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Relativistic Holonomic Fluids

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The notion of holonomic fluid in relativity is reconsidered. An intrinsic characterization of holonomic fluids, involving only the unit velocity, is given, showing that in spite of its dynamical appearance the notion of holonomic fluid is a kinematical notion. The relations between holonomic and thermodynamic perfect fluids are studied.

1. INTRODUCTION

(a) In classical mechanics, there is a large class of fluids whose equations of motion admit a relative (Poincaré) integral invariant—namely, the velocity. For them, the vorticity becomes an absolute integral invariant, and it is well-known the important rôle played by these two invariants in the development of hydrodynamics: integration of the equations of motion, study of particular fluid flows, (Helmholtz) properties of the vorticity, etc.

In relativity, the analogue of the above class of fluids was considered many years ago by Lichnerowicz [1], who called them *holonomic fluids.* Holonomic fluids also admit a relative integral invariant, but it is not, as it could seem, the unit velocity of the fluid: from the point of view of the theory of integral invariants, the relativistic analogue of the classical velocity is the *current* which differs from the unit velocity by a scalar factor called the *index function.* The corresponding absolute integral invariant is then given by the exterior derivative of the current which is called the *dynamical vorticity tensor.* For the particular case of barotropic perfect fluids, these appelations are due to Synge [2], who was the first to develop systematically the relativistic theory of such fluids.

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(b) Extending a result by Eisenhart [3], Lichnerowicz [1] showed an important property of holonomic fluids: their stream lines are the extremes of a distance conformal to the space-time distance. In other words: the (unit) velocity of a holonomic fluid is a geodesic vector field for a metric conformally related to the space-time metric. The vector fields verifying this property (or their congruence of integral curves) are called *conformally geodesic*.[3] Killing, conform-killing, or irrotational vector fields, beside barotropic perfect fluid velocities, are usual examples of conformally geodesic fields. However, how does one recognize a given vector field as belonging to this class? We shall give here necessary and sufficient conditions for a vector field to be conformally geodesic.

Also, as a converse of the above Lichnerowicz result, we shall show that any conformally geodesic velocity makes any conservative fluid holonomic.

In many known examples of fluid flows, the index function has been expressed as a function of the (thermo-)dynamical variables, showing that the relativistic relative integral invariant is a dynamical quantity, in contrast with the classical one, which is clearly kinematical. In spite of the fact that the existence of such a dynamical invariant is the main feature of holonomic fluids, our above results show that holonomic fluids admit a purely kinematical characterization.

Barotropic perfect fluids are holonomic fluids, but holonomic perfect fluids are not, in general, barotropic. We shall study the relations of both concepts.

(c) Holonomic fluids are interesting in relativity because of (i) the conformally geodesic character of their stream lines, (ii) the known [5] relations between their invariant integrals and their first integrals, and (iii) the conservation laws for the vorticity that take place for them [1]. For these reasons, we think that the holonomic character of a fluid may be usefully taken into account in the obtainment of explicit solutions to the Einstein equations as well as in the study of particular classes of motions and the analysis of certain conjectures on barotropic perfect fluid space-times (such as the Lichnerowicz's conjecture[4] on spherical symmetry under appropiate asymptotic conditions, or the Treciokas–Ellis conjecture[5] on vorticity-free or expansion-free consequences under distorsion-free conditions).

[3] Conformal geodesics have been considered recently in a different relativistic context (see [4]).

[4] See, e.g., [6] and references therein.

[5] See [7] or the more recent analysis by Collins [8].

(d) The definition of holonomic fluids and some of their properties are recalled in Section 2. Section 3 is devoted to the characterization of the conformally geodesic congruences, and Section 4 relates them to holonomic fluids. Finally, Section 5 analyzes the relation between barotropic and holonomic perfect fluids.

The results (without proof) of this paper were communicated at the Spanish relativistic meeting E.R.E. 88 [9].

2. HOLONOMIC FLUIDS

(a) Let (V_4, g)[6] be the space-time, $\mathrm{sig}(g) = -2$. Vector and tensor fields, and the expressions relating them, unless otherwise stated, will be given in their covariant form.

Fluids may be considered here as general continuous media because, for our purposes, we need to know only their energy tensor, no matter what their constitutive equations might be. Furthermore, no other material restrictions will be imposed on it (such as symmetry, energy conditions, etc.), and the fluid will be supposed to evolve only under external gravitational forces. Thus: *a fluid is here given by a conservative second rank tensor* T: *its energy tensor.*

To a physical fluid one may associate Lagrangian (or intrinsic) and Eulerian (or relative) velocities. Lagrangian velocities are those which may be univocally obtained from the constitutive equations and the intrinsic state variables; for example, the *proper* velocity (unit time-like eigenvector of T, if it exists and is unique), the *mass* velocity (unit colineal vector to conservative Eckart's mass momentum, which becomes proper when the heat flow vanishes), etc. Eulerian velocities are those which, generically, need additional, external criteria to be univocally obtained; for example, *permanence* velocities (unit vector fields for which the state variables are invariant; if they exist and are not unique), *freely falling* velocities, etc. It is the nature of every particular problem which dictates the choice of the velocities to be considered. However, we are not concerned here with this choice, so that both, Lagrangian and Eulerian velocities will be indistinctly called *associated velocities.*

(b) Let T and u be, respectively, the energy tensor of a fluid and an associate (unit) velocity. Consider a decomposition of T of the form:

$$T = mu \otimes u + \Pi \qquad (1)$$

[6] All the results of this paper may be generalized to higher dimensions.

where $m \neq 0$ and Π are, respectively, a scalar and a second-order tensor, and denote by J the *specific divergence* of Π:[7]

$$\delta\Pi = mJ \tag{2}$$

The conservation of T, $\delta T = 0$, is equivalent to the system:

$$\delta(mu) = mi(u)\,J, \qquad a = \perp(u)\,J \tag{3}$$

where a is the *acceleration* of u, $a \equiv i(u)\,\nabla u$.

Definition 2.1. (Lichnerowicz).[8] A fluid is said to be holonomic (with respect to the associate velocity u) if the corresponding specific divergence J is exact: $J = dF$. Then, F is called the *holonomy potential* and m the *holonomy density*; $f = e^F$ is called the *index function*.

For a holonomic fluid, the conservation equations (2) become:

$$a = dF - \dot{F}u \tag{4}$$

$$\theta = \dot{F} - \dot{M} \tag{5}$$

where θ is the *expansion* of u, $\theta \equiv -\delta u$, and M the logarithmic holonomy density, $M = \ln m$.[9] Their solutions will be noted (u, F, m).

(c) Lichnerowicz [1] showed that the current $C \equiv fu$ is a relative integral invariant for the equations of motion of the holonomic fluids and that, consequently, its exterior differential dC is an absolute integral invariant. That is, if D and D' are two 2-chains on the same world-lines tube of u, one has:

$$\int_{\delta D} C = \int_{\delta D'} C, \qquad \int_D dC = \int_{D'} dC \tag{6}$$

It is not difficult to prove the differential version of these properties. In fact, it is sufficient to show that $\mathcal{L}(\mu u)\,dC$[10] vanishes for any function μ: this follows from the development

$$\mathcal{L}(\mu u)\,dC = d\{\mu i(u)\,d(fu)\} = d\{\mu f i(u)[dF \wedge u + a]\} = d\{\mu f[\dot{F} - dF + a]\}$$

and relations (4).

[7] δ, $i(u)$, $\perp(u)$, ∇, and d denote, respectively, the divergence, contraction (interior product), orthogonal projection, covariant derivative, and exterior differentiation operators.

[8] His definition in Ref. [1] differs slightly from ours: see our comment below (Section 4b).

[9] Newtonian notation has been used for time-like derivatives: $\dot{x} \equiv i(u)\,\nabla x$, x being any tensorial quantity.

[10] $\mathcal{L}(X)$ denotes the Lie derivative with respect to the vector field X.

3. CONFORMAL GEODESICS

(a) When dealing with conformal structures, conformal geodesics are considered as the solutions to a system of ordinary differential equations that generalizes the geodesic one in such a way [4] that its space of solutions is invariant under conformal transformations of the metric.

Here, being directly concerned with fluid flows in *given* space-times, we shall adopt another point of view: we will look for a metric-dependent characterization of the congruences of conformal geodesics by the differential system satisfied by their tangent unit vector field.

(b) In the space-time (V_4, g), let u be a unit (time-like) vector field. The vector field[11] $u' \equiv e^{\alpha}u$ is a unit vector field for the conformal metric $g' = e^{2\alpha}g$, and its acceleration a' is related to the acceleration a of u by:

$$a' = a - d\alpha + \dot{\alpha}u \tag{7}$$

When there exists a function α such that $a' = 0$, we say the vector field u (or the congruence of its integral curves) is *conformally geodesic*. The Eisenhart theorem [3] follows: *a unit vector field u is conformally geodesic iff. there exist a function α, called the acceleration potential, such that*

$$d\alpha = a + \dot{\alpha}u \tag{8}$$

(c) Unless its acceleration is already given in the form (8), it is not simple to know whether a field u admits an acceleration potential. To answer this problem one must find the *conditional system*[12] in u attached to the differential system (8); that is to say, the necessary and sufficient conditions that u must satisfy to insure that the system (8) in α admits a solution.

By exterior differentiation of (8), we have

$$da + d\dot{\alpha} \wedge u + \dot{\alpha}\, du = 0$$

so that, taking the exterior product by u, we obtain

$$u \wedge da + \dot{\alpha}u \wedge du = 0 \tag{9}$$

Let $w = *(u \wedge du)$ be the *vorticity* of u, $*$ being the (Hodge) dual operator. When w does not vanish, applying the operator $i(w)*$ to (9) we find

$$i(w) * (u \wedge da) + \dot{\alpha}w^2 = 0$$

[11] Remember that according to our covariant convention, u and u' denote the corresponding g-associated 1-forms.

[12] Some conditional systems for Maxwell equations and barotropic fluids may be found, respectively, in [10] and [11].

which gives us the value of $\dot{\alpha}$ in terms of u and its derivatives; let us denote by $\beta[u]$ this function of u:

$$\beta[u] \equiv -(1/w^2)\, i(w) * (u \wedge du) \tag{10}$$

According to (8), if u is conformally geodesic, it verifies

$$d(a + \beta u) = 0 \tag{11}$$

Conversely, suppose that u is such that (11) takes place for the scalar β defined by (10); then a function α exists such that $d\alpha = a + \beta u$ and, applying $i(u)$, it follows $\dot{\alpha} = \beta$: u is conformally geodesic. When w vanishes, we have $u = \tau\, dt$, and putting $\gamma \equiv -\ln \tau$, it follows:

$$d\gamma = a + \dot{\gamma} u \tag{12}$$

which, by the Eisenhart result, insures that u is conformally geodesic. On the other hand, if α is a function verifying (8), it results from (2):

$$d(\alpha - \gamma) \wedge u = 0 \leftrightarrow \alpha - \gamma = H(t) \leftrightarrow \tau \cdot e^{\alpha} = h(t)$$

that is, $e^{-\alpha}$ is an integrating factor. Consequently, we have shown:

Proposition 3.1. A unit vector field u of acceleration a and vorticity w is conformally geodesic if, and only if, $w = 0$ or

$$d(a + \beta u) = 0 \tag{13}$$

with

$$\beta = -(1/w^2)\, i(w) * (u \wedge du) \tag{14}$$

Concerning the corresponding acceleration potentials (which fix the conformal transformations making geodesic the conformally geodesic field), we have:

Proposition 3.2. Let u be conformally geodesic. (i) If $w = 0$, to every integrating factor τ of u corresponds an acceleration potential $\alpha = -\ln \tau$. (ii) If $w \neq 0$, the acceleration potential α is determined, up to an additive constant by $d\alpha = a + \beta u$.

(d) Equations (13) and (14), or $w = 0$, constitute the conditional system in u for the differential system (8) in α. Conversely, we may consider equations (8) as a differential system in u, and ask for the corresponding conditional system in α. The answer is easy to obtain in this case; in fact, for a given function α, g being the space-time metric, every time-like geodesic congruence of the metric $g' = e^{2\alpha} g$ is such that its g-unit tangent vector verifies Eq. (8). Thus, any function α is an acceleration potential of a family of conformal geodesics.

4. HOLONOMIC FLUIDS AND CONFORMALLY GEODESIC VELOCITIES

(a) As we already said, the solutions to the conservation equations (4–5) for a holonomic fluid will be noted by (u, F, m). According to the Eisenhart theorem, the first of these equations expresses the conformally geodesic character of the associated velocity, and shows that the acceleration and holonomy potentials differ at most by an additive constant. Suppose now, given a conformally geodesic unit vector field u as solution of the system (13–14), we let F be an acceleration potential for it; the equation (5) in the unknown m always admits a solution, determined up to a u-invariant factor, so that we have a solution (u, F, m) to the conservation equations (4–5). With this solution (u, F, m), let us consider a conservative energy tensor T and define $\Pi \equiv T - mu \otimes u$; one then has:

$$\delta \Pi = \delta T + \dot{m} u + m\theta - ma = m[(\dot{M} + \theta)\, u - a] = m\, dF$$

that is to say, T is a holonomic energy tensor with associated velocity u, holonomy potential F, and holonomy density m. Therefore, taking into account Proposition 3.2, we have:

Proposition 4.1. To every conformally geodesic velocity u can be associated a family of solutions $\{(u, F, m)\}$ to the conservation equations for holonomic fluids, such that: (i) the Fs are determined up to an additive constant if $w \neq 0$ and up to a function of the potential of u if $w = 0$. (ii) For every pair u, F, the ms are determined up to a u-invariant factor. (iii) For every solution (u, F, m), any conservative energy tensor T is a holonomic fluid with associated velocity u, holonomy potential F, and holonomy density m.

(b) In his definition of holonomic fluids (see footnote 8) Lichnerowicz considered the associated velocity u as being an eigenvector of the energy tensor T, but he never used this fact in his development of the theory. For this reason, we have excluded it in our definition 2.1, which we give respect to any associated velocity. In doing so, we are able to separate the features which are necessary and sufficient for the existence of the integral invariants, from the features related to the particular character of the associated velocities. Here we are only considering the first ones, and it is to be understood that our results must be constrained by the definition equations of the specific Lagrangian- or Eulerian-associated velocities, when they are previously given.

From this point of view, assertion (iii) of Proposition 4.1 says that a fluid is holonomic if, and only if, its associated velocity is conformally geodesic. For this reason, and whenever hydrodynamics is concerned, the conformally geodesic velocities will be equivalently called *holonomic velocities*.

The different characterizations of the holonomic velocities that we have seen, are collected in the following proposition.

Proposition 4.2. The following statements are equivalent:

i. The unit vector field u is conformally geodesic.

ii. There exists a function F such that $a = dF - \dot{F}u$.

iii. There exists a function f such that $C \equiv fu$ is a relative integral invariant.

iv. There exists a function f such that $d(fu)$ is an absolute integral invariant.

v. The vector field u is either vorticity-free or a solution to our system (13–14).

vi. The vector field u is the associated velocity of a holonomic fluid.

(c) It is well-known, and easy to see, that a *barotropic perfect fluid*, $T = (\rho + p)\, u \otimes u - pg$, $\rho = \rho(p)$, is a holonomic fluid with respect to u, with holonomy potential π such that $dp = (\rho + p)\, d\pi$. Thus, the variables u, π, $\rho + p$ are a solution $(u, \pi, \rho + p)$ to the conservation equations (4–5), and this solution is particular in the sense that π and $\rho + p$ are functionally dependent.

This property can be generalized: a solution (u, F, m) to the conservation equations (4–5) will be called *barotropic* if $dF \wedge dm = 0$, correspondingly, the velocities u giving rise to barotropic solutions (u, F, m) will be called *barotropic velocities*.

Any trivial barotropicity (i.e., $F = F_0$, const.) corresponds to a geodesic velocity and, conversely, every geodesic velocity u may be associated to a family of solutions $\{(u, F_0, m)\}$ with F_0 const., the holonomy density m being determined, up to a u-invariant factor, by $\dot{M} + \theta = 0$. When $dF \neq 0$, by virtue of $dF \wedge dm = 0$, we may write $\dot{F} - \dot{M} = \dot{F}[1 - M'(F)]$, so that we have

$$\theta = g(F)\, \dot{F} \tag{15}$$

Conversely, if F is a holonomic potential verifying (15) for some $g(F)$, let M be a solution to the equation $M'(F) = 1 - g(F)$; then (u, F, m) is a solution to the conservation system (4–5) such that $dF \wedge dm = 0$. Thus, we have:

Proposition 4.3. The necessary and sufficient condition for a non-geodesic unit vector field u to be a barotropic velocity is the existence of a function F such that

$$a = dF - \dot{F}u, \qquad \theta = g(F)\dot{F} \tag{16}$$

For every solution u, F to this system, the set (u, F, m), with $m = \exp\{\int [1 - g(\pi)]^{-1} d\pi\}$ is a barotropic solution to the conservation system for holonomic fluids.

Proposition 3.1 characterizes intrinsically the conformally geodesic fields (i.e., the holonomic velocities). In an analogous way, the conditions on u for the existence of a function F verifying (16) will characterize intrinsically the barotropic velocities. However, the equations (16) are identical to those for the velocities of barotropic perfect fluids, so that barotropic velocities are the velocities of barotropic perfect fluids, and have been already characterized elsewhere [11].

5. PERFECT FLUIDS AND HOLONOMY

(a) Suppose $T = (\rho + p)\, u \otimes u - pg$ is a holonomic fluid with respect to u, with holonomy density $\rho + p$; we have $H = -pg$, hence $\delta H = dp$ and, consequently, $(\rho + p)\, dF = dp$, where F is the holonomy potential. It follows $d\rho \wedge dp = 0$ and, thus, T is barotropic. As the converse has been seen in the above section, we have:

Proposition 5.1. A perfect fluid $T = (\rho + p)\, u \otimes u - pg$ is holonomic with respect to the proper velocity u, with holonomy density $\rho + p$ if, and only if, it is barotropic.

In what follows, we shall suppose the perfect fluid to be holonomic with respect to u but with holonomy density different from $Q^{-1} \equiv \rho + p$.

The components orthogonal to u of the conservation equations $\delta T = 0$ may be written

$$Q\, dp = a + Q\dot{p}u \tag{17}$$

which, by differentiation and taking exterior product by u, gives rise to

$$dQ \wedge dp \wedge u = u \wedge da + Q\dot{p}u \wedge du \tag{18}$$

(b) In this paragraph we will consider vorticity-free flows of thermodynamic fluids. When a fluid admits a thermodynamic scheme,[13] ρ is decomposed in the form $\rho = r(1+\varepsilon)$, and the thermodynamic *closure* of the fluid is obtained by requiring that: (i) the 1-form $\sigma \equiv d\varepsilon + p\,d(1/r)$ be integrable: $\sigma \wedge d\sigma = 0$; and (ii) the current ru be conserved: $\delta(ru) = 0$. In terms of the variables u, ρ, p, a perfect fluid admits a thermodynamic scheme iff the differential equation in $h\mathscr{L}^{\rho}(u)\,h + \theta = 0$ admits a solution of the form $h = h(\rho, p)$ [14] and, in the case $\dot\rho \neq 0$, it occurs iff. $\dot p/\dot\rho$ is a function of state, that is $\dot p/\dot\rho = \psi(\rho, p)$ [15].

We already know that, in the vorticity-free case, $u = \tau\, dt$, u is conformally geodesic (Proposition 3.1) with holonomy potential $\gamma = -\ln\tau$:

$$a = d\gamma - \dot\gamma u \tag{19}$$

Then, from (17–19) it follows that either the fluid is barotropic or the functions t and γ lie in the thermodynamical 2-plane $\pi(d\rho, dp)$.

When $a \neq 0$, it follows that τ and t are functionally independent; then we have $\pi(d\rho, dp) \equiv \pi(dt, d\tau) \equiv \pi(u, a)$ and, consequently, $J(\rho, p; t, \tau) \neq 0$, where J denotes the Jacobian. Under this condition, it is easy to see that the fluid admits a thermodynamic scheme iff. $d(\dot\tau/\dot t) \wedge dt \wedge d\tau = 0$; but as u is a unit vector field, we have $\dot t = \tau^{-1}$, so that, this condition may be written $d\dot\tau \wedge dt \wedge d\tau = 0$, that is, $d\dot\gamma \wedge dt \wedge d\tau = d\dot\gamma \wedge u \wedge a = 0$. On the other hand, because of (19), it results $da + d\dot\gamma \wedge u + \dot\gamma \cdot du = 0$, so that as u is vorticity-free, we have $a \wedge da = d\dot\gamma \wedge u \wedge a$. Thus, we have shown:

Proposition 5.2. A (nonbarotropic) perfect fluid with vorticity-free and nongeodesic proper velocity u admits a thermodynamic scheme iff. its acceleration a is an integrable 1-form: $a \wedge da = 0$. Then, the velocity, acceleration, and holonomy potentials are functions of state.

When $a = 0$, we have $u = dt$ and, from (17), it follows $dp = \dot p\, dt$, that is, $\dot p = p'(t)$. Consequently, $\dot p/\dot\rho$ is a function of ρ and p if, and only if, $\dot\rho$ does. Therefore, by virtue of eq. (4), and to the last characterization of a thermodynamic scheme, we have:

Proposition 5.3. A (nonbarotropic) perfect fluid with vorticity-free and geodesic velocity u admits a thermodynamic scheme if, and only if, $d\theta \wedge d\rho \wedge dp = 0$.

(c) Now, let us consider a perfect fluid with a nonvanishing vorticity and holonomic (i.e., conformally geodesic) proper velocity, and let F be the holonomy potential. From (4) and (17), we can find a and u as combina-

[13] It is Eckart's scheme [12] (see also [13] and references therein).

tions of dF and dp, and evaluate the elements of integrability of the 2-plane $\pi(u, a)$, that is, $u \wedge a \wedge da$ and $u \wedge a \wedge du$. When $\pi(u, a)$ is not integrable, one has $Q\dot{p} - \dot{F} = 0$ and then $Q\,dp - dF = 0$. Thus, we have:

Proposition 5.4. A perfect fluid with holonomic proper velocity for which the 2-plane $\pi(u, a)$ is not integrable, is barotropic.

On the other hand, from the results of Section 3 we know that the holonomy potential F is such that $\dot{F} = \beta[u]$, where $\beta[u]$ is the function of the velocity given by (14); taking into account (18), it follows:

$$\dot{F} = -(1/w^2)\, i(w) * (u \wedge da) = -(1/w^2)\, i(w) * (dQ \wedge dp \wedge u) + Q\dot{p}$$

From this expression, and the one obtained by solving in u the system (4–17), we obtain:

$$dF = Q\,dp - \mu u, \qquad \text{with } \mu = (1/w^2) * (u \wedge w \wedge dp \wedge dQ) \tag{20}$$

the converse being also easy to show, we have:

Proposition 5.5. Let $T = Q^{-1} u \otimes u - pg$ be a perfect fluid with nonvanishing vorticity w. Then, u is a holonomic velocity iff the 1-form $Q\,dp - \mu u$ is exact, where $\mu = (1/w^2) * (u \wedge w \wedge dp \wedge dQ)$.

Applying the operator $i(u) * d$ to the 1-form $Q\,dp - \mu u$ one obtains, by Proposition 5.4, $w = (\mu Q^2)^{-1} * (u \wedge d\rho \wedge dp)$: a perfect fluid with holonomic proper velocity has a nonvanishing vorticity in so far as u separates from the 2-plane $\pi(d\rho, dp)$.

Finally, let us study the thermodynamical dependence of the holonomy potential. In the vorticity-free case, we have shown (Proposition 5.2) that the holonomy potential F is a function of state, $F = F(\rho, p)$ say. Such a generic dependence is incompatible for nonvanishing vorticity: from (20) it follows that u lies in the 2-plane $\pi(d\rho, dp)$. Thus $\mu = 0$ and, consequently, $F = F(p)$:

Proposition 5.6. If a perfect fluid has holonomic proper velocity with nonvanishing vorticity, and if its holonomy potential is a function of state, then it is barotropic.

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