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Local thermal equilibrium and ideal gas Stephani universes

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Abstract The Stephani universes that can be interpreted as an ideal gas evolving in local thermal equilibrium are determined. Five classes of thermodynamic schemes are admissible, which give rise to five classes of regular models and three classes of singular models. No Stephani universes exist representing an exact solution to a *classical* ideal gas (one for which the internal energy is proportional to the temperature). But some Stephani universes may approximate a classical ideal gas at first order in the temperature: all of them are obtained. Finally, some features about the physical behavior of the models are pointed out.

Keywords Thermodynamics · Inhomogeneous cosmological models

1 Introduction

The second member of Einstein equations describes the energy content of the medium, but not the medium itself. So, exact solutions to Einstein equations correspond to prescribed forms of energy tensors, but they do not fix the nature of the particular (class of) physical fluid(s) that creates them.

Generically [1], the physical interpretation of an exact solution to Einstein equations, needs more than that of their *energy quantities*, i.e. more than that of the quantities that appear in the energy tensor: it is also necessary to 'close the context'. Let us see what this means.

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A non vacuum physical space-time is the history of *one* of the possible evolutions of a physical medium. But there are in general many physical media that can have in common one or some particular evolutions (for example, many fluids admit a rigid and static evolution with same energy density and pressure). In fact, from the evolution point of view, the definition of a medium is tantamount to the knowledge of the set of *all* its possible physical evolutions. Of course, this knowledge does not need to be *explicit*, it is sufficient that it be *implicit*, i.e. that it be done by the *set of equations* whose space of solutions is precisely the whole set of all its possible physical evolutions.

But according to a universal determinism requirement in Physics, such a set of equations has to be (causally) *closed*, i.e. has to admit a well posed initial value problem.

In general the conservation equations imposed by Einstein equations on the energy tensor (energy conservation equations) do not constitute a closed system of equations for the energy quantities. This is why we say that their physical interpretation is not sufficient to define the medium. To ‘close the context’ means here just to choose a *closure* for the energy conservation equations, i.e. to complete these last ones with some other physically meaningful equations in such a way to obtain a closed system.

The closure of the energy conservation equations usually [2] obliges to introduce new quantities, related at least in part to the Duhem-Gibbs balance equation, and called by this fact *thermodynamic quantities*.

Thus, for a *perfect energy tensor*,

$$T = (\rho + p)u \otimes u - pg, \quad (1)$$

where ρ is the *energy density*, p the *pressure* and u the (unit) *velocity* of the fluid, the usual closure [3] corresponds to the hypothesis of *local thermal equilibrium* (also called *standard thermodynamic scheme*). The local thermal equilibrium evolution of a fluid is thus determined by the following four conditions:

- Energy-momentum conservation: $\nabla \cdot T = 0$.
- The energy density ρ is decomposed in terms of the *matter density* r and the *specific internal energy* ϵ : $\rho = r(1 + \epsilon)$.
- The equation of conservation of matter is required: $\nabla \cdot (ru) = 0$.
- The thermodynamic quantities *temperature* T and *specific entropy* s are related by equations of state compatible with the thermodynamic Duhem-Gibbs relation: $Tds = d\epsilon + pd(1/r)$.

It is clear that this standard thermodynamic closure is a strict constraint on the whole space of solutions to the Einstein Equations for perfect energy tensors, i.e. *not all* the solutions of this last space can be interpreted as being local thermal equilibrium evolutions of a perfect fluid [4].

Observe that this constraint notably increases the number of variables because it involves not only the starting energy quantities (u, ρ, p) but also the new thermodynamic ones (r, ϵ, T, s). So that an important question arises: *is it possible to know, by the sole inspection of the energy quantities of a perfect energy tensor, if it describes the local thermal equilibrium evolution of a perfect fluid?*

As we showed some years ago [5], the answer is positive. Its precise formulation (denoting by $u(x)$ the directional derivative, with respect to u , of the quantity x) is given by the following

Theorem χ (local thermal equilibrium) *A divergence-free perfect energy tensor evolves in local thermal equilibrium if, and only if, the space-time function $\chi \equiv u(p)/u(\rho)$ depends only on the variables p and ρ : $d\chi \wedge dp \wedge d\rho = 0$.*

This result is strikingly interesting because it shows that, in spite of its name, of its historic origin and of its usual conceptualization, the notion of *local thermal equilibrium* for a perfect fluid is a *purely hydrodynamic*, not thermodynamic, notion [6].

The function χ is called the *indicatrix of local thermal equilibrium*. To every physical fluid in local thermal equilibrium, whatever its evolution be, corresponds a sole, particular, indicatrix (particular equation of state) which physically represents the square of the sound velocity in this fluid [7]. So that the direct application of theorem χ not only allows

- i) to detect, among the known families of solutions to Einstein equations for a perfect energy tensor, those sub-families that describe evolutions in local thermal equilibrium of physical fluids, but also
- ii) to separate these last sub-families in sub-sub-families that describe different (local thermal equilibrium) evolutions of *the same* physical fluid.

Conversely, theorem χ generates the first general method to know whether a solution for a perfect energy tensor is the evolution of a previously given fluid: it is sufficient to verify if the indicatrix function χ of the solution coincides with the square of the thermodynamic expression of the sound velocity of the given fluid expressed as a function of the variables p and ρ . In [7] we pointed out these applications and gave a concrete example studying the thermodynamic class II Szekeres–Szafron space-times.

In the present paper we consider, on one hand, the paradigmatic *ideal gas* as the previously given particular fluid and, on the other hand, the Stephani universes [8] as starting family of solutions to Einstein equations for a perfect energy tensor, both because of their relative generality and their frequent applications in cosmology and stellar interiors.

The Stephani universes admitting a generic thermodynamic scheme were obtained by Bona and Coll [9] and later revisited by Krasinski et al. [10]. In a recent paper, Sussman [11] has analyzed a family of spherically symmetric Stephani universes that may be interpreted as either a classical mono-atomic ideal gas or a matter-radiation mixture.

Here we are interested in the deductive and systematic determination of *all* Stephani universes that represent the evolution in local thermal equilibrium of a *generic* ideal gas and, among them, of those that admit a first order contact in the temperature with the *classical* ideal gas [12].

The main results of this work are based on previous ones: firstly, on our hydrodynamic characterization of an ideal gas in local thermal equilibrium [13], that we shorten here in Sect. 2; secondly, on the above cited work by Bona and Coll [9] about the thermodynamic schemes in Stephani universes, that we summarize in Sect. 3. In Sect. 4 we apply an algorithm introduced in Sect. 2 to a canonical form for the thermodynamic Stephani universes presented in Sect. 3 in order to obtain: i) a theorem (Theorem 1) that characterizes the generic ideal gas Stephani universes and ii) the associated indicatrix function and five classes of compatible ideal gas thermodynamic schemes. In Sect. 5 we integrate, up to a Friedmann-like

equation, the ideal gas equations obtained in theorem 1; this integration leads to different models with thermodynamics of a specific class. In Sect. 6 we study the Stephani ideal gas models that approximate a classical ideal gas at first order in the temperature. Finally, Sect. 5 is devoted to analyze how complementary physical requirements, like energy and compressibility conditions, restrict the domain where the models have a good physical behavior. A part of the results of this paper were stated without proof in [14].

2 Local thermal equilibrium evolution of an ideal gas

As already mentioned, Theorem χ provides an easy and general method (in fact the first known one) to answer if whether or not a solution to Einstein equations for a perfect energy tensor corresponds with a given thermodynamic fluid: to check that, as functions of p and ρ , the indicatrix function of the solution and the square of the sound velocity of the fluid coincide.

Elsewhere [13] we have given the hydrodynamic characterization of the sound velocity for some representative classes of fluids. We present here in two propositions the part of these results about ideal gases that we will use in this paper.

Remember that a perfect energy tensor is said *barotropic* if p and ρ are not independent, $dp \wedge d\rho = 0$, and that it is said *isoenergetic* if $u(\rho) = 0$. As we shall see in the next section, for the class of space-times analyzed here, the Stephani universes, the barotropic case leads to the already well known Friedmann-Robertson-Walker models, meanwhile the isoenergetic case is not possible because of the non vanishing expansion of these fluids.

A (generic) ideal gas satisfies the equation of state $p = krT$. For it we have shown the following hydrodynamic characterization [13]:

Proposition 1 *The necessary and sufficient condition for a non barotropic and non isoenergetic divergence-free perfect energy tensor (1) to represent the local thermal equilibrium evolution of an ideal gas is that its indicatrix function $\chi \equiv u(p)/u(\rho)$ be a non identity function of the variable $\pi \equiv p/\rho$:*

$$d\chi \wedge d\pi = 0, \quad \chi \neq \pi \quad (2)$$

Proposition 2 *A non barotropic and non isoenergetic divergence-free perfect fluid energy tensor verifying (2) represents the local thermal equilibrium evolution of the ideal gas with specific internal energy ϵ , temperature T , matter density r , and specific entropy s given by*

$$\epsilon(\rho, p) = \epsilon(\pi) \equiv e(\pi) - 1, \quad T(\rho, p) = T(\pi) \equiv \frac{\pi}{k} e(\pi), \quad (3)$$

$$r(\rho, p) = \frac{\rho}{e(\pi)}, \quad s(\rho, p) = k \ln \frac{f(\pi)}{\rho}, \quad (4)$$

$e(\pi)$ and $f(\pi)$ being, respectively,

$$e(\pi) = e_0 \exp \left\{ \int \psi(\pi) d\pi \right\}, \quad \psi(\pi) \equiv \frac{\pi}{(\chi(\pi) - \pi)(\pi + 1)}; \quad (5)$$

$$f(\pi) = f_0 \exp \left\{ \int \phi(\pi) d\pi \right\}, \quad \phi(\pi) \equiv \frac{1}{\chi(\pi) - \pi}. \quad (6)$$

The above two propositions provide a complete algorithm, in four steps, to detect and characterize, in any given family of divergence-free perfect energy tensors $\mathbf{T} = \{\mathbf{T} \equiv [u^\alpha(x^\beta), \rho(x^\beta), p(x^\beta)]\}$, those that represent an ideal gas evolving in local thermal equilibrium:

- Step 1** Calculate the coordinate dependence of the space-time functions $p/\rho = \pi(x^\beta)$ and $u(p)/u(\rho) = \chi(x^\beta)$
- Step 2** Determine the ideal gas subset of $\mathbf{T}$ by imposing the ideal gas hydrodynamic condition (2): $d\chi \wedge d\pi = 0$.
- Step 3** Obtain, in this subset, the explicit expression of the indicatrix function: $\chi = \chi(\pi)$.
- Step 4** Calculate, from $\chi(\pi)$, the *generating functions* $e(\pi)$ and $f(\pi)$ given in (5) and (6), and obtain from them the thermodynamic variables by using (3) and (4).

3 Thermodynamic Stephani universes

The conformally flat perfect energy tensor solutions to Einstein equations with nonzero expansion were first considered by Stephani [8] and are usually called *Stephani universes*. They were rediscovered by Barnes [15], who obtained them as the conformally flat class of irrotational and shear-free perfect fluid space-times with nonzero expansion.

In order to generalize the cosmological principle, Bona and Coll [9] looked for space-times admitting an iso-invariant synchronization, and without any hypothesis on the energy tensor, they likewise found the Stephani universes.

An iso-invariant synchronization amounts the existence of an irrotational and shear-free observer u with non zero expansion such that the induced metrics on the orthogonal hypersurfaces are of constant curvature. Then, there exists an adapted coordinate system, $u = (1/\alpha)\partial_t$, such that the line element takes the form:

$$ds^2 = -\alpha^2 dt^2 + \Omega^2 \delta_{ij} dx^i dx^j \quad (7)$$

where

$$\alpha \equiv R \partial_R \ln \Omega, \quad \Omega \equiv \frac{R(t)}{1 + 2 \bar{b}(t) \cdot \bar{r} + \frac{1}{4} K(t) r^2} \quad (8)$$

$R(t)$, $\bar{b}(t)$ and $K(t)$ being five arbitrary functions of time.

The metric (7) is a solution to the perfect energy tensor of the form (1), with energy density and pressure given by

$$\rho = \frac{3}{R^2} (\dot{R}^2 + K - 4b^2), \quad p = -\rho - \frac{R}{3} \frac{\partial_R \rho}{\alpha} \quad (9)$$

Moreover, the expansion of u and the curvature of the spatial metric are homogeneous and respectively given by

$$\theta(t) = \frac{3\dot{R}}{R} \neq 0, \quad \kappa(t) = \frac{1}{R^2} (K - 4b^2) \quad (10)$$

The conditions under which Stephani universes describe the evolution of a fluid in local thermal equilibrium, were directly obtained by Bona and Coll in [9],

and later considered by Krasinski et al. [10]. The results that we need from [9] are here summarized in the following lemmas.

Lemma 1 *The necessary and sufficient condition for a Stephani universe to represent the evolution of a fluid in local thermal equilibrium is that it admit a three-dimensional isometry group.*

A particular class of evolutions in local thermal equilibrium is that of *barotropic evolutions*; in our case, they are easily determined:

Lemma 2 *A Stephani universe represents the barotropic evolution of a fluid iff it admits a six-dimensional isometry group (Friedmann–Robertson–Walker space-time).*

The following lemma states, in thermodynamical terms, that Stephani universes do not admit isometry groups of dimension 4 or 5.

Lemma 3 *The dimension of the maximal isometry group of a Stephani universe that represents the non barotropic evolution of a fluid is three.*

This paper deals with Stephani universes describing non barotropic evolutions of fluids in local thermal equilibrium. In what follows, they will be called for short *thermodynamic Stephani universes*.

A canonical form for them has also been obtained in [9]. From it, and matching the two space-like coordinates r and z in a sole variable, say w , a straightforward calculation leads to:

Proposition 3 *The metric of the thermodynamic Stephani universes may be written*

$$ds^2 = -\alpha^2 dt^2 + \Omega^2 \delta_{ij} dx^i dx^j \quad (11)$$

where

$$\begin{aligned} \alpha &\equiv R \partial_R \ln L, & \Omega &\equiv \frac{w}{2z} L \\ L &\equiv \frac{R(t)}{1 + b(t)w}, & w &\equiv \frac{2z}{1 + \frac{\varepsilon}{4} r^2} \end{aligned} \quad (12)$$

$R(t)$ and $b(t)$ being two arbitrary functions of time.

Its symmetry group is spherical, plane or pseudospherical depending on ε to be 1, 0 or -1 and the Friedmann–Robertson–Walker limit occurs when $b = \text{constant}$.

Furthermore, the energy density and pressure are given by

$$\rho = \frac{3}{R^2} (\dot{R}^2 + \varepsilon - 4b^2), \quad p = -\rho - \frac{R}{3} \frac{\partial_R \rho}{\alpha} \quad (13)$$

4 Ideal gas Stephani universes

Among the thermodynamic Stephani universes, what are those corresponding to ideal gases?

In order to determine them, we shall study the restrictions that our above ideal gas condition (2) imposes on the functions $R(t)$ and $b(t)$ that characterize the general thermodynamic Stephani universes (11) (12), following step by step the algorithm presented at the end of Sect. 2.

Step 1 Coordinate dependence of π and χ

From the second equation in (13), a direct calculation leads to:

$$\frac{p}{\rho} = \pi(R, w) = \frac{a}{\alpha} - 1 \quad (14)$$

where α is given in (12) and a is defined by

$$a = a(R) \equiv -\frac{R\partial_R \rho}{3\rho}, \quad (15)$$

Now we look for the indicatrix function χ . From (14) and (15), we have

$$\chi \equiv \frac{u(p)}{u(\rho)} = \frac{\partial_R p}{\partial_R \rho} = \frac{1}{\partial_R \rho} \partial_R(\rho\pi) = \pi - \frac{R}{3a} \partial_R \left(\frac{a}{\alpha} \right) \quad (16)$$

but, from the definition of α in (12) it follows

$$\alpha = \alpha(R, w) \equiv \frac{1 + (b - Rb')w}{1 + bw}, \quad (17)$$

where x' stands for dx/dR for any $x = x(R)$. Then, from (14) and (17) it results:

$$\partial_R \alpha = \left(\frac{a}{\pi + 1} - 1 \right) \left(\frac{b''}{b'} + \frac{a}{R(\pi + 1)} \right)$$

and substituting in (16), we obtain:

Lemma 4 *For the thermodynamic Stephani universes, the thermodynamic variable $\pi \equiv p/\rho$ and the indicatrix function $\chi \equiv u(p)/u(\rho)$ take the expressions:*

$$\pi = \pi(R, w) = \frac{a(1 + bw)}{1 + (b - Rb')w} - 1, \quad (18)$$

$$\chi = \chi(\pi, R) \equiv \pi + \frac{1}{3} - \frac{1}{3}(\pi + 1) \left[\frac{a'R}{a^2} + \frac{1}{a} + (\pi + 1 - a) \frac{Rb''}{a^2 b'} \right] \quad (19)$$

where a is given by (15) and the prime indicates derivative with respect to the variable R .

Step 2 Ideal gas hydrodynamic condition: $d\chi \wedge d\pi = 0$

Expression (19) shows that, in agreement with the local thermal equilibrium condition, the indicatrix function χ depends on the only energetic variables ρ and p , $\chi = \chi(\rho, p)$. Indeed, (19) can be written:

$$\chi = \chi(\pi, R) \equiv \pi + \frac{1}{3} + \frac{1}{3}(\pi + 1)[(\pi + 1)A_1(R) + A_2(R)] \quad (20)$$

where $\pi = p/\rho$ and where

$$A_1(R) \equiv -\frac{Rb''}{a^2 b'}, \quad A_2(R) \equiv \frac{Rb''}{ab'} - \frac{a'R}{a^2} - \frac{1}{a} \quad (21)$$

Thus, ρ being an effective function of R , the functions A_1 and A_2 can be considered as depending on ρ . Every choice of these two functions determines different indicatrix functions that correspond to different media. How we must take A_1 and A_2 in the ideal gas case?

The thermodynamic Stephani universes describing non barotropic evolutions of fluids in local thermal equilibrium are such that the function $\pi(R, w)$ in (18) satisfies $\partial_w \pi \neq 0$. Consequently condition $d\chi \wedge d\pi = 0$ is equivalent to $\partial_R \chi(\pi, R) = 0$. From (20) this equation may be written:

$$A'_1(R)(\pi + 1) + A'_2(R) = 0 \quad (22)$$

with $A_1(R)$ and $A_2(R)$ given by (21).

But, π and R being independent variables, from (22) it follows that the functions in (21) are constant, $A_1(R) = c_1$, $A_2(R) = c_2$. From now on, c_1 and c_2 will be called the *principal constants*.

Moreover we can eliminate the function $b(R)$ in the second of equations (21) by using its values from the first one. Thus, in a first step, we obtain two equations for the two functions $b(R)$, and $\rho(R)$, this last one being related to $a(R)$ by (15). In a second step, we put every solution $b(R)$, $\rho(R)$ for these equations in the expression (13) of the energy density ρ in order to obtain an equation for the expansion factor $R(t)$, which determines then the corresponding Stephani universe. More precisely, we can state:

Theorem 1 *A thermodynamic Stephani universe (11) (12) represents an ideal gas if, and only if, the metric function $b(R)$ and the energy density $\rho(R)$ satisfy the two equations:*

$$Ra' = -a(c_1 a^2 + c_2 a + 1), \quad b''R = -c_1 a^2 b' \quad (23)$$

where $a(R)$ is given in (15) and the principal constants c_1 and c_2 are arbitrary. Then, in terms of $b(R)$ and $\rho(R)$, the expansion factor $R(t)$ satisfies the generalized Friedmann equation:

$$\rho(R) = \frac{3}{R^2}[\dot{R}^2 + \varepsilon - 4b^2(R)] \quad (24)$$

A first glance at Eqs. (23), (24) allows us to know that the ideal gas Stephani universes are, generically, a family of solutions depending on seven parameters, namely: the two principal constants c_1 and c_2 (we will see below that they determine five classes with different thermodynamical properties), the initial values a_0 , ρ_0 and R_0 for the functions $a(R)$, $\rho(R)$ and $R(t)$ and, finally, two integration constants b_1 and b_2 giving the general solution $b(R)$ to the second equation in (23). In next section we study in detail Eqs. (23), (24) and we will analyze the properties of the models that we obtain depending on these seven parameters.

Step 3 Indicatrix function: $\chi = \chi(\pi)$

Now we look for the explicit expression of the indicatrix function. Assuming Eqs. (23), the indicatrix (19) becomes a second degree polynomial in π whose coefficients depend on the principal constants c_1 and c_2 . More precisely, we have:

Proposition 4 *The indicatrix function $\chi(\pi)$ of the ideal gas Stephani universes is of the form:*

$$\chi(\pi) = \beta\pi^2 + \gamma\pi + \delta, \quad (25)$$

where β, γ and δ are constants related to c_1 and c_2 by

$$\beta = \frac{c_1}{3}, \quad \gamma = 1 + \frac{1}{3}(c_2 + 2c_1), \quad \delta = \frac{1}{3}(c_1 + c_2 + 1). \quad (26)$$

Step 4 Thermodynamic variables

Finally, the expression (25) of the indicatrix function $\chi(\pi)$ determines, according to Proposition 2, the other thermodynamic variables, and so, the thermodynamical properties of the fluid. Expressions (3) and (4) show that the thermodynamic variables are determined by the generating functions $e(\pi)$ and $f(\pi)$ given by (5) and (6) respectively. Consequently, the thermodynamic scheme depends on the reduced indicatrix $\bar{\chi}$:

$$\bar{\chi}(\pi) \equiv \chi(\pi) - \pi = \beta\pi^2 + \bar{\gamma}\pi + \delta, \quad (27)$$

with $\bar{\gamma} \equiv \gamma - 1$, and β, γ and δ given by (26). Then, depending on the degree and the roots of $\bar{\chi}(\pi)$ we find five different classes of ideal gas Stephani universes and, for every class, the integrals (5) and (6) admit simple analytic expressions. We summarize the results in the following:

Proposition 5 *From a thermodynamical point of view and according to the values of the principal constants c_1 and c_2 , the ideal gas Stephani universes may belong to five classes. For every one of them, the reduced indicatrix $\bar{\chi}(\pi)$ and the generating functions $e(\pi)$ and $f(\pi)$ are given by*

$$\text{CLASS 1 } (c_1 = c_2 = 0) \quad \bar{\chi}(\pi) = \frac{1}{3}$$

$$f(\pi) = f_0 \exp\{3\pi\}, \quad e(\pi) = \frac{e_0}{(\pi + 1)^3} \exp\{3\pi\} \quad (28)$$

$$\text{CLASS 2 } (c_1 = 0, c_2 \neq 0) \quad \bar{\chi}(\pi) = \frac{1}{3}[(c_2(\pi + 1) + 1)]$$

$$\begin{aligned} f(\pi) &= f_0 [(c_2(\pi + 1) + 1)]^{\frac{3}{2}} \\ e(\pi) &= \frac{e_0}{(\pi + 1)^3} [(c_2(\pi + 1) + 1)]^{3(1 + \frac{1}{c_2})} \end{aligned} \quad (29)$$

$$\text{CLASS 3 } (\Delta \equiv c_2^2 - 4c_1 = 0, c_1 \neq 0) \quad \bar{\chi}(\pi) = \frac{c_2}{12}[(c_2(\pi + 1) + 2)]^2$$

$$\begin{aligned} f(\pi) &= f_0 \exp \left\{ \frac{-12}{c_2[c_2(\pi + 1) + 2]} \right\} \\ e(\pi) &= e_0 \frac{[c_2(\pi + 1) + 2]^3}{(\pi + 1)^3} \exp \left\{ \frac{-6(c_2 + 2)}{c_2[c_2(\pi + 1) + 2]} \right\} \end{aligned} \quad (30)$$

CLASS 4 ($\Delta \equiv c_2^2 - 4c_1 > 0$, $c_1 \neq 0$) $\bar{\chi}(\pi) = \frac{c_1}{3}(\pi - \pi_+)(\pi - \pi_-)$

$$\begin{aligned} f(\pi) &= f_0 \left(\frac{\pi - \pi_+}{\pi - \pi_-} \right)^{\frac{3}{\sqrt{\Delta}}}, \quad \pi_{\pm} \equiv -1 + \frac{1}{2c_1}(-c_2 \pm \sqrt{\Delta}) \\ e(\pi) &= e_0 \frac{(\pi - \pi_+)^{\lambda_+} (\pi - \pi_-)^{\lambda_-}}{(\pi + 1)^3}, \quad \lambda_{\pm} \equiv \frac{3}{2} \left[1 \pm \frac{c_2 + 2}{\sqrt{\Delta}} \right] \end{aligned} \quad (31)$$

CLASS 5 ($\Delta \equiv c_2^2 - 4c_1 < 0$) $\bar{\chi}(\pi) = \frac{1}{3}[c_1\pi^2 + (2c_1 + c_2)\pi + c_1 + c_2 + 1]$

$$\begin{aligned} f(\pi) &= f_0 \exp \left\{ \frac{2}{3\sqrt{-\Delta}} \arg \tan \frac{2c_1(\pi + 1) + c_2}{\sqrt{-\Delta}} \right\} \\ e(\pi) &= e_0 \frac{[\bar{\chi}(\pi)]^{3/2}}{(\pi + 1)^3} \exp \left\{ \frac{c_2 + 2}{3\sqrt{-\Delta}} \arg \tan \frac{2c_1(\pi + 1) + c_2}{\sqrt{-\Delta}} \right\} \end{aligned} \quad (32)$$

The complete space-time ideal models, which will be obtained in next section by integrating Eqs. (23), depend on seven parameters. The above proposition shows that, among these seven parameters, the principal constants c_1 and c_2 are the two thermodynamical ones.

5 Ideal gas Stephani models

The above section has obtained and explored the indicatrix function χ for ideal gases in Stephani universes, and constructed the generating functions $f(\pi)$ and $e(\pi)$ that directly give the thermodynamic variables. Here we complete the study by the exploration of the conditions (23) for the existence of the indicatrix, which involve the metric functions.

The study of these conditions (23) leads to different models that depend, generically, on seven parameters as pointed out above. The principal constants c_1 and c_2 distinguish the five classes of ideal gas Stephani universes already obtained in Proposition 5. But the analysis of the equations also distinguishes two families of models defined by the fact that the function $a(R)$ given in (15) may be constant or not. Now we consider these two cases separately.

5.1 Singular models: $a'(R) = 0$

If $a(R) = a_0 \neq 0$, the first equation in (23) implies that the parameter a_0 of the model depends on the thermodynamic parameters, the principal constants c_1 and c_2 . In these cases, we can easily integrate (15) with $a(R) = a_0$ to obtain $\rho(R)$. On the other hand, the second equation in (23) leads to $b' = CR^{-c_1 a_0^2}$, C being a constant. Then we obtain:

Theorem 2 *In the singular ideal gas Stephani models, $a(R) = a_0$,*

i) *the energy density is given by:*

$$\rho(R) = \rho_0 \left(\frac{R_0}{R} \right)^{3a_0} \quad (33)$$

where ρ_0 and R_0 are constants, and

ii) the metric function $b(R)$ is given by one of the following expressions

$$b(R) = b_1 + b_2 \left(\frac{R}{R_0} \right)^\mu \quad \text{if } \mu \neq 0 \quad (34)$$

$$b(R) = b_1 + b_2 \ln \left(\frac{R}{R_0} \right) \quad \text{if } \mu = 0. \quad (35)$$

where $\mu \equiv 1 - c_1 a_0^2$, and $b_1, b_2 \neq 0$ are arbitrary constants.

As a consequence of (23), the relationship between the constants a_0, μ and c_1, c_2 is given by:

$$c_1 = \frac{1}{a_0^2}(1 - \mu), \quad c_2 = \frac{1}{a_0}(\mu - 2) \quad (36)$$

These expressions show that both, a_0 and μ , are thermodynamical parameters. Moreover, one can easily see how the above cases are related to the five classes of ideal gas Stephani universes established in Proposition 5.

When $\mu = 0$ one has $\Delta \equiv c_2^2 - 4c_1 = 0$, and these cases belong to Class 3. When $\mu = 1$ one has $c_1 = 0, c_2 \neq 0$, and they belong to Class 2. All the other singular cases belong to Class 4, because $\Delta = (\mu/a_0)^2 > 0$. Thus, if we take into account Proposition 5 and (36) we can state:

Proposition 6 *The singular ideal gas models, $a(R) = a_0$, given in Theorem 2, belong to the following classes:*

$$\begin{aligned} &\text{if } \mu = 1, \text{ to Class 2, with } c_2 = -\frac{1}{a_0}. \\ &\text{if } \mu = 0, \text{ to Class 3, with } c_2 = -\frac{2}{a_0}. \\ &\text{if } 0 \neq \mu \neq 1, \text{ to Class 4, with } c_1 \text{ and } c_2 \text{ given by (36)}. \end{aligned}$$

5.2 Regular models: $a'(R) \neq 0$

Equations (23) are equivalent to Eqs. (21) with $A_1 = c_1$ and $A_2 = c_2$; differentiating the first of these equations and considering the second one, one finds:

$$\frac{a''}{a'} - \frac{2a'}{a} - \frac{b''}{b'} = 0 \quad (37)$$

From this equation one can obtain b as a function of a . But this relation and the first equation in (23) imply the second one. Thus, we have:

Proposition 7 *The ideal gas Stephani universes with $a'(R) \neq 0$ are those for which*

$$Ra' = -a(c_1 a^2 + c_2 a + 1), \quad b(R) = \frac{b_2}{a(R)} + b_1 \quad (38)$$

where c_1, c_2, b_1 and $b_2 \neq 0$ are arbitrary constants.

Consequently, for regular ideal gas models, one needs to solve the first order differential Eq. (38) for the function $a(R)$. The solutions can be obtained in an implicit form $h(a, R) = 0$ depending on the degree and roots of the polynomial $q(a) \equiv c_1 a^2 + c_2 a + 1$. But an implicit expression for $a(R)$ does not allow to determine explicit expressions for $\rho(R)$ (solution to (15)) and $b(R)$, and these expressions are necessary in order to establish the Friedmann Eq. (24) and to obtain the metric tensor by solving it. We can overcome this shortcoming by a change of variables. Indeed, a been an effective function of R , the Eq. (15) can be written

$$\frac{1}{\rho} \frac{d\rho}{da} = -\frac{3a}{R} \frac{dR}{da} = \frac{3}{c_1 a^2 + c_2 a + 1}, \quad (39)$$

and consequently, the energy density takes the expression

$$\rho(a) = \rho_0 Q^3(a) \quad (40)$$

where

$$Q(a) \equiv \exp \left\{ \int \frac{da}{q(a)} \right\}, \quad q(a) \equiv c_1 a^2 + c_2 a + 1 \quad (41)$$

On the other hand, the first Eq. in (38) takes the form $aq(a)dR/da = R$, and its solutions depend on a simple function of $q(a)$ and $Q(a)$. Finally, Friedmann Eq. (24) may be considered as a differential equation on $a(t)$ by changing:

$$\dot{R} = \frac{dR}{da} \dot{a} = \frac{R}{aq(a)} \dot{a} \quad (42)$$

All these considerations lead to the following:

Theorem 3 *The regular ideal gas Stephani models, $a'(R) \neq 0$, are those for which the metric functions R and b are of the form*

$$R(a) = \frac{R_0}{a} \sqrt{|q(a)Q^{c_2}(a)|}, \quad b(a) = \frac{b_2}{a} + b_1 \quad (43)$$

where $Q(a)$ and $q(a)$ are given by

$$Q(a) \equiv \exp \left\{ \int \frac{da}{q(a)} \right\}, \quad q(a) \equiv c_1 a^2 + c_2 a + 1 \quad (44)$$

and $a = a(t)$ is a solution to the generalized Friedmann Eq.

$$\rho_0 Q^3(a) = \frac{3\dot{a}^2}{a^2 q^2(a)} + \frac{3}{R^2(a)} [\varepsilon - b^2(a)], \quad (45)$$

c_1, c_2, b_1 and $b_2 \neq 0$ being arbitrary constants.

The analytic expression of the function $Q(a)$ depends on the polynomial $q(a)$. Thus we can distinguish five different models depending on the values of the constants c_1 and c_2 . These cases coincide with the five classes of ideal gas Stephani universes established in Proposition 5. The integration of these five cases leads to simple analytic expressions for the function $Q(a)$:

Proposition 8 *According to the class to which the ideal gas Stephani universe belongs, the functions $q(a)$ and $Q(a)$, that determine by (43) the metric function R when $a'(R) \neq 0$, are given by:*

$$\text{CLASS 1 } (c_1 = c_2 = 0) \quad q(a) = 1$$

$$Q(a) = \exp\{a\} \quad (46)$$

$$\text{CLASS 2 } (c_1 = 0, c_2 \neq 0) \quad q(a) = c_2 a + 1$$

$$Q(a) = (c_2 a + 1)^{1/c_2} \quad (47)$$

$$\text{CLASS 3 } (\Delta \equiv c_2^2 - 4c_1 = 0, c_1 \neq 0) \quad q(a) = \frac{1}{4}(c_2 a + 2)^2$$

$$Q(a) = \exp\left\{\frac{4}{c_2(c_2 a + 2)}\right\} \quad (48)$$

$$\text{CLASS 4 } (\Delta \equiv c_2^2 - 4c_1 > 0, c_1 \neq 0) \quad q(a) = c_1(a - a_+)(a - a_-)$$

$$Q(a) = \left(\frac{a - a_+}{a - a_-}\right)^{\frac{1}{\sqrt{\Delta}}}, \quad a_{\pm} \equiv \frac{1}{2c_1}(-c_2 \pm \sqrt{\Delta}) \quad (49)$$

$$\text{CLASS 5 } (\Delta \equiv c_2^2 - 4c_1 < 0) \quad q(a) = c_1 a^2 + c_2 a + 1$$

$$Q(a) = \exp\left\{\frac{2}{\sqrt{-\Delta}} \arg \tan \frac{3(2c_1 a + c_2)}{\sqrt{-\Delta}}\right\}. \quad (50)$$

6 Approximate classical ideal gas

Until now we have considered *generic* ideal gases characterized by the equation of state $p = krT$. In this section we consider *classical* ideal gases which also satisfy the energetic equation of state $\epsilon = c_v T$, and we analyze when a Stephani universe represents a classical ideal gas in local thermal equilibrium. This energetic equation of state restricts the indicatrix function $\chi(\pi)$ of the ideal gases. More precisely, we have shown elsewhere [7]:

Proposition 9 *The indicatrix function of a classical ideal gas takes the expression:*

$$\chi(\pi) = \frac{\gamma_a \pi}{1 + \pi} \quad (51)$$

where $\gamma_a = 1 + \frac{k}{c_v}$, is the adiabatic index, $1 < \gamma_a < 2$.

If a perfect fluid has the indicatrix function (51), then it represents a classical ideal gas with specific energy:

$$e(\pi) = \frac{\gamma_a - 1}{\gamma_a - 1 - \pi} \quad (52)$$

From this result and Proposition 4 we can state: *a thermodynamic Stephani universe never represents in a exact way a classical ideal gas in local thermal equilibrium*. Nevertheless we should take into account that, in a relativistic framework, the classical ideal gas approximation works for low temperatures. Thus, we can look for ideal gas schemes that satisfy the energetic equation of state of a classical ideal gas at first order in T . This fact can be analyzed by studying the expressions (3) and their first and second derivatives for small T . Then, taking into account (5) and Proposition 9, we obtain

Proposition 10 *The necessary and sufficient condition for an ideal gas to have, at first order in the temperature, the energetic equation of state of a classical ideal gas, $\epsilon(T) = c_v T + o(T^2)$, is that the indicatrix function $\chi(\pi)$ and the specific energy $e(\pi)$ satisfy:*

$$\chi(0) = 0, \quad \chi'(0) = \gamma_a; \quad e(0) = 1 \quad (53)$$

Now we can study the compatibility between (53) and the Stephani ideal gas indicatrix function (25) at first order in π , and we easily obtain $\delta = 0$ and $\gamma = \gamma_a$. Moreover, this means that the adiabatic index γ fixes the parameters c_1 and c_2 of the classical ideal gas models. More precisely we have:

Proposition 11 *If an ideal gas Stephani universe represents a classical one at first order in the temperature, then the indicatrix function can be written:*

$$\chi(\pi) = \left(\gamma - \frac{2}{3} \right) \pi^2 + \gamma \pi, \quad 1 < \gamma < 2 \quad (54)$$

and the principal constants are the following functions of γ :

$$c_1 = 3\gamma - 2 > 1, \quad c_2 = 1 - 3\gamma < -2 \quad (55)$$

This proposition implies that, if we impose on the thermodynamic parameters c_1 and c_2 the conditions (55), the ideal gas indicatrix function $\chi = \chi(\pi)$ satisfies (53). Then, only the models of Class 4 in Proposition 5 are admitted in this case. Indeed, from (55) we obtain:

$$c_1 \neq 0, \quad \Delta = (c_1 - 1)^2 = 9(\gamma - 1)^2 > 0 \quad (56)$$

which agree with the conditions of Proposition 5 that define the ideal gas Stephani universes of Class 4. Moreover, if we determine the parameters $\pi_{\pm}$ and $\lambda_{\pm}$ of the scheme (31) in terms of the adiabatic index γ and we fix the constant e_0 in order to have $e(0) = 1$, we can state:

Proposition 12 *For the Stephani universes, the ideal gas thermodynamic schemes which approximate, at first order in the temperature, a classical one with adiabatic index γ , are generated by the functions*

$$f(\pi) = f_0 \left[\frac{\pi}{(\gamma - 2/3)\pi + (\gamma - 1)} \right]^{\frac{1}{\gamma-1}}, \quad e(\pi) = \left[1 + \frac{\pi}{3(\gamma - 1)(\pi + 1)} \right]^3 \quad (57)$$

6.1 Classical singular models

As a consequence of Proposition 6 the singular models of Class 4 satisfy $0 \neq \mu \neq 1$. Then, the relations (36) and the restrictions (55) lead to either $a_0 = 1$ and $\mu = 3(1 - \gamma) < 0$, or $a_0 = 1/(3\gamma - 2)$ and $\mu = 1 - a_0 > 0$. Then, if we take into account Theorem 2, we can state:

Proposition 13 *There are two families of singular models that approximate, at first order in the temperature, a classical ideal gas with adiabatic index γ :*

MODELS $a_0 = 1$

$$\rho(R) = \rho_0 \left(\frac{R_0}{R} \right)^3, \quad b(R) = b_1 + b_2 \left(\frac{R_0}{R} \right)^{3(\gamma-1)} \quad (58)$$

MODELS $a_0 \neq 1$

$$\rho(R) = \rho_0 \left(\frac{R_0}{R} \right)^{\frac{1}{\gamma-2/3}}, \quad b(R) = b_1 + b_2 \left(\frac{R}{R_0} \right)^{\frac{\gamma-1}{\gamma-2/3}} \quad (59)$$

6.2 Classical regular models

Proposition 8 and relations (56) imply that the models of Class 4 are the sole regular models compatible with the classical ideal gas approximation in question. If we write the parameters of the model in terms of γ by using (55), we obtain the following result:

Proposition 14 *For the regular models in Theorem 3 that approximate, at first order in the temperature, a classical ideal gas with adiabatic index γ , the functions $q(a)$, $Q(a)$, and $R(a)$ take the expression:*

$$\begin{aligned} q(a) &= (a-1)[(3\gamma-2)a-1], \quad Q(a) = \left[\frac{a-1}{(3\gamma-2)a-1} \right]^{\frac{1}{3(\gamma-1)}}, \\ R(a) &= \frac{R_0}{a} \left[\frac{[(3\gamma-2)a-1]^{3\gamma-2}}{a-1} \right]^{\frac{1}{3(\gamma-1)}} \end{aligned} \quad (60)$$

7 On the physical behavior of the models

In this work we have presented Stephani models that describe evolutions in local thermal equilibrium of a generic ideal gas (Sect. 5) or a classical ideal gas (Sect. 6). The space-time domain where these models have a good physical behavior are defined by complementary physical requirements.

Firstly, we have the Plebański energy conditions that, for a perfect fluid, state $-\rho < p \leq \rho$. In terms of the hydrodynamic variable $\pi = p/\rho$ these conditions take the form $-1 < \pi \leq 1$. Nevertheless, in the ideal gas case it is reasonable to consider positive pressures p . Thus we should impose on π :

$$0 < \pi \leq 1 \quad (61)$$

On the other hand, the relativistic compressibility conditions [16] should be also required. Elsewhere [13] we have shown that these conditions can be written in terms of hydrodynamic variables by means of the indicatrix function $\chi(\rho, p)$, and that for an ideal gas they become [13]:

$$\frac{\pi}{2\pi + 1} < \chi(\pi) < 1, \quad (\chi(\pi) - \pi)\chi'(\pi) > -\frac{2\chi(\pi)(1 - \chi(\pi))}{(\pi + 1)} \quad (62)$$

These conditions imposed on the indicatrix function (25) of the ideal gas Stephani universes imply the restriction of the parameters c_1 and c_2 which determine the thermodynamical properties of the models. Moreover, given a pair of values $(\tilde{c}_1, \tilde{c}_2)$ physically compatible, the compressibility conditions (62) only hold for values of π on a subset of the interval $]0, 1]$. This means that the corresponding thermodynamic scheme is only well defined for these values of π . Thus, for example, for a model of class 1 (that is, $c_1 = c_2 = 0$), conditions (62) hold if $\pi \in]0, \frac{2}{3}]$.

For a classical ideal gas the compressibility conditions (62) hold for every $\pi \in]0, 1[$ if we assume that the adiabatic index $\gamma \in]1, 2[$. Nevertheless, for the Stephani models studied in Sect. 6 which approximate a classical ideal gas at first order in the temperature, these conditions only hold for $\pi \in]0, \pi_m[$, where the maximum value of π depends on the adiabatic index as $\pi_m = \frac{2}{\gamma + \sqrt{\gamma^2 + 4(\gamma - 2/3)}} < 1$. It is worth remembering that this approximation is valid only for small values of π . Thus for π near of the maximum value π_m , these models have a good physical behavior but they do not approximate a classical ideal gas.

Once obtained the admissible values of the hydrodynamic variable π for the different classes, we can look for the space-time domains where every model is physically reasonable. We should analyze the expression (18) that gives π in terms of the spatial coordinate ω and the metric function $R(t)$. It is worth remarking that ω , as defined in (12), is a bounded coordinate in the case of spherical symmetry ($\varepsilon = +1$) and, otherwise, it is not bounded. On the other hand, depending on the behavior of $R(t)$ we could have models only valid at early or at present times. But all these analysis will be considered elsewhere.

Finally, we want to comment about the relationship between the results by Sussman in [11] and ours. His models approximate a classical ideal gas in some specific cases that, in our notation, correspond to the spherically symmetric ($\varepsilon = 1$) singular case (58) with $a_0 = 1$ and $b_1 = 0$. In spite of this restriction, Sussman's paper presents a wide analysis on the space-time domains and observational features concerning the cases of a classical mono-atomic ideal gas ($\gamma = 5/3$) and a matter-radiation mixture ($\gamma = 4/3$).

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References

1. Generically means here, in a technical sense, *at least for all the energy tensors containing more than four variables*, as it will become clear in the next paragraph
2. All the causal closures that have been proposed in the literature for arbitrary fluids are (or may be related to) thermodynamical closures, originated in relativity from many different

approaches and giving rise to very different results. They essentially started with Eckart's third paper on thermodynamics of irreversible processes, and their contrast with Landau and Lifchitz's point of view produced a great number of new propositions, more or less decanted over the works by Israel, Steward and Marle, and subsequently over the relativistic version of extended thermodynamics

3. Apart from the *barotropic* one. The *barotropic closure* for a perfect fluid energy tensor states that the energy density ρ is a function of the sole pressure p , $\rho = f(p)$. It remains that, very frequently, this relation is considered as an *equation of state* consequence of the rheology of the fluid (what is of relatively little interest), better than as a particular evolution restriction (which offers a rich range of interest). See [13] for this concept
4. By this and the other reasons mentioned above, in the rest of the paper to the appellations 'perfect fluid space-times' or 'perfect fluid solutions' (to Einstein equations), we prefer those of 'perfect energy tensor space-times' or 'perfect energy tensor solutions'
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6. An important consequence of the purely hydrodynamic physical meaning of the local thermal equilibrium is that it suggests a precise *hierarchy of irreversibility levels*, the lowest one being the usual evolution in local thermal equilibrium and the successive other levels corresponding to increasing irreversibility. But this aspect will be analysed elsewhere
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