_2\!:\ \underline{\tilde{\tau}}_0\phi_2 + \underline{\tilde{\tau}}_2\phi_0 = 0, \tag{1}$$

where the τ's are given, in the Newman–Penrose notation, by

$$\begin{aligned}\tau_0 &\equiv (D - \epsilon + \bar{\epsilon} - 2\rho - \bar{\rho})(\Delta + \mu - 2\gamma)\\ &\quad - (\delta - \beta - \bar{\alpha} - 2\tau + \bar{\pi})(\bar{\delta} + \pi - 2\alpha),\\ \underline{\tau}_0 &\equiv (\bar{\delta} + 3\pi - \bar{\beta} - \alpha)(\bar{\delta} + \pi - 2\alpha),\\ \underline{\tau}_2 &\equiv (D + 3\rho + \epsilon - \bar{\epsilon})(D - \rho + 2\epsilon),\end{aligned} \tag{2}$$

and $\sim$ is the operator which permutes separately the real and complex vectors of the null tetrad. The two uncoupled equations in (1), T_0 and T_2, were first given by Teukolsky[2]; the remaining two, $\underline{T}_0$ and $\underline{T}_2$, by Teukolsky and Press.[3]

The Teukolsky–Press relations were the starting point to show[3] that the Maxwell equations can be integrated by separation of variables in perturbed Kerr geometries. For this reason, they play an important role in many problems related to the Kerr space-times. It is the case, in particular, in the problem of the perturbations of a Kerr black hole by incident electromagnetic waves, first considered by Starobinsky and Churilov,[4] which could be studied in detail (see Chandrasekhar[5]).

But, in spite of their simple derivation, the Teukolsky–Press relations are not easy to interpret: derived from the Maxwell equations, one does not know, conversely, to what extent the Maxwell equations are implied by them.

On the other hand, some authors[6,7] have given Teukolsky-Press-like relations in the Kerr-Newman space-times, but the precise conditions under which the Teukolsky-Press relations may be generalized to other space-times have not yet been found.

This paper answers both problems: we find a rigorous geometric interpretation of the Teukolsky–Press relations and their connection with the Maxwell equations, and we give their generalizations to arbitrary null tetrads and arbitrary space-times.

(b) For this task, we need two important notions: those of *Maxwellian structure* and of *conditional system* associated to a given differential system.

It is well known that a electromagnetic field (arbitrary two-form) selects algebraically, at every point of the space-time, a pair of orthogonal two-planes which, in the regular case, define a $2+2$ almost-product structure.[8–10] The *Maxwellian structures* are the $2+2$ almost-product structures defined by the regular solutions to the vacuum Maxwell equations.

On the other hand, let $D_1(\phi_0,\phi_1,\phi_2)$ and $D_2(\phi_0,\phi_2)$ be two differential systems in the ϕ's. We shall say that D_2 is a *conditional system* for D_1 if all their solutions (ϕ_0,ϕ_2) may be completed to solutions (ϕ_0,ϕ_1,ϕ_2) of D_1 and if, conversely, all the solutions (ϕ_0,ϕ_1,ϕ_2) of D_1 are such that (ϕ_0,ϕ_2) are solutions of D_2.

We shall see here that *the Maxwell equations always admit a conditional system in* (ϕ_0,ϕ_2) that is, generically, of third order. Moreover, *this system degenerates to a second-order system if, and only if, the basic null tetrad is associated naturally to a Maxwellian structure.*

The principal null tetrads of the type D vacuum space-time are associated to a Maxwellian structure. Consequently, the conditional system admitted there by the Maxwell equations is a second-order one. Then, its comparison with Eqs. (1) and (2) shows that, up to a missing equation, *the Teukolsky–Press relations on the type D vacuum space-times are nothing but the conditional system in* (ϕ_0,ϕ_2) *admitted by the Maxwell equations.*

The missing equation in the Teukolsky–Press relations

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is identically verified on the solutions of the Maxwell equations which are invariant under the isometry group of the Kerr metric. This is, perhaps, the reason why this equation has been omitted up to now: the Maxwell solutions usually considered in this context belong essentially to this class.

On the basis of our preceding results, the generalization of the Teukolsky–Press relations to any null tetrad in any space-time must be considered as given by the conditional system in (ϕ_0,ϕ_2) associated there to the Maxwell equations.

It will be then easy to characterize the type D nonvacuum space-times in which the first two Teukolsky–Press equations remain uncoupled.

(c) The paper is organized as follows: in Sec. II we introduce "*à la* Rainich" the notion of Maxwellian structure and then, give its version in the complex formalism. Section III is devoted to finding the conditional systems in (ϕ_0,ϕ_2) admitted by the Maxwell equations, and Sec. IV gives its explicit expression in terms of the spin coefficients. Finally, in Sec. V, we compare them with the Teukolsky–Press relations and discuss the remainder of the results stated in the precedent paragraph (b).

This paper contains some results published elsewhere,[11] but here we consider the general Maxwell equations (with source term), obtain the explicit form of the third-order conditional system, and give detailed proofs of our statements.

II. MAXWELLIAN STRUCTURES

(a) Let Ω be a domain of the space-time (V_4,g), g being a Lorentzian metric of signature -2. To every two-form F is associated the *Minkowski stress-energy tensor* T, given by $2T\equiv F^2+(*F)^2$ with $F^2\equiv F\times F$, $\times$ being the cross product,[12] and $*$ denoting the Hodge dual operator.[13] The tensor T verifies $T^2=\chi^2 g$, where χ is nonzero if, and only if, F is *regular*.[14] In this section we shall consider only regular two-forms, so that the tensor $P\equiv\chi^{-1}T$ defines a $2+2$ almost-product structure. Let G be the simple unit two-form characterizing the field of timelike two-planes of the structure

$$\tfrac{1}{2}\mathrm{tr}\, G^2=1,\quad \mathrm{tr}\, *G\times G=0,\quad P=G^2+(*G)^2, \tag{3}$$

tr being the trace operator; the field of spacelike two-planes is then characterized by $*G$, and one has[15]

$$F=e^{\phi+*\psi}G=e^{\phi}(\cos\psi G+\sin\psi *G), \tag{4}$$

where $2\phi=\ln 2\chi$. Every regular two-form F is thus biunivocally characterized by its *components* $\{G,\phi,\psi\}$. Note that, given the *geometric component G*, the *energetic component* ϕ determines the norm of the eigenvalues of T, and both, G and ϕ, characterize T itself. Finally, among all the two-forms associated with a given T, the *Rainich component* ψ selects, by a duality rotation, the particular two-form F.

(b) In terms of these components, the vacuum Maxwell equations for F,

$$\delta F=0,\quad \delta *F=0, \tag{5}$$

may be written[9]

$$d\phi=\Phi,\quad d\psi=\Psi, \tag{6}$$

where the one-forms Φ and Ψ are functionals of the sole geometric component G:

$$\begin{aligned}\Phi&\equiv*(\delta G\wedge *G+\delta *G\wedge G),\\ \Psi&\equiv*(\delta G\wedge G-\delta *G\wedge *G),\end{aligned} \tag{7}$$

δ being the codifferentiation operator and "$\wedge$" denoting the exterior product.[16]

From (6), the *Rainich Theorem*[8] follows: A simple and unitary two-form G is the geometric component of a (local) solution to the vacuum Maxwell equations if, and only if, it verifies the equations

$$d\Phi=0,\quad d\Psi=0. \tag{8}$$

The almost-product structures defined by such simple and unitary solutions to Eq. (8) will be called *Maxwellian structures.*

To every Maxwellian structure, say G, the first of relations (6), $d\phi=\Phi$, associates a (one-parameter, additive) family of energetic components ϕ, characterizing a (homothetic) family of energy tensors T. In fact, it may be shown that this relation is strictly equivalent to the conservation equation $\delta T=0$ (Ref. 17). In a similar way, the second of the relations (6), $d\psi=\Psi$, associates with G a (one-parameter, additive) family of Rainich components ψ, characterizing (up to a homothecy) a family of two-forms F related by a constant duality rotation. In fact, it may be shown that this relation is strictly equivalent to the Rainich's complexion equation. The set $\{G,\phi,\psi\}$ then defines the two-parameter family of solutions to the Maxwell equations having the same almost-product structure.

(c) The form (6) of the Maxwell equations may be easily obtained in the complex vectorial formalism.[18] Let us consider the complex two-forms Z^I $(I=0,1,2)$ given by

$$Z^0=\bar{m}\wedge n,\quad Z^1=n\wedge l-\bar{m}\wedge m,\quad Z^2=l\wedge m,$$

where $\{l,n,m,\bar{m}\}$ is a complex null tetrad; since the Z^I's are self-duals, $*Z^I=iZ^I$, the basis $\{Z^I,\bar{Z}^I\}$ of the complex two-forms separates invariantly the self-dual and anti-self-dual parts of every two-form W: $W=W_I Z^I+\underline{W}_I\bar{Z}^I$. In particular, for every real two-form F, the complex two-form $\hat{F}\equiv F-i*F$ is self-dual and its components in the basis $\{Z^I\}$ will be designed by ϕ_I: $\hat{F}=\phi_I Z^I$ $(I=1,2,3)$.

The *general Maxwell equations* $\delta F=J$, $\delta *F=0$, now may be written in the form

$$J=\delta\hat{F}=\delta\{\phi_I Z^I\}=\phi_1\delta Z^1-i(d\phi_1)Z^1+\delta H.$$

Contracting by Z^1 and taking into account that $Z^1\times Z^1=g$, one finds that *the general Maxwell equations are equivalent to the system*

$$d\phi_1=\phi_1 h+\omega, \tag{9}$$

where

$$h\equiv i(\delta Z^1)Z^1,\quad \omega\equiv i(\delta H-J)Z^1,\quad H\equiv\phi_0 Z^0+\phi_2 Z^2. \tag{10}$$

In order to formulate the Rainich Theorem in this formalism, let us consider the almost-product structure associated to a null tetrad, defined by the element Z^1 of the corresponding self-dual basis, $Z^1=G-i*G$. The two-form G is the geometric component of every two-form F_0 having the expression (4), and one has $\hat{F}_0=\phi_1^0 Z^1$ with $\phi_1^0=e^{\phi+i\psi}$, that is $H=0$: The vacuum Maxwell equations for F_0 are then

$$d \ln \phi_1^0 = h, \tag{11}$$

and their (local) integrability condition is

$$dh = 0. \tag{12}$$

Expressing Z^1 in terms of G in the definition (10) of h, and taking into account that $*(v \wedge A) = -(-1)^p i(v) * A$ for every one-form v and every p-form A, one finds $h = \Phi + i*\Psi$, where Φ and Ψ are the functionals of G given by (7). We then have the following.

Proposition 1 (Rainich Theorem): An almost-product structure Z^1 is (locally) Maxwellian iff the one-form $h \equiv i(\delta Z^1) Z^1$ is closed: $dh = 0$.

Let us consider the component $(dh)_1$ of dh in the basis $\{Z^I, \bar{Z}^I\}$. From the identity $(A, dv) = \delta i(v) A + i(v) \delta A$ and the orthogonal properties of the Z^I's, one has

$$-2(dh)_1 = (Z^1, dh) = \delta i(h) Z^1 + i(h) \delta Z^1$$
$$= \delta^2 Z^1 + i^2(\delta Z^1) Z^1 = 0,$$

and thus we have the following.

Proposition 2: The differential system $dh = 0$ characterizing the Maxwellian structures consists of five second-order complex equations in Z^1.

Considered as equations on the spin coefficients of a null complex tetrad compatible with the Maxwellian structure, they are first-order equations; their explicit expression may be found elsewhere.[19]

III. CONDITIONAL SYSTEMS FOR THE MAXWELL EQUATIONS

(a) The differential system (8) defining the Maxwellian structures is satisfied by the component G of all the solutions (ϕ, ψ, G) to the vacuum Maxwell equations (6) and, conversely, all his solutions G may be completed to solutions (ϕ, ψ, G) to the Maxwell system. In other words, in order that the Maxwell equations, considered as an (overdetermined) system in the two unknowns ϕ and ψ, be compatible, it is necessary and sufficient that the system (8) in G holds. We give the following definition.

Definition: Let $D_1(x,y)$ and $D_2(y)$ be two differential systems in p unknowns x and q unknowns y; let $S_1 \subset F^{p+q}$ and $S_2 \subset F^q$ be their corresponding spaces of solutions, and let π: $F^{p+q} \to F^q$, $(x,y) \mapsto (y)$ be the natural projection. We shall say that D_2 is a conditional system in the y's for D_1 if $\pi(S_1) = S_2$.

Thus, the Rainich Theorem may be equivalently enounced by saying that *the Maxwell equations admit a second-order conditional system in G.*

(b) Let us now consider the general Maxwell equations (9) in the unknowns ϕ_I. By differentiation, one has

$$0 = d\phi_1 \wedge h + \phi_1 \, dh + d\omega,$$

and, taking into account (9), it follows that

$$\Omega + \phi_1 \, dh = 0, \tag{13}$$

where

$$\Omega \equiv d\omega + \omega \wedge h. \tag{14}$$

Thus when dh does not vanish, a necessary condition for the existence of ϕ_1 is that the two-forms Ω and dh be proportional:

$$\Omega \otimes dh = dh \otimes \Omega. \tag{15}$$

In such a case, the sufficient conditions are obtained by imposing that the proportionality factor between both two-forms be effectively a solution of Eq. (9). These conditions are first-order equations in Ω or, from (14), third-order equations in (ϕ_0, ϕ_2). When dh vanishes, we have (locally) $h = d \ln \phi_1^0$ and Eqs. (9) may be written $d(\phi_1/\phi_1^0) = \omega/\phi_1^0$, whose integrability conditions are $\Omega = 0$. We have shown the following.

Theorem 1: The Maxwell equation always admit a conditional system in (ϕ_0, ϕ_2). It is, generically, a third-order system, and it reduces to a second-order one if, and only if, the almost-product structure associated to the self-dual basis is Maxwellian: $dh = 0$. In such a case, the system is given by

$$\Omega(\phi_0, \phi_2) = 0. \tag{16}$$

For every solution (ϕ_0, ϕ_2) to (16), there exists a family of functions which complete it to solutions to the Maxwell equations. If ϕ_1 is such a function, all the others are of the form $\phi_1 + \phi_1^0$ where ϕ_1^0 is the general solution for the electromagnetic fields admitting Z^1 as the complex geometric component.

(c) Now consider a non-Maxwellian geometry and let X be any two-form such that $(X, dh) \neq 0$. If Eqs. (15) are verified, then, according to (13), we have

$$\phi_1 = -(\Omega, X)/(dh, X), \tag{17}$$

and Eqs. (9) impose

$$(\Omega, X) d(dh, X) - (dh, X) d(\Omega, X)$$
$$= -(\Omega, X)(dh, X) h + (dh, X)^2 \omega.$$

After rearranging terms, these equations may be written in the form

$$i(X) i'(X) \{\Omega \otimes \nabla \, dh - dh \otimes \nabla \Omega$$
$$+ \Omega \otimes h \otimes dh - dh \otimes \omega \otimes dh\}$$
$$+ \{i(\Omega) i'(dh) - i(dh) i'(\Omega)\}(X \otimes \nabla X) = 0, \tag{18}$$

where $i(\ )$ and $i'(\ )$ denote, respectively, contraction over the first and last two-form elements of the tensorial basis. From Eqs. (15) and their covariant derivatives, it follows, respectively, that the term in $X \otimes \nabla X$ vanishes and that the tensor between brackets, which is in $\Lambda^2 \otimes T* \otimes \Lambda^2$, is symmetric in their antisymmetric components. Thus, as (18) must be verified for any two-form X, we have the following.

Theorem 2: The third-order conditional system in (ϕ_0, ϕ_2) for the Maxwell equations is given by

$$\Omega \otimes \nabla dh - dh \oplus \{\nabla \Omega - h \otimes \Omega + \omega \otimes dh\} = 0. \tag{19}$$

To every solution (ϕ_0, ϕ_2) to this system, corresponds a unique solution (ϕ_0, ϕ_1, ϕ_2) to the Maxwell equations, the ϕ_1 being given by (17).

IV. THE SECOND-ORDER CONDITIONAL SYSTEM IN THE SPIN COEFFICIENTS' FORMALISM

(a) Let $\{\Omega_I, \underline{\Omega}_I\}$ be the components of the two-form Ω with respect to the chosen self-dual basis $\{Z^I, \bar{Z}^I\}$. From the orthogonality properties $(Z^0, Z^2) = 1, (Z^1, Z^1) = -2,$ and the definition (14) of Ω, we have, for the component Ω_1,

$$-2\Omega_1 = (Z^1,\Omega) = (Z^1,\omega\wedge h) + (Z^1,d\omega)$$
$$= -i(\omega)i(h)Z^1 + \delta i(\omega)Z^1 + i(\omega)\delta Z^1,$$

where the adjoint character of $i(\ )$ (resp. δ) with respect to $\wedge$ (resp. d) has been taken into account. But, from $Z^1\times Z^1 = g$ and the definition (10) of h and ω, it follows $i(h)Z^1 = \delta Z^1$ and $i(\omega)Z^1 = \delta H - J$ so that we have

$$-2\Omega_1 = -i(\omega)\delta Z^1 + \delta(\delta H - J) + i(\omega)\delta Z^1 = 0.$$

Consider now the component Ω_0 and Ω_2: the second-order terms in ϕ_0 and ϕ_2 come from $d\omega$ or, according to the definitions (10) of ω and H and the orthogonality properties of the basis Z^I, from the antisymmetrization of $\nabla d\phi_0\times Z^0 + \nabla d\phi_2\times Z^2$; but $Z^0\times Z^0 = Z^2\times Z^2 = 0$, $\Omega_0 = (\Omega,Z^2)$, and $\Omega_2 = (\Omega,Z^0)$ so that Ω_0 (resp. Ω_2) does not depend on the second-order derivatives of ϕ_2 (resp. ϕ_0). On the other hand, it is clear that Ω depends at most on the first derivatives of J, and, finally, denoting by $\sim$ the operator which permutes separately the real and complex vectors of the null tetrad, $\sim^2 = \text{Id}$, $\tilde{Z}^1 = -Z^1$, $\tilde{Z}^0 = -Z^2$, one has $\tilde{J} = J$, $\tilde{\phi}_0 = -\phi_2$ so that, from the definitions (10) of h and ω, it follows that $\tilde{h} = h$ and $\tilde{\omega} = -\omega$ and, consequently, $\tilde{\Omega} = -\Omega$. Taking into account all these results, we have the following.

Proposition 3: The second-order conditional system in (ϕ_0,ϕ_2) for the general Maxwell equations is of the form

$$\begin{aligned}
&-\Omega_0 \equiv D_0\phi_0 + D_2\phi_2 + \mathscr{J}_0 = 0,\\
&-\underline{\Omega}_0 \equiv \underline{D}_0\phi_0 + \underline{D}_2\phi_2 + \underline{\mathscr{J}}_2 = 0,\\
&\Omega_2 \equiv \tilde{D}_0\phi_2 + \tilde{D}_2\phi_0 - \tilde{\mathscr{J}}_0 = 0,\\
&\underline{\Omega}_2 \equiv \tilde{\underline{D}}_0\phi_2 + \tilde{\underline{D}}_2\phi_0 - \tilde{\mathscr{J}}_2 = 0,\\
&\tfrac{1}{2}(\Omega_1 + \underline{\Omega}_1) \equiv D_1\phi_2 - \tilde{D}_1\phi_0 - \mathscr{J}_1 = 0,
\end{aligned} \tag{20}$$

where D_2 is a first-order derivation operator and the $\mathscr{J}_I$'s are functions of J and its first derivatives.

(b) In order to obtain the explicit expression for the components (20) of the two-form Ω, in terms of the spin coefficients and the directional derivatives associated to the null tetrad, we need of some intermediate expressions. The evaluation of the codifferentials of the Z^I's, which may be easily performed using Ref. 18, gives

$$\delta Z^0 = 2i(\sigma_1)Z^0 + i(\sigma_2)Z^1,$$
$$\delta Z^1 = -2i(\sigma_0)Z^0 + 2i(\sigma_2)Z^2,$$
$$\delta Z^2 = -i(\sigma_0)Z^1 - 2i(\sigma_1)Z^2,$$

where the σ_I's denote the following one-forms[18]:

$$\sigma_0 = \tau l + \kappa n - \rho m - \sigma\overline{m},$$
$$\sigma_1 = \gamma l + \epsilon n - \alpha m - \beta\overline{m},$$
$$\sigma_2 = \nu l + \pi n - \lambda m - \mu\overline{m}.$$

The codifferentials of the tetrad one-form are

$$\delta l = -(\epsilon + \bar{\epsilon}) + (\rho + \bar{\rho}),\quad \delta n = (\gamma + \bar{\gamma}) - (\mu + \bar{\mu}),$$
$$\delta m = -\bar{\pi} + \tau + \bar{\alpha} - \beta,\quad \delta\overline{m} = -\pi + \bar{\tau} + \alpha - \bar{\beta},$$

and the action of the operator $\sim$ on the σ_I's is

$$\tilde{\sigma}_1 = -\sigma_1,\quad \tilde{\sigma}_0 = -\sigma_2.$$

Following Crossman and Fackerell,[6] we write

$$D^{rs}_{pq} = D + (p-1)\epsilon - (q+1)\rho + (r-1)\bar{\epsilon} - s\bar{\rho},$$
$$\delta^{rs}_{pq} = \delta + (p-1)\beta - (q+1)\tau - (r-1)\bar{\alpha} + s\bar{\pi},$$

and denote by Δ^{rs}_{pq} and $\bar{\delta}^{rs}_{pq}$, respectively, the transforms of D^{rs}_{pq} and δ^{rs}_{pq} by the operator $\sim$.

Taking into account the above expressions, the computation of relations (20) is not a difficult task; denoting by

$$\tau_0 = \delta^{21}_{01}\bar{\delta}^{10}_{30} - D^{21}_{01}\Delta^{10}_{30},\quad \underline{\tau}_2 = -D^{00}_{22}D^{10}_{30},$$
$$\underline{\tau}_0 = \bar{\delta}^{00}_{22}\bar{\delta}^{10}_{30},\quad \tau_1 = D^{20}_{21}\delta^{10}_{30} - (\tau + \bar{\pi})D^{10}_{30},$$

the second-order operators on the ϕ_I's, the result is the following.

Theorem 3: In any space-time, the second-order conditional system in (ϕ_0,ϕ_2) of the general Maxwell equations, $\Omega = 0$, is of the form (20), where

$$\begin{aligned}
&D_0 = \tau_0 - \kappa\nu + \sigma\lambda,\\
&D_2 = -2\kappa\delta^{3/2\ 1/2}_{3/2\ 1/2} + 2\sigma D^{3/2\ 1/2}_{3/2\ 1/2} - \delta\kappa + D\sigma,\\
&\underline{D}_0 = \underline{\tau}_0 - \lambda D^{00}_{22} - \bar{\sigma}\Delta^{10}_{30} - \bar{\kappa}\nu - D\lambda,\\
&\underline{D}_2 = \underline{\tau}_2 - \kappa\bar{\delta}^{00}_{22} - \bar{\kappa}\delta^{10}_{30} - \sigma\bar{\sigma} - \bar{\delta}\kappa,\\
&D_1 = \tau_1 + \kappa\Delta^{20}_{21} + \sigma(\pi + \bar{\tau}) + \Delta\kappa,\\
&\mathscr{J}_0 = \kappa J^1 + \delta^{21}_{01}J^2 + \sigma J^3 + D^{21}_{01}J^4,\\
&\mathscr{J}_2 = \bar{\kappa}J^1 + \bar{\sigma}^{00}_{22}J^2 - D^{00}_{22}J^3 - \bar{\sigma}J^4,\\
&\mathscr{J}_1 = D^{20}_{21}J^1 + \Delta^{20}_{21}J^2 + (\bar{\pi} + \tau)J^3 - (\pi + \bar{\tau})J^4.
\end{aligned} \tag{21}$$

V. THE TEUKOLSKY–PRESS RELATIONS AND THEIR GENERALIZATIONS

(a) Let us consider, on any type D vacuum space-time, the null tetrads associated to the Bel directions[20] (*principal null tretrads*). In the Newman–Penrose formalism,[21] we have

$$\begin{aligned}
&\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0,\\
&\kappa = \nu = \sigma = \lambda = 0,
\end{aligned} \tag{22}$$

and the Bianchi identities become

$$d\Psi_2 = 3\Psi_2\cdot h. \tag{23}$$

Then one has $dh = 0$ and thus, according to Proposition 1, the almost-product structure associated to the null tetrads is Maxwellian.

For such space-times, the four Teukolsky–Press relations may be written in the form (1) with the values (2) of the τ's. On the other hand, the evaluation of Eqs. (20) under the hypothesis (22), leads, *in the source free case* $J = 0$, to the equations

$$\Omega_A = T_A,\quad \underline{\Omega}_A = \underline{T}_A, \tag{24}$$

for $A = 0,2$ and

$$\tfrac{1}{2}(\Omega_1 + \underline{\Omega}_1) = \tau_1\phi_2 - \tilde{\tau}_1\phi_0 = 0, \tag{25}$$

for $A = 1$. Thus we have the following.

Theorem 4: On type D vacuum space-times, the conditional system in (ϕ_0,ϕ_2) for the source-free Maxwell equations, associated with the principal null tetrads, consists of the Teukolsky–Press relations (1) completed with the relation (25).

From (20) and (21) it is easy to see that (24) holds iff relations (22) hold. For *any* type D space-time, we have the following.

Proposition 4: The first two Teukolsky–Press relations

T_0 and T_2 decouple if, and only if, the Bel directions of the space-time are geodesic, are shear-free, and define a Maxwellian structure.

(b) In the particular case of the Kerr metric, the first two equations (1), in addition to being uncoupled in ϕ_0 and ϕ_2, may be separated into radial and angular parts relative to the Boyer–Linquist coordinates for the electromagnetic fields which are invariant[22] under the action of the two-dimensional isometry group. For these fields, the fifth equation (25) is identically satisfied when the first four equations (24) hold. This is perhaps the reason why Eq. (25) has not been (apparently) considered up to now. But if, in the same geometric context, one wishes to consider, for example, nonperiodic time-dependent electromagnetic fields, then Eq. (25) must be necessarily added to the usual Teukolsky–Press relations (24) in order to insure the existence of ϕ_1.

(c) Theorem 4 shows that, *once completed, the natural geometric generalization of the Teukolsky–Press relations is our conditional system in* (ϕ_0,ϕ_2). This is a manifold generalization: the second-order conditional system (20) extends the validity of the Teukolsky–Press relations, step by step, to noninvariant fields, to nonprincipal tetrads, to non-source-free Maxwell equations, and to arbitrary space-times. Finally, when the chosen null tetrads do not define a Maxwellian structure, the third-order conditional system (19) must be used instead of the second-order one.

[1]In preceding papers (see Ref. 11) we called them *Teukolsky relations*. The present appellation seems more correct.
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