Tempered stable GARCH models

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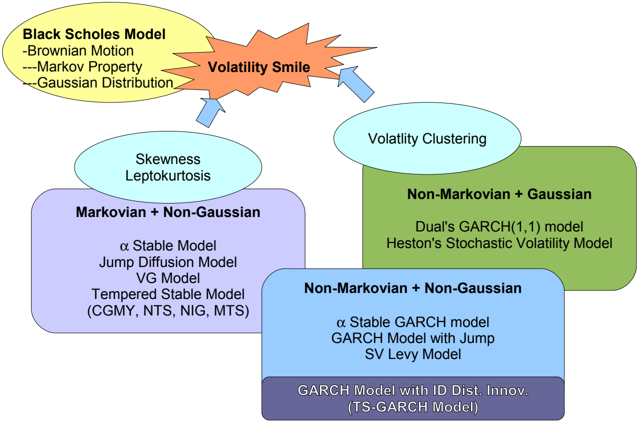
Overview


Black Scholes Model
-Brownian Motion
---Markov Property
---Gaussian Distribution
Volatility Smile
Skewness
Leptokurtosis
Volatlity Clustering
Markovian + Non-Gaussian
α Stable Model
Jump Diffusion Model
VG Model
Tempered Stable Model
(CGMY, NTS, NIG, MTS)
Non-Markovian + Gaussian
Dual's GARCH(1,1) model
Heston's Stochastic Volatility Model
Non-Markovian + Non-Gaussian
α Stable GARCH model
GARCH Model with Jump
SV Levy Model
GARCH Model with ID Dist. Innov.
(TS-GARCH Model)

α-stable and ID distributions

The characteristic functions:

- α-stable $\begin{cases} \exp\left(imu - \sigma^\alpha |u|^\alpha \left(1 - i\beta \mathrm{sgn}(u) \tan\left(\frac{\pi\alpha}{2}\right)\right)\right) & \text{if } \alpha \neq 1 \\ \exp\left(imu - \sigma|u| \left(1 + i\beta \mathrm{sgn}(u) \left(\frac{2}{\pi}\right) \ln(|u|)\right)\right) & \text{if } \alpha = 1 \end{cases}$
- CTS: $\exp[ium + \Gamma(-\alpha)\{C_1((\lambda_+ - iu)^\alpha - \lambda_+{}^\alpha) + C_2((\lambda_- + iu)^\alpha - \lambda_-{}^\alpha)\}]$
- MTS: $\exp(imu + G_R(u; \alpha, C, \lambda_+, \lambda_-) + G_I(u; \alpha, C, \lambda_+, \lambda_-))$
- KR-TS: $\exp(ium + H_\alpha(u; k_+, r_+, p_+) + H_\alpha(-u; k_-, r_-, p_-))$

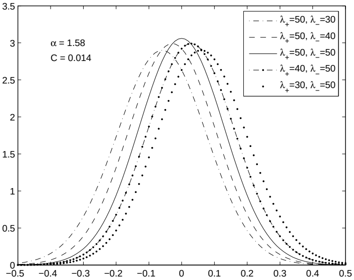
$$G_R(u; \alpha, C, \lambda_+, \lambda_-) = \sqrt{\pi} C 2^{-(\alpha+3)/2} \Gamma\left(-\frac{\alpha}{2}\right) \left((\lambda_+^2 + u^2)^{\alpha/2} - \lambda_+^\alpha + (\lambda_-^2 + u^2)^{\alpha/2} - \lambda_-^\alpha\right)$$

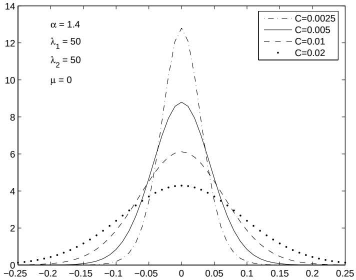
$$G_I(u; \alpha, C, \lambda_+, \lambda_-) = \frac{iuC\Gamma\left(\frac{1-\alpha}{2}\right)}{2^{\frac{\alpha+1}{2}}} \left(\lambda_+^{\alpha-1} F\left(1, \frac{1-\alpha}{2}; \frac{3}{2}; -\frac{u^2}{\lambda_+^2}\right) - \lambda_-^{\alpha-1} F\left(1, \frac{1-\alpha}{2}; \frac{3}{2}; -\frac{u^2}{\lambda_-^2}\right)\right)$$

$$H_\alpha(u; x, y, p) = \frac{x\Gamma(-\alpha)}{p} \left(F(p, -\alpha; 1+p; iyu) - 1\right)$$

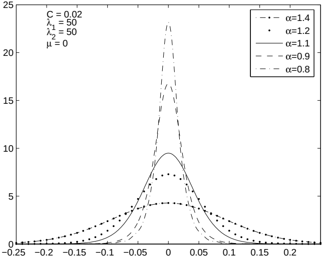
$C, C_1, C_2, \lambda_\pm, r_\pm, k_\pm > 0$, $p_\pm > -1/2$, $\alpha \in (0, 2)$, $m \in \mathbb{R}$, $\sigma > 0$ and $\beta \in [-1, 1]$.

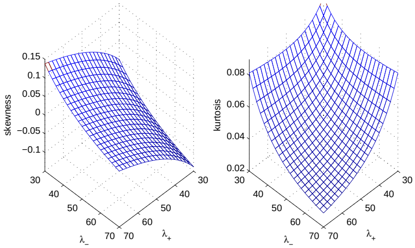
Example: MTS distributions





Example: MTS distributions



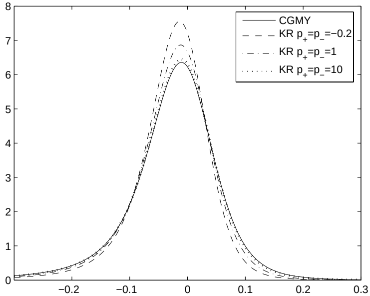


Properties of the tempered stable distributions

- They (CTS, MTS, and KR) have finite moments of all orders.
- They have finite exponential moments on some real interval.
- They have heavier tails than the normal distribution, and thinner than α stable distribution.
- By give appropriate values to parameters, they can have zero means and unit variance : standard CTS, standard MTS, standard KR distributions.
- The innovation process of GARCH model can be assumed to be the standard CTS (MTS, or KR) distributions : CTS-GARCH, MTS-GARCH, and KR-GARCH models.

Properties of the tempered stable distributions

The KR distribution converges weakly to the CTS distribution as $p_\pm \to \infty$ provided that $k_\pm = c(\alpha + p_\pm)r_\pm^{-\alpha}$ where $c > 0$.



standard TS distribution

- standard CTS distribution :

$$C = C_1 = C_2 := \left(\Gamma(2-\alpha)(\lambda_+{}^{\alpha-2} + \lambda_-{}^{\alpha-2})\right)^{-1}$$

$$m := -\Gamma(1-\alpha)(C_1\lambda_+{}^{\alpha-1} - C_2\lambda_-{}^{\alpha-1})$$

- standard MTS distribution :

$$C := 2^{\frac{\alpha+1}{2}} \left(\sqrt{\pi}\Gamma\left(1-\frac{\alpha}{2}\right)\left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right)\right)^{-1}$$

$$m := -\frac{\Gamma\left(\frac{1-\alpha}{2}\right)\left(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}\right)}{\sqrt{\pi}\Gamma\left(1-\frac{\alpha}{2}\right)\left(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}\right)}$$

- standard KR distribution:

$$m := -\Gamma(1-\alpha)\left(\frac{k_+r_+}{p_+ + 1} - \frac{k_-r_-}{p_- + 1}\right)$$

$$k_+ := c(\alpha + p_+)r_+^{-\alpha}, k_- := c(\alpha + p_-)r_-^{-\alpha},$$

where

$$c = \frac{1}{\Gamma(2-\alpha)}\left(\frac{\alpha + p_+}{2 + p_+}r_+^{2-\alpha} + \frac{\alpha + p_-}{2 + p_-}r_-^{2-\alpha}\right)^{-1}$$

GARCH Model with Infinitely Divisible distributed innovations

Stock Price Dynamics:

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r_t - d_t + \lambda_t\sigma_t - L(\sigma_t) + \sigma_t\varepsilon_t,$$
$$\sigma_t^2 = (\alpha_0 + \alpha_1\sigma_{t-1}^2\varepsilon_{t-1}^2 + \beta_1\sigma_{t-1}^2) \wedge \rho, \quad \varepsilon_0 = 0,$$

- $L(x) = \log(E[e^{x\varepsilon_t}])$ defined on the interval $(-a, b)$ and $0 < \rho < b^2$.
- Normal GARCH model : ε_t has the standard normal distribution.
- CTS(MTS, KR)-GARCH model : ε_t has the standard CTS(MTS, KR) distribution.

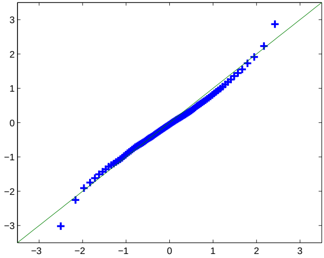
Historical parameter estimation

Table: Statistic of the goodness of fit tests

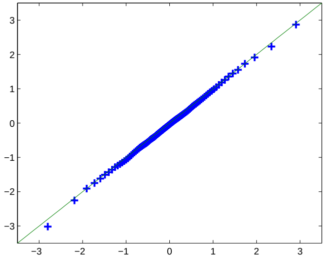
Ticker	Model	KS(*p*-value)	χ^2(*p*-value)	AD
SPX	normal-GARCH	0.0311 (0.0158)	323.4665 (0.4979)	401.9810
	CTS-GARCH	0.0278 (0.0414)	304.2689 (0.6123)	0.6835
	MTS-GARCH	0.0276 (0.0436)	310.1418 (0.5191)	0.7354
	KR-GARCH	0.0277 (0.0426)	307.9739 (0.5378)	**0.0595**
IBM	normal-GARCH	0.0547 (0.0000)	413.5590 (0.0000)	52009.9121
	CTS-GARCH	0.0223 (0.1656)	262.0601 (0.1462)	0.4819
	MTS-GARCH	0.0194 (0.3022)	237.5442 (0.4595)	0.5646
	KR-GARCH	0.0217 (0.1892)	262.2524 (0.1248)	**0.0459**
INTC	normal-GARCH	0.0239 (0.1150)	352.4107 (0.0003)	133934.5383
	CTS-GARCH	0.0221 (0.1746)	300.8394 (0.0169)	0.3973
	MTS-GARCH	0.0221 (0.1718)	307.0700 (0.0090)	0.3715
	KR-GARCH	0.0220 (0.1765)	299.7535 (0.0152)	**0.0496**
MSFT	normal-GARCH	0.0330 (0.0086)	329.1551 (0.0009)	12223.5241
	CTS-GARCH	0.0143 (0.6838)	226.5456 (0.6761)	0.9734
	MTS-GARCH	0.0149 (0.6331)	229.7076 (0.6383)	1.2923
	KR-GARCH	0.0121 (0.8558)	227.6621 (0.6222)	**0.0540**
AMZN	normal-GARCH	0.0894 (0.0000)	367.7290 (0.0000)	204.7783
	CTS-GARCH	0.0222 (0.5718)	181.5525 (0.1791)	0.3310
	MTS-GARCH	0.0181 (0.8090)	176.6796 (0.2531)	0.3729
	KR-GARCH	0.0213 (0.6229)	178.3598 (0.1943)	**0.0583**

Normal-GARCH vs TS-GARCH

Normal

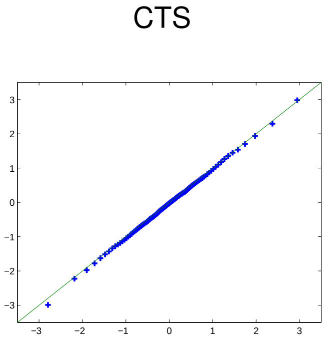


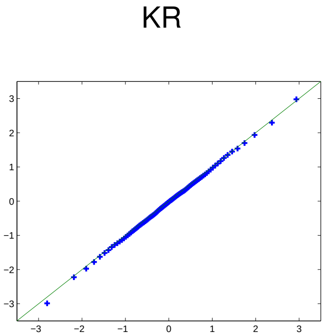
MTS



Data : Daily return for IBM

Normal-GARCH vs TS-GARCH

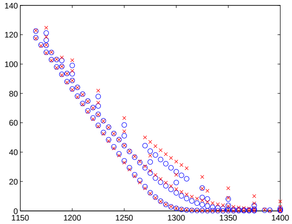




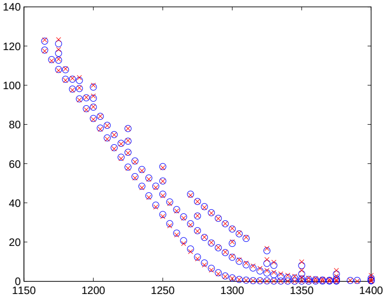
Option prices

Call prices (March 10, 2006) with multi-maturities

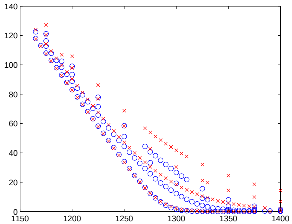
1. Normal-GARCH



2. CTS-GARCH



3. KR-GARCH



Implied volatility

Errors between the market and model prices

[Time to maturity : 1 week, date : March 10, 2006]

	APE	AAE	RMSE	ARPE
Normal-GARCH	0.014214	0.47237	0.60438	0.27075
CTS-GARCH	0.014994	0.49829	0.62068	0.30319
KR-GARCH	0.0098485	0.3273	0.38034	0.28453