General Relativity and Gravitation, Vol. 36, No. 11, November 2004 (© 2004)

Aligned Electric and Magnetic Weyl Fields

Joan Josep Ferrando[1] and Juan Antonio Sáez[2]

Received February 26, 2004

The results on the non-existence of purely magnetic solutions are extended to the wider class of spacetimes which have homothetic electric and magnetic Weyl fields. This class is a particularization of the spacetimes admitting a direction for which the relative electric and magnetic Weyl fields are aligned. We give an invariant characterization of these metrics and study the properties of their Debever null vectors. The directions 'observing' aligned electric and magnetic Weyl fields are obtained for every Petrov-Bel type.

KEY WORDS: electric and magnetic Weyl fields.

1. INTRODUCTION

The electric and magnetic Weyl tensors are gravitational quantities E and B attached to any observer and playing an analogous role to the electric and magnetic fields [1, 2]. Some classes of spacetimes can be defined by imposing the existence of an observer for which the electric and magnetic parts of the Weyl tensor satisfy some restriction. These properties imposed on the electric and magnetic Weyl fields imply integrability conditions which are sometimes very restrictive. Thus, although a lot of physically interesting purely electric ($B = 0$) solutions are known, severe restrictions appear in dealing with purely magnetic ones ($E = 0$) (see references in [3, 4]). We want to remark here that there are no vacuum solutions with a purely magnetic type D Weyl tensor [5], and McIntosh *et al.* [6] have conjectured that a similar restriction could take place for a wide range of type I spacetimes.

[1] Departament d'Astronomia i Astrofísica, Universitat de València, E-46100 Burjassot, València, Spain; e-mail: joan.ferrando@uv.es
[2] Departament de Matemàtica Econòmico-Empresarial, Universitat de València, E-46071 València, Spain; e-mail: juan.a.saez@uv.es.

0001-7701/04/1100-2497/0

In the literature we can find significant steps to support the McIntosh *et al.* conjecture. It was shown for a shear-free observer [7, 8] and, recently, Van der Bergh has shown the conjecture provided that the observer defines a normal congruence [9] or for a freely falling observer [10]. Moreover, the conjecture is also true under weaker conditions on the shear and vorticity tensors that trivially hold when the shear or the vorticity vanish [11]. These results are also valid for non vacuum solutions with a vanishing Cotton tensor [11].

It is worth pointing out that an extension of the conjecture is known for type D metrics. Indeed, elsewhere [12] we have shown that not only the purely magnetic solutions are forbidden, but also a wider class of type D solutions. More precisely, we have shown [12]: *if a spacetime with a vanishing Cotton tensor has a type D Weyl tensor with (complex) eigenvalues of constant argument, then it is a purely electric solution*. This means that the constant argument, necessarily, takes the values 0 or π.

For type D metrics, a constant eigenvalue means that the electric and magnetic Weyl fields with respect to a principal observer are homothetic, $E = kB$, k being a real constant. In the present paper we obtain a similar extension for algebraically general spacetimes. More precisely, we show that when a type I metric with vanishing Cotton tensor has homothetic electric and magnetic Weyl fields with respect to a shear-free or a vorticity-free observer, then the spacetime is purely electric. Furthermore, we will show this result under the weaker kinematic restrictions obtained in [11]. Recently Barnes [4] has presented the result concerning a shear-free observer. He has also generalized a result by Hall [5] by showing that there are no type II vacuum solutions with a Weyl eigenvalue of constant argument.

The spacetimes with homothetic electric and magnetic Weyl fields are a particular case of those for which the electric and magnetic Weyl fields are aligned (they are proportional tensors). This class of spacetimes that we call Weyl-aligned has already been considered for the case of a time-like congruence [7, 13] and here we also consider the metrics which have this property for a null or a space-like congruence. Moreover, for every Petrov-Bel type we determine all the directions (without restricting its causal character) for which the associated electric and magnetic fields are aligned. We also point out that this alignment condition has the following interpretation: the Weyl tensor can be obtained from a purely electric Weyl-like tensor by means of a duality rotation. This rotation is constant in the homothetic case.

In this work we also analyze the relation between the Weyl-aligned metrics and some classes of 'degenerate' type I spacetimes defined by McIntosh and Arianrhood [13]. They used the dimensionless complex scalar $M = \frac{a^3}{b^2} - 6$, where $a = \mathrm{tr}\,\mathcal{W}^2$ and $b = \mathrm{tr}\,\mathcal{W}^3$ are, respectively, the quadratic and the cubic Weyl symmetric scalar invariants. The scalar M is related to the Penrose cross-ratio invariant [14] and it governs the geometry defined by the Debever null directions: $M = 0$ in Petrov-Bel type D and, in type I, M is real positive or infinite when the four Debever directions span a 3–plane [13]; the case M real negative occurs when the Penrose-Rindler [15] disphenoid associated with the Debever directions has two equal edges [16]. Elsewhere [17] we have presented an alternative approach to analyzing this Debever geometry using the complex angle between the principal bivectors and the unit Debever bivectors. Here we show that the case M negative can be reinterpreted in terms of permutability properties with respect to the metric tensor of a frame [18] built with the Debever null vectors. This result was presented without proof at the Spanish Relativity Meeting 1998 [19].

The purely electric and purely magnetic conditions depend on the observer and, consequently, they are not invariant a priori. Nevertheless, McIntosh *et al.* [6] showed that the Weyl-electric and Weyl-magnetic spacetimes admit an intrinsic characterization in terms of some scalar invariants: M must be real positive or infinite, and a must be real, positive in the electric case and negative in the magnetic case. Consequently, the Debever directions of a type I purely electric and purely magnetic Weyl tensor span a 3–plane. Elsewhere [3], we have generalized the purely electric and purely magnetic concepts by considering electric and magnetic Weyl fields with respect to an arbitrary direction. These generalized Weyl-electric or Weyl-magnetic spacetimes also permit the scalar M to take real negative values, the cubic scalar b being real or purely imaginary, respectively. Thus, the new classes of gravitational fields that we have considered in [3] admit a partially symmetric frame built with Debever vectors.

All the results quoted above show that the spacetimes where the invariant M is a real function have Debever directions with special properties, and their subfamily where b^2 is a real function can be identified in terms of the electric and magnetic Weyl fields. In this paper we will characterize the other metrics for which M is a real function in terms of the relative electric and magnetic Weyl fields. More precisely, we will show that the necessary and sufficient conditions for M to be real is that the spacetime admits a (not necessarily time-like) direction for which the electric and magnetic parts are aligned.

The article is organized as follows. In section 2 we present the basic formalism and we define the concepts of Weyl-aligned spacetime and Weyl-aligned direction. In section 3 we determine, for every Petrov-Bel type, the Weyl-aligned directions and we characterize the full class of Weyl-aligned metrics intrinsically, as well as, some specific subclasses. In section 4 we analyze in detail the type I Weyl-aligned spacetimes by studying the properties of the Debever null vectors. Finally, section 5 is devoted to extending the results on the non existence of purely magnetic solutions to spacetimes with homothetic electric and magnetic fields.

2. WEYL-ALIGNED SPACETIMES

Let W be the Weyl tensor of an oriented and time-oriented spacetime (V_4, g) of signature $\{-, +, +, +\}$. We can associate with any unit vector field v ($v^2 = \epsilon$, $\epsilon = \pm 1$) the electric and magnetic Weyl fields:

$$E = E[v] \equiv W(v; v), \qquad B = B[v] \equiv *W(v; v) \tag{1}$$

where $*$ is the Hodge dual operator and we denote $W(v; v)_{\alpha\gamma} = W_{\alpha\beta\gamma\delta} v^\beta v^\delta$. The electric and magnetic fields (1) with respect to a spacelike or timelike congruence determine the Weyl tensor fully. This fact was pointed out years ago for the timelike case [1, 2], and also holds for a spacelike congruence [3]. When v is a null vector we can also define the electric and magnetic fields (1) but, in this case, they do not determine the Weyl tensor [3]. Nevertheless, here we also consider the electric and magnetic parts with respect to a null direction. In this work we will use the following definitions.

Definition 1. A metric is Weyl-aligned at a point of spacetime when there is a vector v for which the associated electric and magnetic Weyl fields are aligned at this point. Then, the angle $\phi \in [0, \pi[$ such that $\cos\phi B[v] + \sin\phi E[v] = 0$ is called the rotation index associated with v.

Definition 2. We say that v is a Weyl-aligned vector if the attached electric and magnetic Weyl fields are aligned for some rotation index ϕ.

These definitions extend the concepts of generalized Weyl-electric and Weyl-magnetic spacetimes and Weyl-electric and Weyl-magnetic directions given in [3]. The purely electric (resp. magnetic) case corresponds to the rotation index taking the value 0 (resp. $\pi/2$). On the other hand, the rotation index has the following interpretation: the Weyl tensor W can be written:

$$W = \cos\phi W_0 + \sin\phi * W_0 \tag{2}$$

W_0 being a Weyl-like tensor which is purely electric for the vector v. That is, ϕ plays the role of a duality rotation.

From here on we work in the self-dual complex formalism. A self-dual 2–form is a complex 2–form $\mathcal{F}$ such that $*\mathcal{F} = \mathrm{i}\mathcal{F}$. We can associate biunivocally to every real 2–form F the self-dual 2–form $\mathcal{F} = \frac{1}{\sqrt{2}}(F - \mathrm{i} * F)$. The endowed metric on the 3-dimensional complex space of the bivectors is $\mathcal{G} = \frac{1}{2}(G - \mathrm{i}\,\eta)$, G being the usual metric on the 2–form space, $G = \frac{1}{2} g \wedge g$, $(g \wedge g)_{\alpha\beta\mu\nu} = 2(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu})$, and η being the metric volume element. A $\mathcal{G}$-unit bivector $\mathcal{U} = \frac{1}{\sqrt{2}}(U - \mathrm{i} * U)$ corresponds to every timelike unit simple 2–form U ($(U, *U) = 0, (U, U) = -1$), and $\mathcal{H} = \frac{1}{\sqrt{2}}(H - \mathrm{i} * H)$ is a null bivector for $\mathcal{G}$ when H is singular ($(H, H) = (H, *H) = 0$).

A unit bivector $\mathcal{U}$ defines a timelike 2–plane with volume element U and its orthogonal spacelike 2–plane with volume element $*U$. We denote these *principal 2–planes* as their volume element. The null directions $l_{\pm}$ in the 2–plane U are the (real) eigendirections of $\mathcal{U}$ and they are called *principal directions*. These principal directions may be parameterized in such a way that $U = l_{-} \wedge l_{+}$. On the other hand a null bivector $\mathcal{H}$ defines two null *fundamental 2–planes*, with volume elements H and $*H$, which cut in the unique (real) eigendirection l that $\mathcal{H}$ admits. Just one parametrization of the null vector l exists such that it is future-pointing and $H = l \wedge e_2$, where e_2 is a spacelike unit vector orthogonal to l, and fixed up to change $e_2 \hookrightarrow e_2 + \mu l$. With this parametrization we name l *fundamental vector* of $\mathcal{H}$.

The algebraic classification of the Weyl tensor W can be obtained by studying the traceless linear map defined by the self-dual Weyl tensor $\mathcal{W} = \frac{1}{2}(W - \mathrm{i} * W)$ on the bivectors space. We can associate to the Weyl tensor the complex scalar invariants

$$a \equiv \mathrm{tr}\, \mathcal{W}^2 = \rho_1^2 + \rho_2^2 + \rho_3^2, \qquad b \equiv \mathrm{tr}\, \mathcal{W}^3 = \rho_1^3 + \rho_2^3 + \rho_3^3 = 3\rho_1\rho_2\rho_3 \tag{3}$$

where ρ_i are the eigenvalues. It will also be useful to consider the dimensionless scalar invariant [6, 13]:

$$M \equiv \frac{a^3}{b^2} - 6 = \frac{2(\rho_1 - \rho_2)^2(\rho_2 - \rho_3)^2(\rho_3 - \rho_1)^2}{9\rho_1^2\rho_2^2\rho_3^2} \tag{4}$$

The invariant M is well defined for the Petrov-Bel types I, D or II if we permit it to be infinite in the case of a type I metric with $b = 0$. In types D and II, M is identically zero, and we extend its validity by considering that it also takes the zero value for type N and type III metrics.

In terms of the invariants a and b the characteristic equation reads $x^3 - \frac{1}{2}ax - \frac{1}{3}b = 0$. Then, Petrov-Bel classification follows taking into account both the eigenvalue multiplicity and the degree of the minimal polynomial. The algebraically regular case (type I) occurs when $6b^2 \neq a^3$ and so the characteristic equation admits three different roots. If $6b^2 = a^3 \neq 0$, there is a double root and a simple one and the minimal polynomial distinguishes between types D and II. Finally, if $a = b = 0$ if all the roots are equal and so zero, and the Weyl tensor is of type O, N or III, depending on the degree of the minimal polynomial.

The electric and magnetic Weyl fields (1) associated to a unit vector field v give, respectively, the real and imaginary parts of the Petrov matrix $\mathcal{W}(v; v)$:

$$2\mathcal{W}(v; v) = W(v; v) - \mathrm{i} *W(v; v) \equiv E[v] - \mathrm{i}\, B[v] \tag{5}$$

On the other hand, there are four scalars built with the electric and magnetic Weyl fields which are independent, up to sign, of the unit vector v ($v^2 = \epsilon$) [1, 2]. In fact they are the real and imaginary parts of the complex scalar invariants a and b:

$$a = (\mathrm{tr}\, E^2 - \mathrm{tr}\, B^2) - 2\,\mathrm{i}\,\mathrm{tr}(E \cdot B), \tag{6}$$

$$b = -\epsilon[(\mathrm{tr}\, E^3 - 3\,\mathrm{tr}(E \cdot B^2)) + \mathrm{i}(\mathrm{tr}\, B^3 - 3\,\mathrm{tr}(E^2 \cdot B))] \tag{7}$$

3. WEYL-ALIGNED DIRECTIONS

Here we determine for every Petrov-Bel type: (i) the conditions for the spacetime to be Weyl-aligned, (ii) the rotation index ϕ for which this condition holds, and (iii) the Weyl-aligned vectors corresponding to every rotation index ϕ. We will express these vectors in terms of ϕ and the canonical frames or other geometric elements associated with the Weyl tensor. As the richness of these frames depends on the Petrov-Bel type [2, 20], we will consider every algebraic class separately.

In order to determine the Weyl-aligned directions we do not need to solve any equations because we can use the results obtained in [3] on the Weyl electric directions. Indeed, taking into account that ϕ gives the duality rotation (2), where W_0 is purely electric, we have the following:

Lemma 1. *The necessary and sufficient condition for v to be a Weyl-aligned vector for W with associated rotation index ϕ is that v is a Weyl-electric vector for $W_0 = \cos\phi W - \sin\phi * W$, that is, $e^{-\mathrm{i}\phi}\mathcal{W}(v; v)$ is real.*

Thus, we can use the results of [3] by changing $\mathcal{W}$ by $e^{-\mathrm{i}\phi}\mathcal{W}$ in every Petrov-Bel type.

Every type N or type III Weyl tensor has associated the *fundamental 2–planes* H and $*H$ and the *fundamental vector* l (which determines the quadruple or the triple Debever direction, respectively) [2, 20]. In type III metrics a unit bivector U is also outlined. Then, an oriented and ortochronous null real frame $\{l, l', e_2, e_3\}$ exists such that $U = \pm\, l \wedge l'$, $H = l \wedge e_2$.

From lemma 1, the condition that a vector v must satisfy to be Weyl-aligned with associated rotation index ϕ follows from the Weyl-electric solutions in [3] replacing H by $\cos\frac{\phi}{2} H - \sin\frac{\phi}{2} * H$ in the type N case and by $\cos\phi H - \sin\phi * H$ in the type III case. In this way, we have:

Proposition 1. *Every type N or type III spacetime is Weyl-aligned, the rotation index ϕ being arbitrary.*

For the type N metrics the Weyl-aligned vectors with associated rotation index ϕ are those on the planes $\cos\frac{\phi}{2} H \pm \sin\frac{\phi}{2} * H$. These Weyl-aligned vectors are the fundamental vector l (which satisfies $E[l] = B[l] = 0$) and the other (spacelike) vectors lying on the null 3–plane orthogonal to l.

For the type III metrics the Weyl-aligned vectors with associated rotation index ϕ are the triple Debever direction l (which satisfies $E[l] = B[l] = 0$) and the spacelike direction $\cos\phi\, e_3 - \sin\phi\, e_2$.

A Weyl tensor of Petrov-Bel type D or type II admits a simple eigenvalue and a double one $\rho = -\frac{b}{a}$. The eigenbivector associated with the simple eigenvalue defines two *principal 2–planes* U and $*U$. In type D metrics the principal directions $l_\pm$ of the timelike plane U are the double Debever directions. In type II metrics one of the principal directions of U is the fundamental direction of the null eigenbivector H associated with the double eigenvalue and it coincides with the double Debever vector l. In this case an oriented and orthochronous null real frame $\{l, l', e_2, e_3\}$ exists such that $U = \pm\, l \wedge l'$ and $H = l \wedge e_2$ [2, 20] .

If we take into account lemma 1, the conditions for the spacetime to be Weyl-aligned follows from the results in [3] on the purely electric metrics, by just replacing ρ for $e^{-\mathrm{i}\phi}\rho$. For the type II metrics we also must change H by $\cos\frac{\phi}{2} H - \sin\frac{\phi}{2} * H$. In this way, we get:

Proposition 2. *Every type D or type II spacetime is Weyl-aligned and the rotation index is $\phi = \theta$ (mod π), θ being the argument of a Weyl eigenvalue.*

For the type D metrics the Weyl-aligned vectors are the principal ones, that is, $v \in U$ or $v \in *U$. The only null Weyl-aligned directions are the double Debever directions $l_\pm$. In the 2–plane U there are timelike and spacelike Weyl-aligned directions. Every $v \in *U$ is a spacelike Weyl-aligned direction.

For the type II metrics the Weyl-aligned vectors are the double Debever direction l and the spacelike directions $\cos\frac{\theta}{2} e_2 + \sin\frac{\theta}{2} e_3$ *and* $-\sin\frac{\theta}{2} e_2 + \cos\frac{\theta}{2} e_3$.

Finally let us consider the algebraically general case. The Weyl tensor of a type I spacetime admits three different eigenvalues ρ_j. The associated eigenbivectors $\mathcal{U}_i$ determine six *principal 2–planes* U_i and $*U_i$ which cut in the four orthogonal *principal directions* that a type I metric admits. The unit principal vectors define the Weyl canonical frame $\{e_\alpha\}$ that satisfies $U_i = e_0 \wedge e_i$ [2, 20].

Taking into account lemma 1, the results of [3] can be applied changing ρ_j for $e^{-\mathrm{i}\phi}\rho_j$. These three complex numbers are real if, and only if, the ratio between two eigenvalues is real or infinity, the argument of every one being either ϕ or $\pi + \phi$. On the other hand, $e^{-\mathrm{i}\phi}\rho_j$ are complex conjugated for two values of j if, and only if, the two eigenvalues ρ_j have the same modulus, and the argument θ of the third eigenvalue is either ϕ or $\pi + \phi$. So, we have

Proposition 3. *A type I spacetime is Weyl-aligned if, and only if, one of the two following conditions hold:*

(1) The ratio between every two eigenvalues is real (or infinity).

(2) Two of the eigenvalues have the same modulus.

If condition (1) holds, then the rotation index is $\phi = \theta \pmod{\pi}$, *θ being the argument of a Weyl eigenvalue. Moreover, the Weyl-aligned directions are the timelike Weyl principal direction* e_0 *and the spacelike Weyl principal directions* e_i.

If condition (2) holds and the third eigenvalue has different modulus, say $|\rho_1| = |\rho_2| \neq |\rho_3|$, *then the rotation index is* $\phi = \theta_3 \pmod{\pi}$, *$\theta_3$ being the argument of the eigenvalue* ρ_3. *Moreover, the Weyl-aligned directions are the spacelike directions* $e_1 \pm e_2$.

If condition (2) holds and we have equimodular eigenvalues, then there are three rotation index given by $\phi_i = \theta_i \pmod{\pi}$, *$\theta_i$ being the argument of every Weyl eigenvalue. Moreover, the Weyl-aligned directions with associated rotation indices* ϕ_i *are the spacelike directions* $e_j \pm e_k$, i, j, k *taking different values.*

Once the rotation index and the Weyl-aligned directions have been found for an arbitrary Weyl tensor by considering the different Petrov-Bel types, we study and characterize some classes of Weyl-aligned spacetimes. We begin by considering some direct consequences of the results above.

Corollary 1. *If a metric is Weyl-aligned for a timelike or a null direction with associated rotation index* ϕ, *then it is Weyl-aligned for a spacelike direction with the same rotation index.*

A null direction is Weyl-aligned if, and only if, it is a multiple Debever direction. Consequently, a spacetime is Weyl-aligned for a null direction if, and only if, it is algebraically special.

Every timelike Weyl-aligned direction is a Weyl principal direction.

On the other hand, we also recover the following result suggested by Barnes [7]:

Corollary 2. *If a spacetime is Weyl-aligned for a timelike direction, then the Weyl tensor is Petrov-Bel type I, D or O, and this direction is a Weyl principal one.*

We have also shown that an algebraically special spacetime is always Weyl-aligned, but the more degenerate the Petrov-Bel type is, the richer the number of rotation indexes that exist. More precisely, we have:

Corollary 3. *Every algebraically special spacetime is Weyl-aligned.*

For Petrov-Bel types N and III ($a = b = 0$) *the rotation index* ϕ *is arbitrary. For Petrov-Bel types D and II* ($a^3 = 6b^2 \neq 0$) *the rotation index is* $\phi = \theta \pmod{\pi}$, *θ being a Weyl eigenvalue.*

In next section we will analyze in detail the Weyl-aligned type I spacetimes and we will show that they can be characterized in terms of the properties of their Debever directions. From this study we will get the following invariant characterization of the Weyl-aligned spacetimes:

Theorem 1. *A spacetime is Weyl-aligned if, and only if, the Weyl invariant scalar M defined in (4) is real. Moreover:*

(i) $M = 0$ *if, and only if, the spacetime is algebraically special.*
(ii) $M > 0$ *if, and only if, the spacetime is Petrov-Bel type I and it is Weyl-aligned for a timelike direction; this direction is the principal one and the metric is also Weyl-aligned for the three spacelike principal directions.*
(iii) $M < 0$ *if, and only if, the spacetime is Petrov-Bel type I and it is Weyl-aligned for the bisectors $e_i \pm e_j$ of a spacelike principal 2-plane.*

4. DEBEVER VECTORS IN WEYL-ALIGNED TYPE I SPACETIMES

Let us now consider a Type I spacetime. Then, the self-dual Weyl tensor takes the canonical form:

$$\mathcal{W} = -\sum_{j=1}^{3} \rho_j \, \mathcal{U}_j \otimes \mathcal{U}_j \tag{8}$$

where $\{\mathcal{U}_j\}$ are the unit eigenbivectors associated with the simple eigenvalues ρ_j. We have already shown [17] that, for every Weyl eigenvalue, say ρ_3, we can consider the unit bivectors $\mathcal{V}_\epsilon$, $\epsilon = \pm 1$:

$$\mathcal{V}_\epsilon = \cos\Omega\, \mathcal{U}_1 + \epsilon \; \sin\Omega\, \mathcal{U}_2 \tag{9}$$

where the complex Weyl invariant Ω is given by

$$\cos 2\Omega = \frac{3\rho_3}{\rho_2 - \rho_1} \tag{10}$$

The bivectors $\mathcal{V}_\epsilon$ are *unit Debever bivectors* [20] , that is, their principal directions are the four simple Debever directions that a type I Weyl tensor admits.

These expressions have been obtained privileging ρ_3. A similar argument with the other two eigenvalues leads to other pairs of Debever bivectors and gives us other angles Ω_1 and Ω_2. These angles are not independent, and from (10) it is easy to show that

$$\cos^2\Omega_1 = \frac{1}{\sin^2\Omega}, \qquad \cos^2\Omega_2 = -\tan^2\Omega \tag{11}$$

Writing $\Omega = \phi - \mathrm{i}\psi$, we can calculate the principal directions of the bivectors (9), and we obtain the following expression for the Debever directions [17]:

$$l_{\epsilon\pm} = \cosh\psi\, e_0 \pm \cos\phi\, e_1 \pm \epsilon \sin\phi\, e_2 + \epsilon \sinh\psi\, e_3, \qquad (\epsilon = \pm 1) \tag{12}$$

On the other hand, taking into account (10), the invariant M given in (4) can be expressed in terms of Ω. So, for every M, this expression poses a cubic equation for $\cos 2\Omega$, every solution being associated with one of the angles Ω_i quoted in (11). More precisely, we have for $k = 0, 1, 2$:

$$\cos 2\Omega = \sqrt{3}\left\{N + \tfrac{\beta}{2}(N - \mathrm{i})\left[\beta\, e^{\frac{2\pi k}{3}\,\mathrm{i}} + e^{\frac{-2\pi k}{3}\,\mathrm{i}}\right]\right\}, \qquad \beta \equiv \sqrt[3]{\tfrac{N+\mathrm{i}}{N-\mathrm{i}}}, \qquad N \equiv \sqrt{\frac{6}{M}} \tag{13}$$

Let's go on now the Weyl-aligned type I metrics. We start with the first subclass pointed out in Proposition 3: the ratio between every two eigenvalues is real (or infinity). This case implies that the M given in (4) is real positive (or infinity), and this condition leads to $\cos 2\Omega$ being a real function if we take into account (13). But if $\cos 2\Omega$ is real, the ratio between two eigenvalues is real (or infinity) as a consequence of (10). Thus, we have three equivalent conditions.

On the other hand, if $\Omega = \phi - \mathrm{i}\psi$, $\cos 2\Omega$ is real when $\mathrm{senh}\psi \cos\phi \,\mathrm{sen}\phi = 0$. But if we take into account the expression (12), this condition states that the four Debever directions are linearly dependent and they span the 3–plane orthogonal to e_j. Moreover, accordingly to (10), ρ_j is the shortest eigenvalue. We can summarize these results, which complete those of McIntosh *et al.* [6, 13] (see also [17]), as:

Theorem 2. *In a type I spacetime the following statements are equivalent:*

(i) The metric is Weyl-aligned for a principal direction (and then for every principal direction).
(ii) M is real positive or infinite.
(iii) $\cos 2\Omega$ *is real.*
(iv) The ratio between every two eigenvalues is real (or infinite).
(v) The Debever directions span a 3–plane.

Moreover, if one of the above conditions hold, the 3-plane that Debever directions span is orthogonal to e_j, ρ_j being the shortest eigenvalue. The case $M = \infty$ corresponds to $b \equiv tr\mathcal{W}^3 = 0$.

Let us consider the second subclass in Proposition 3: two of the eigenvalues have the same modulus. This means that the ratio between these two eigenvalues lies on the unit circle, $\frac{\rho_2}{\rho_1} = e^{i\theta}$, $\theta \in (0, 2\pi)$, and then the invariant M given in (4) is a real negative or infinity. This condition implies that one of the solutions in (13) is a purely imaginary function or zero. But if $\cos 2\Omega_3$ is purely imaginary or zero, the eigenvalues ρ_2 and ρ_1 have the same modulus as a consequence of (10). Thus, we have three equivalent conditions and, taking into account proposition 3, we have established a similar result to the four first statements of the previous theorem.

Now we look for a description of this case in terms of the Debever directions. Elsewhere [16] the Penrose-Rindler [14] disphenoid has been used for this purpose. Nevertheless, here we interpret this case in terms of permutability properties of a frame built with the Debever null vectors.

When the Debever directions $\{l_a\}_{a=1}^4$ are independent, they become a null frame. It is said that two vectors $\{l_1, l_2\}$ of a null frame are permutable (or that the frame is P_2) if $(l_1, l_b) = (l_2, l_b)$ $(b = 3, 4)$, that is, if we can not distinguish between l_1 and l_2 by making the product with the other two vectors [18]. A remarkable property is that if a null frame is P_2, then we can reparameterize the vectors of the frame to make the other two directions permutable too, that is, we can get a $P_2 \times P_2$ frame. In the same way, it is said that all the vectors are permutable (or that the frame is P_4) if all the products (l_a, l_b) $(a \neq b)$ are equal [18].

From (12) it is easy to show that if the Debever directions are independent, they admit a reparametrization to a $P_2 \times P_2$ frame if, and only if, $\cos 2\Omega$ is purely imaginary. As we have seen, this means that there are two eigenvalues, say ρ_1 and ρ_2 $(\rho_1 \neq \pm\rho_2)$ with the same modulus. Then, the pair of permutable Debever directions are those which are the principal directions of V_ϵ constructed privileging ρ_3, that is the principal directions of the Debever bivectors such that their bisectors are $\mathcal{U}_1$ and $\mathcal{U}_2$.

The particular case of the three eigenvalues having the same modulus leads to $M = -6$ $(a = 0)$ and the three solutions of (13) are $\cos \Omega_k = \mathrm{i}\sqrt{3}$. Moreover then, and only then, a reparametrization of the Debever vectors exists such that we can built a P_4-frame. All these results can be summarized as:

Theorem 3. *In a type I spacetime the following statements are equivalent:*

(i) The metric is Weyl-aligned for a non principal direction.
(ii) M is real negative or infinity.
(iii) $\cos 2\Omega$ *is purely imaginary or zero.*
(iv) There exist two eigenvalues such that their ratio lies in the unit circle (have the same modulus).
(v) The frame of Debever vectors can be reparameterized to be $P_2 \times P_2$.

Moreover, if ρ_1 and ρ_2 have the same modulus, the pairs of permutable vectors are the principal directions of the Debever bivectors given in (9). In this case V_ϵ and $*V_{-\epsilon}$ cut each other in the bisectors $e_1 \pm e_2$ which are the Weyl-aligned directions of point (i).

The frame of Debever vectors can be reparameterized to be P_4 if, and only if, all the eigenvalues have the same modulus, that is when $M = -6$ $(a = 0)$. In this case the metric is Weyl-aligned for the bisectors $e_i \pm e_j$ of every spacelike principal plane.

5. HOMOTHETIC ELECTRIC AND MAGNETIC WEYL FIELDS IN VACUUM: KINEMATIC RESTRICTIONS

A number of known results restrict the existence of purely magnetic spacetimes. From the initial one by Hall [5] which showed that there are no purely magnetic type D vacuum solutions, some studies are known that extend this result in different ways. On one hand, the extension for Type I metrics conjectured by McIntosh *et al.* [6] has been shown when the observer is: (i) shear-free [7, 8], (ii) vorticity-free [9], (iii) geodesic [10]. The vorticity-free and shear-free conditions have been weakened recently [11] by means of first-order differential conditions which hold trivially when $\sigma = 0$ or $\omega = 0$. In this last work another kind of progression has been acquired: the restriction is also valid for non vacuum solutions with a vanishing Cotton tensor. This extension has also been shown for type D spacetimes in a paper [12] where a third kind of generalization is obtained: not only the purely magnetic solutions are forbidden, but also those whose Weyl eigenvalue has a constant argument other than 0 or π.

In this section we will give a similar extension for type I spacetimes. Indeed, as we have shown in section 3, every type D metric is Weyl-aligned and the rotation index is given by the argument of the Weyl eigenvalue. Thus, the extension for type D spacetimes quoted above applies when, for an observer, the electric and magnetic Weyl fields satisfy $E = kB$, k being a constant factor. Now we generalize the kinematic restrictions obtained in [11] to the type I spacetimes with this *homothetic* property. We start by giving the following

Definition 3. We will say that the electric and magnetic Weyl fields E and B with respect to an observer u are homothetic if they are aligned with a constant rotation index ϕ, that is, $\cos\phi B + \sin\phi E = 0$, $\mathrm{d}\phi = 0$.

The interpretation given in section 2 for the rotation index allows us to describe the homothetic condition in this way: the Weyl tensor can be obtained from a purely electric Weyl-like tensor by means of a constant duality rotation. The case $\phi = \pi/2$ corresponds to the purely magnetic case which has been analyzed in [11]. We will now show that homothetic spacetimes are subjected to similar restrictions as the purely magnetic ones on the kinematic coefficients of the observer.

Under the hypothesis of a vanishing Cotton tensor, the Bianchi identities take the same expression as in the vacuum case [11]. Thus they may be written in the $1 + 3$ formalism [21]:

$$\begin{aligned}
&\operatorname{div} E = -3\, B(\omega) + [\sigma, B] \\
&\operatorname{div} B = -\,[\sigma, E] + 3\, E(\omega) \\
&\hat{E} - \operatorname{curl} B = -\theta\, E + 3\, E \hat{\times} \sigma -\, \omega \wedge E + 2\,\, a \wedge B \\
&\hat{B} + \operatorname{curl} E = -\theta B + 3\, \sigma \hat{\times} B - \omega \wedge B - 2\, a \wedge E
\end{aligned} \tag{14}$$

where D is the covariant spatial derivative, div and curl are, respectively, the covariant spatial divergence and curl operators, $\wedge$ and [,] are the generalized covariant vector products and ˆ means the projected trace-free symmetric part (see for example [21] for more details). Now, if ϕ is constant and $\phi \neq 0$, then $E = -\cot\phi B$, and removing E from the equations above, a straightforward calculation leads to:

$$[\sigma, B] = 3B(\omega) \tag{15}$$

$$\operatorname{div} B = 0 \tag{16}$$

$$\operatorname{curl} B = -2a \wedge B \tag{17}$$

But these are the same restrictions as those we have used in [11] for the $E = 0$ case. Then, taking into account the results in [11], we can state:

Theorem 4. *In a spacetime with vanishing Cotton tensor if the electric and magnetic fields are homothetic with respect to an observer u satisfying one of the following conditions:*

(i) $\operatorname{tr}(\operatorname{curl}\sigma)^2 - 3\operatorname{tr}(\hat{D}\omega + 2a\widehat{\otimes}\omega)^2 \neq 2(\operatorname{curl}\sigma, \hat{D}\omega + 2a\widehat{\otimes}\omega)$
(ii) $\operatorname{tr}(\hat{D}\omega + 2a\widehat{\otimes}\omega)^2 = 0$
(iii) $\operatorname{tr}(\hat{D}\omega + 2a\widehat{\otimes}\omega)^2 \geq \operatorname{tr}(\operatorname{curl}\sigma)^2$

Then, the spacetime is purely electric and u is a Weyl principal direction.

From here, a corollary follows.

Corollary 4. *In a spacetime with vanishing Cotton tensor if the electric and magnetic fields are homothetic with respect to a shear-free or a vorticity-free observer u, then the spacetime is purely electric and u is a Weyl principal direction.*

The result which states that the vacuum solutions with electric and magnetic Weyl fields proportional for a shear-free observer are, necessarily, purely electric has been also presented recently by Barnes [4].

ACKNOWLEDGMENTS

This work has been supported by the Spanish Ministerio de Ciencia y Tecnología, project AYA2003-08739-C02-02 (partially financed by FEDER funds).

REFERENCES

[1] Matte, A. (1953). *Can. J. Math.* **5**, 1.
[2] Bel, L. (1962). *Cahiers de Physique* **16**, 59; English translation see : Bel, L. (2000). *Gen. Rel. Grav.* **32**, 2047.
[3] Ferrando, J. J., and Sáez, J. A. (2002). *Class. Quant. Grav.* **19**, 2437.

[4] Barnes, A. (2004). In *Proc. Spanish Relat. Meet.–03*, Ed. Universitat d'Alacant (gr-qc/0401068).
[5] Hall, G. S. (1973). *J. Phys. A* **6**, 619.
[6] McIntosh, C. B. G., Arianrhod, R., Wade, S. T., and Hoenselaers, C. (1994). *Class. Quant. Grav.* **11**, 1555.
[7] Barnes, A. (1984). In *Proc. Conf. Class. Gen. Relat. Lond.-1983*, Cambridge University Press, Cambridge.
[8] Haddow, B. M. (1995). *J. Math. Phys.* **36**, 5848.
[9] Van der Bergh, N. (2003). *Class. Quant. Grav.* **20**, L1.
[10] Van der Bergh, N. (2003). *Class. Quant. Grav.* **20**, 165.
[11] Ferrando, J. J., and Sáez, J. A. (2003). *Class. Quant. Grav.* **20**, 2835.
[12] Ferrando, J. J., and Sáez, J. A. (2004). *J. Math. Phys.* **45**, 652.
[13] McIntosh, C. B. G., and Arianrhod, R. (1990). *Class. Quant. Grav.* **7**, L213.
[14] Penrose, R. (1960). *Ann. Phys.* **10**, 171.
[15] Penrose, R., and Rindler, W. (1986). *Spinors and Spacetime* Vol. 2, Cambridge University Press, Cambridge.
[16] Arianrhod, R., and McIntosh, C. B. G. (1992). *Class. Quant. Grav.* **9**, 1969.
[17] Ferrando, J. J., and Sáez, J. A. (1997) *Class. Quant. Grav.* **14**, 129.
[18] Coll, B., and Morales, J. A. (1991). *J. Math. Phys.* **32**, 2450.
[19] Ferrando, J. J., and Sáez, J. A. (1999). In *Proc. Spanish Relat. Meet.–98*, World Scientific, Singapore.
[20] Ferrando, J. J., Morales, J. A., and Sáez, J. A. (2001). *Class. Quant. Grav.* **18**, 4969.
[21] Maartens, R., and Bassett, B. A. (1998). *Class. Quant. Grav.* **15**, 705.