

Physics of Relativistic Perfect Fluids

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Abstract

A criterion is presented and discussed to detect when a divergence-free perfect fluid energy tensor in the space-time describes an evolution in *local thermal equilibrium*.

This criterion is applied to the class II Szafron-Szekeres perfect fluid space-times solutions, giving a very simple characterization of those that describe such thermal evolutions. For all of them, the significant thermodynamic variables are explicitly obtained.

Also, the specific condition is given under which the divergence-free perfect fluid energy tensors may be interpreted as an *ideal gas*.

1 Physical fluids and energetic evolutions. The inverse problem

In the absence of nuclear or chemical reactions, and independent of the external constraints to which it may be submitted, a material medium is considered here as *physically characterized* by the specification of its molecular components.

In a domain Ω of the space-time, and in the absence of exterior constraints, every initial configuration of a material medium gives rise to a particular evolution. And, to every one of these evolutions in Ω an *energy tensor* [1] T_f corresponds *univocally*. Due to the absence of exterior constraints, the energy tensors so obtained are divergence-free, $\nabla \cdot T_f = 0$.

Thus, a set $\mathbf{T}_f \equiv \{T_f\}$ of *energy tensors* is associated to every medium f , namely those of all its possible unconstrained evolutions in the space-time domain Ω . We have the diagram of Figure 1.

The energetic description of a particular evolution of the medium f by the energy tensor T_f , consists, and only consists, of the specification of the *energy density* ρ_f , the *energy flow density* q_f and the *stress tensor* t_f of this evolution of f in Ω .

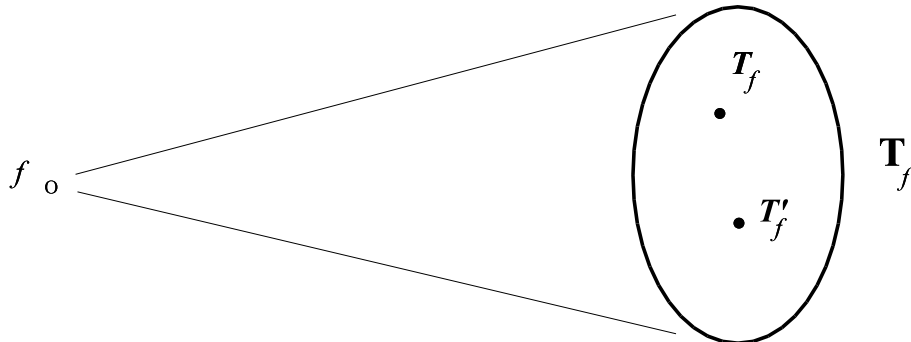


Figure 1:

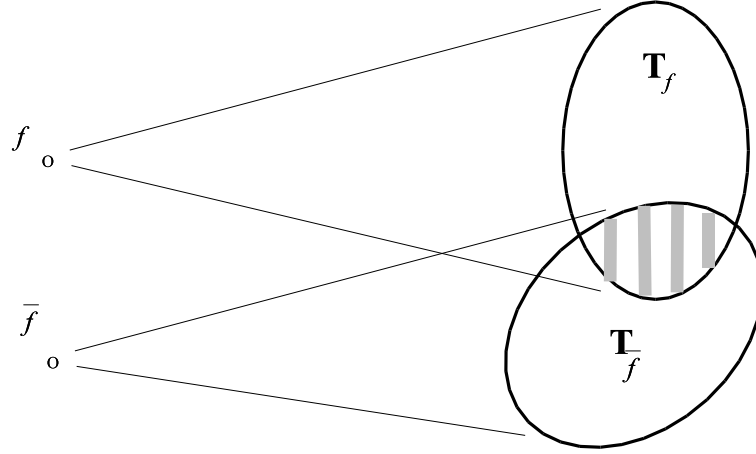


Figure 2:

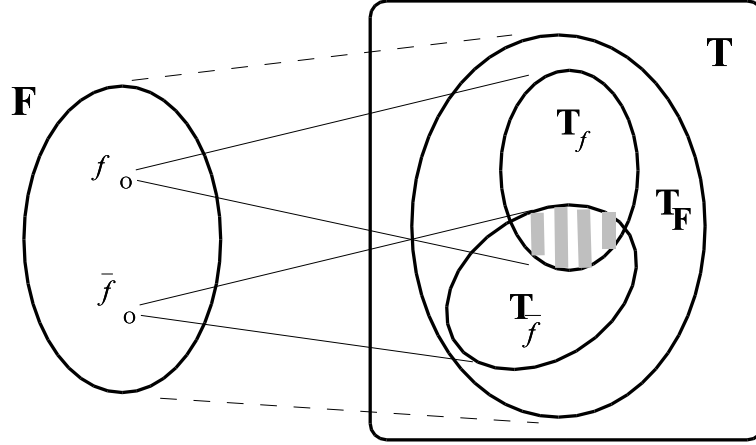


Figure 3:

Consequently, the energetic description by T_f *does not explicitly contain* the physical characterization of the medium, that is, the specification of its molecular components.

Thus, a given divergence-free energy tensor T_f could describe the particular evolution of more than one medium. In other words, the sets \mathbf{T}_f and $\mathbf{T}_{\bar{f}}$ of all possible unconstrained evolutions of two mediums f and \bar{f} are not necessarily disconnected as the diagram of Figure 2 shows.

On the other hand, divergence-free energy tensors T may be generated theoretically in many ways, which may or may not correspond to particular evolutions of a physical medium. Thus, if \mathbf{T} denotes the set of all divergence-free energy tensors T , $\mathbf{T} \equiv \{T/\nabla \cdot T = 0\}$, \mathbf{F} the set of all mediums f , $\mathbf{F} \equiv \{f\}$, and $\mathbf{T}_{\mathbf{F}}$ that of all divergence-free energy tensors T describing all particular evolutions of all mediums f , $\mathbf{T}_{\mathbf{F}} \equiv \{T \in \mathbf{T} / \exists f : T = T_f, f \in \mathbf{F}\}$, we have the diagram of Figure 3.

Usually, in many (theoretical or experimental) physical situations one starts from known elements of \mathbf{F} and looks for some of the corresponding elements of \mathbf{T} . Here we are interested in the *inverse problem* that inquires about the existence and characterization of mediums f in \mathbf{F} corresponding to a given energy tensor T of \mathbf{T} (see Figure 4).

In this paper we shall restrict the analysis of such an inverse problem to the subclass \mathbf{T}^* of \mathbf{T}

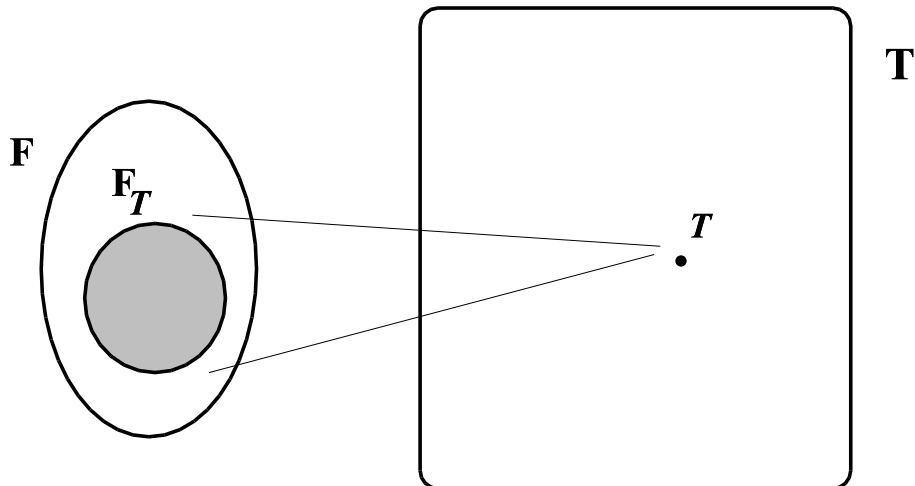


Figure 4:

constituted by the set of *perfect fluid energy tensors*, that is to say, by those divergence-free energy tensors,

$$\nabla \cdot T = 0 \leftrightarrow \nabla_\alpha T^{\alpha\beta} = 0 \quad (1)$$

of the form

$$T = (\rho + p)u \otimes u + pg \quad (2)$$

In spite of its apparent clarity, it is worthwhile asking the following question: is this inverse problem physically meaningful?

In Section 2 we shall see that additional assumptions are necessary in order to obtain physically meaningful answers, and the assumption selected here will be that of *local thermal equilibrium (l.t.e.)*. The necessary and sufficient condition for an energy tensor T of \mathbf{T}^* to be in *l.t.e.* will be presented, and some important properties will be mentioned.

In Section 3 we shall show how our condition applies in a particular example: among the Szekeres-Szafron class II space-time solutions, our condition easily allows us to obtain *all* the perfect fluid solutions which are in *l.t.e.* and gives the corresponding matter density, specific entropy and temperature.

Finally, in Section 4, as a further application, we shall present the necessary and sufficient conditions for a perfect fluid to be an *ideal gas*.

The proof of all our statements will be given elsewhere [2], where we will present a deeper analysis of the ideas related to the "physical meaning" of perfect fluids.

2 Local thermal equilibrium

As described above, the set T_f of energy tensors associated in Ω to a fluid f gives the energetic description of the set of unconstrained evolutions that correspond to the different *initial configurations* that the fluid may adopt. As a consequence, it is *meaningless* to try to find a physical interpretation to a perfect fluid energy tensor T in Ω submitted to the sole divergence-free condition. The reason is

that the system $\{(1), (2)\}$ is not causal, so that not only one energy tensor corresponds to every initial configuration, but a partially arbitrary family of such tensors, which include arbitrarily advanced-retarded-like solutions as well as Maxwell's demonds-like perturbations of the physical evolution [3]. In order to select a physical (causal) evolution, one has to complete the above system conveniently [4]. It is known [5] that its completion with a *l.t.e.* scheme is causal.

Here we shall choose to study the inverse problem for a divergence-free perfect fluid energy tensor under the hypothesis that its evolution takes place in *l.t.e.*.

Let us remember that a perfect fluid is in *l.t.e.* if and only if **i)** the intensive variables of the fluid are related by *equations of state*, **ii)** these equations are compatible with the thermodynamic relation $Tds = d\epsilon + p dv$, **iii)** the *matter density* $r \equiv 1/v$ is conserved, $\nabla \cdot (ru) = 0$.

As already stated, the *l.t.e.* scheme thus defined is a causal completion for the system $\{(1), (2)\}$. But it is not a *minimal* causal completion in the sense that it introduces a clearly superabundant number, not only of equations, but also of unknowns. Because the system $\{(1), (2)\}$ consists of four equations for five unknowns, the following question can be asked: does an *equivalent* formulation of the *l.t.e.* scheme exists that is reduced to the *addition* to the system of a *unique equation* relating the five given unknowns?

This question is far from being academic. If the answer were positive, not only would we dispose of an easy deductive algorithm to detect whether or not a given divergence-free perfect fluid energy tensor evolves in *l.t.e.*, but we would also dispose of an equational and conceptual simplified version of *l.t.e.*: a version that would show that, contrary to what is indicated by general opinion and historic development, the concept of *l.t.e.* does not need a *thermodynamic*, but only a *dynamic*, purely *energetic* background.

In fact, in a very different conceptual situation, we already answered this question positively some years ago [6]. The result is the following theorem:

Theorem (Coll-Ferrando, 1989): *A divergence-free perfect fluid energy tensor evolves in l.t.e. if, and only if, the space-time function*

$$\chi \equiv \frac{\dot{p}}{\dot{\rho}} \quad (3)$$

called the indicatrix of l.t.e., depends only on the variables p and ρ :

$$d\chi \wedge dp \wedge d\rho = 0 \quad (4)$$

We believe that this simple result is very interesting from the physical point of view because of the following remarks:

i) This theorem says, in other words, that a relation of the form

$$\dot{p} = \chi(p, \rho) \dot{\rho} \quad (5)$$

is a *minimal causal completion* to the equations $\nabla \cdot T = 0$ for a perfect fluid.

ii) Condition (4), $d\chi \wedge dp \wedge d\rho = 0$, is a *deductive* condition of *l.t.e.* for the *hydrodynamic* variables u , p and ρ .

iii) The "iff" character of the theorem implies that this condition constitutes an *alternative* definition of *l.t.e.*. And surprisingly enough at first glance, this alternative definition involves *only hydrodynamic, energetic and evolutive* concepts, but *not thermodynamic* ones.

iv) If the conditions of the theorem are verified, that is, if the perfect fluid evolves in *l.t.e.*, then the indicatrix χ becomes a function of state, $\chi(\rho, p)$, and it physically represents the square of the *velocity of the sound* in the fluid [7],

$$\chi(\rho, p) \equiv v_{\text{sound}}^2 \quad (6)$$

v) From the above interpretation, one has the following necessary condition of *physical reality*:

$$0 \leq \chi \leq 1 \quad (7)$$

In the next Section we have selected a general (family of) perfect fluid space-time(s) to show how our theorem is applied in a particular case.

3 Class II Szekeres-Szafron space-times

The Szekeres-Szafron solutions to Einstein Equations constitute one of the largest families of perfect fluid exact solutions with no generic symmetries. A class of them, the so called *class II*, is given by a line element ds^2 of the form [8]

$$ds^2 = dt^2 - R^2 \left\{ (B + P)^2 dx^2 + \frac{dy^2 + dz^2}{\left[1 + \frac{k}{4}(y^2 + z^2)\right]^2} \right\} \quad (8)$$

with

$$P \equiv \frac{\frac{1}{2}(y^2 + z^2)U + yV_1 + zV_2}{1 + \frac{k}{4}(y^2 + z^2)} \quad (9)$$

where $R(t)$, $U(x)$, $V_1(x)$, $V_2(x)$, $V(x)$ are arbitrary functions, $k \in \{0, \pm 1\}$, and $B(x, t)$ is a solution of the equation:

$$\ddot{B} + 3\frac{\dot{R}}{R}\dot{B} - \frac{kB + U}{R^2} = 0 \quad (10)$$

In these coordinates the perfect fluid is comoving, and its pression p and energy density ρ are given by

$$p = -\frac{2R\ddot{R} + \dot{R}^2 + k}{R^2} \quad (11)$$

$$\rho = \frac{2R\dot{R}\dot{B} + 3(B + P)\dot{R}^2 + k(B + 3P) - 2U}{R^2(B + P)} \quad (12)$$

In order to characterize, among these fluids, all those that evolve in *l.t.e.*, it is convenient to first detect the structure of the above function $B(\rho, p)$. Such a structure is given by the following lemma:

Lemma 1: *The general solution of the equation*

$$\ddot{B} + 3\frac{\dot{R}}{R}\dot{B} - \frac{kB + U}{R^2} = 0 \quad (13)$$

is of the form

$$B = \alpha a + (k\beta + 1)b + \beta U \quad (14)$$

where $a(x)$ and $b(x)$ are arbitrary functions and $\alpha(t)$ and $\beta(t)$ verify

$$\ddot{\alpha} = -3\frac{\dot{R}}{R}\dot{\alpha} + \frac{k}{R^2}\alpha \quad (15)$$

$$\ddot{\beta} = -3\frac{\dot{R}}{R}\dot{\beta} + \frac{1}{R^2}(k\beta + 1) \quad (16)$$

The isobaric case $\dot{p} = 0$, which according to expression (11) corresponds to particular choices of $R(t)$, obviously verifies our condition of *l.t.e.*.

In order to select all other solutions that evolve in *l.t.e.*, it is now easy, because $p = p(t)$, to develop expression (4) directly, and thus allows us to neglect time derivatives of ρ and χ . The result is

Theorem 1: *The non isobaric class II Szekeres-Szafron perfect fluids that evolve in l.t.e. are those for which*

$$a(x) = 0 \quad \text{or} \quad a(x) \neq 0, \quad k = 0, \quad U(x) = 0 \quad (17)$$

For such fluids it is then easy, starting from the "old" definition of *l.t.e.*, to find their thermodynamic variables. One obtains:

Theorem 2: *For the non isobaric class II Szekeres-Szafron perfect fluids that evolve in l.t.e., the matter density r is given by*

$$r = \frac{\phi(u)}{R^3(\beta u + 1)}, \quad u \equiv \frac{kb + U}{b + P} \quad (18)$$

for the case $a(x) = 0$, and by

$$r = \frac{\phi(u)}{R^3(\alpha + u)}, \quad u \equiv \frac{b + P}{a} \quad (19)$$

for the case $a(x) \neq 0$, $k = 0$, $U(x) = 0$. The specific entropy s and the temperature T are, respectively

$$s = s(u), \quad T = \frac{1}{s'} \left(\frac{\rho + p}{r} \right)' \quad (20)$$

where the prime denotes derivative with respect to u .

We must note the relatively easy way, and the simple and compact *general* results that may be obtained by direct application of our above theorem on *l.t.e.*. It is worthwhile comparing them with the partial expressions obtained, only for a partial subclass of the class II Szekeres-Szafron solutions, with the method claimed by Quevedo and Sussman in reference 8.

4 When is a perfect fluid an ideal gas?

Results of the above type allow us to easily find *all* the families of perfect fluid solutions of a general class that evolve in *l.t.e.*. But, what does *a family of perfect fluid solutions* mean physically? Is it *a family of physical fluids*, every one of them in a particular evolution state?, is it *a family of evolution states* of one particular physical fluid?, or is it a hybrid family?

There are no known results that allow us to answer this question. Moreover, even for a *unic* given perfect fluid solution one does not dispose of simple criteria to detect its plausible physical meaning.

Here, to show how one can solve these problems, we present a simple criterion for a perfect fluid to be a conventional ideal gas.

Theorem 3: *The necessary and sufficient condition for a divergence-free perfect fluid energy tensor, $T = (\rho + p)u \otimes u + pg$, to represent the l.t.e. evolution of a classical ideal gas,*

$$pv = kT, \quad k \equiv \frac{nR}{m}, \quad (21)$$

with specific internal energy

$$\epsilon = c_v T, \quad (22)$$

is that the indicatrix χ be of the form

$$\chi = \frac{\gamma p}{\rho + p} \quad (23)$$

with

$$1 \leq \gamma \leq 2. \quad (24)$$

Then, the constants k and c_v are related by

$$\gamma = 1 + \frac{k}{c_v} \quad (25)$$

It is important to note the following points:

i) In practice, it is convenient to verify directly that one has

$$d \frac{(\rho + p)\dot{p}}{p\dot{\rho}} = 0 \quad (26)$$

ii) If it is the case, then the constant

$$\gamma \equiv \frac{(\rho + p)\dot{p}}{p\dot{\rho}} \quad (27)$$

fixes, but only fixes, the quotient k/c_v of the gas. Thus, the values $\gamma = 5/3$ and $\gamma = 7/5$ correspond respectively to *monoatomic* and *diatomic* ideal gases.

Other gases, not necessarily ideal ones, will be characterized elsewhere [2].

References

- [1] We follow the physical use of calling the field of energy tensors at every point of Ω *energy tensor* for short.
- [2] B. Coll and J.J. Ferrando, *Relativistic perfect fluids in local thermal equilibrium* to be submitted to the Journal of Mathematical Physics.
- [3] In ref [2] these points are explained in detail.
- [4] Here "conveniently" means "with the necessary and sufficient equations in order to have a *regular* Cauchy problem and with a clear physical meaning for these equations".
- [5] The *l.t.e.* scheme for a relativistic perfect fluid is obtained as the adiabatic and pascalian restriction of the general thermodynamic scheme by C. Eckart, 1940, *Phys.Rev.* **58**, p.919-924. Its causal character was first proved by A. Lichnerowicz, 1965, *C.R.Acad.Sc.Paris* **260**, p.3291-3295.
- [6] B. Coll and J.J.Ferrando, 1989, *J. Math. Phys.* **30**, p. 2918- 2922.
- [7] Let us note that, because equation (5) is a minimal completion, the relation (6) may be *directly* obtained from the study of the propagation of infinitesimal perturbations by the system $\{(1),(2),(5)\}$. From this point of view, the well known relation $\left(\frac{\partial p}{\partial \rho}\right)_s$ appears only *as a constraint* for the *definition* of the entropy s .
- [8] With slight differences, we adopt here the notation of H. Quevedo and A. Sussman, 1995, *Class. Quantum Grav.* **12**, p. 859-874. Nevertheless, attention must be paid to the fact that a lot of their statements and expressions are erroneous; for details, see reference 2. [Note: after the publication of the present paper, these errors were globally corrected in A. Krasinski, H. Quevedo and R.A. Sussmann, 1997, *J. Math. Phys.* **28**, p. 2602-2610, and our "reference 2", which would need a consequent reworking, was never submitted to publication].