Portfolio Choice with Accounting Concerns

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Abstract

The present work analyzes in a dynamic setting the consequences of using different accounting regimes - Historic Cost (HC) *vs.* Fair Value (FV) - for the optimal choice of a financial portfolio, when the owner - a generic Financial Institution - is interested in consumption (dividends) for two periods, and two types of assets are available in the economy: one risky and one risk-free. Comparing with the theoretical optimal portfolio decisions (First Best), we find that both regimes lead to inefficiencies, but FV is ex-ante worse than HC in terms of consumption smoothing and the welfare loss is higher for the companies concerned with long-term business than for those with short-term horizons. Similarly, the ex-ante consumption level for the non-terminal period is worse than the First Best value for both accounting regimes, with FV consumption less than the HC one. When the risky asset is illiquid and/or costs associated with transacting it are relevant to be taken into account, the ex-ante consumption smoothing superiority of the HC regime to the FV one is not always true but depends on the risky asset patterns (expected return and variance) and the transaction costs amount.

Keywords: fair value, accounting regimes, portfolio choice

1 Introduction

The recent introduction of IAS 32, IAS 39 and IFRS 4 affects 7,000 EU listed companies and firms from other 100 countries. This makes the analysis of the Fair Value (FV) measurement concept a crucial research topic. The debate on the attractiveness of the FV regime is still unsolved. On one hand, the standard setters advocate the use of the FV reporting. On the other hand, financial firms, especially banks and insurance companies, defend Historic Cost (HC) accounting. However, there is a shortage of analytical research devoted to the comparative analysis of FV versus HC accounting. This research is strongly needed in order to correctly assess the costs and benefits of these two accounting regimes.

The aim of our research is to investigate whether the adoption of FV accounting has real effects for financial institutions. In particular, we want to investigate whether the choice of the accounting regime (HC vs. FV) affects firms' portfolio selection methodology. Our analysis will enable us to check whether the HC regime encourages the "profit smoothing" activity mentioned by the European Central Bank (2004). Moreover, we will investigate whether the FV system has the capacity to properly reflect the way in which important financial institutions- banks and insurance companies- should manage their business. These institutions should be dedicated especially to long-term decisions and should be less concerned with short-term fluctuations. The adoption of FV accounting can have a negative impact on their activity and shortening their planning horizons (cf. Geneva Association (2004)).

Only few analytical works were dedicated to this topic. O'Hara (1993), Burkhardt and Strausz (2004), Freixas and Tsomocos (2004) and Plantin et al. (2007) have tackled the issue. Using different settings, they all show that the alleged superiority of the "new" FV regime with respect to the "old" HC is highly questionable.

Compared with these studies, our approach differs in two ways: first, we work in a more general setting with financial instruments following commonly accepted patterns in finance theory. For this reason our results apply to a richer class of financial institutions. Second, our analysis is dynamic and allows for firms' reactions to the arrival of new information. This is an essential issue when studying the consequences of adopting different accounting regimes for financial institutions with a long-term orientation.

The structure of the paper is the following: in Section 2 we introduce the general framework. Section 3 describes the First Best (FB) solution. In section 4 we study the consequences of introducing either an HC or a FV accounting regime. Section 5 presents the comparison between the two regimes. Section 6 draws the conclusions.

2 The Model

We assume that there exist a Financial Institution (FI) endowed with I_0 at $T = 0$. The objective of the FI is to maximise its owners' future consumption at dates $T = 1$ and $T = 2$. We represent these consumption level as $c_1$and c_2 respectively[1]. We can think of the FI as an Institutional Investor that has an interest in smoothing the future consumption of the owners[2]. The FI does not receive any new endowment at future dates and is able to generate consumption for its owners only through the investments available in the economy (i.e. it can follow only a self-financing strategy).

At $T = 0$ the FI maximises the following time-separable utility function

[1] This is a simplification of a model with Financial Institutions living for n periods (accounting years)

[2] We abstract from differences between ownership and control in the FI

$$\max \quad E_0\{\delta u(c_1) + \delta^2 u(c_2)\} \tag{1}$$

subject to the different restrictions, in particular those implied by the accounting regime.

The utility $u(.)$ belongs to the general class of CRRA (constant relative risk aversion) utility functions[3]. In particular we work with the *log* utility version of the CRRA utilities family, i.e. $u(c_i) = \log(c_i)$, for $i = 1, 2$. The desirable consequence of using the *log* utility is that it leads to relatively tractable analytical results when coupled with the assumption of log-normality of asset returns.

The time-discount factor $\delta \in (0, 1]$ in the objective function (1) accounts for the relative importance of inter-temporal consumption: a δ close to 0 represents a FI more interested in the short-term, while a δ close to 1 represents a FI equally concerned with consumption for all the periods of its life[4]. As usual with financial assets, storage is not a problem. Hence the FI always prefers (or at least it is not worse) early earnings to late ones. This is modeled by asking the time-discount factor δ to belong to the interval $(0, 1]$.[5]

The general characteristics of our objective function makes our FI particularly interested in consumption smoothing, a concept generally found in the "banking literature". Hence our model mimicks the behavior of these financial institutions[6]. The utility function used chosen implies that $\log(0) = -\infty$, i.e. that the owners ask for a positive amount of consumption at $T = 1$, otherwise "dying" if 0 consumption is provided. In this sense we are dealing with an "impatient" set of owners. We also assume that owners' needs (i.e. the weights δ and δ^2) are known ex-ante. Hence, we are not allowing for surprises (uncertainty) in terms of liquidity needs. As long as we are making an ex-ante analysis, stochastic weights can be easily incorporated in our model instead of constant ones, but this complicates the computations without any intuitive benefits.

Finally, contrary to Burkhardt and Strausz (2004), we do not distinguish explicitly between long-term and short-term projects (assets) in our model, with the long-term ones having a superior rate of return, and then penalizing for their premature liquidation. Our limited rebalancing possibilities are equivalent to costly rebalancing restrictions and can be viewed as a premature liquidation of long-term assets, where both our risky and risk-free assets are a priori long-term.

[3] As Campbell and Viceira (2002) remarks, the CRRA utility functions are "*inherently attractive*" and are "*required to explain the stability of financial variables in the face of secular economic growth*" : *investors are willing to pay almost the same relative costs to avoid given relative risks as they did when they were much poorer, which is possible only if relative risk aversion is almost independent of wealth.*"

[4] The key role played by the parameter δ in interpreting the results is similar to Plantin, Sapra and Shin (2007)

[5] Models with general discount factors not necessary equal with $\frac{1}{1+R_1^f}$ are used in the literature. (see Pliska (1997) e.g.)

[6] See Freixas and Tsomocos (2004)

2.1 Financial Instruments available in the economy

There are only two asset types available for investment in the economy: one risky and one risk-free. As usual, the risky asset is expected to bring a higher expected return than the risk-free one. For simplicity, we assume that the *fair value* of these instruments can be easily determined in our model, i.e. there is a unique available market price for the financial instruments[7].

We take as given the two asset returns, and we assume that their evolution follows an ex-ante known stochastic rule: log-normality[8]. Our FI acquires the financial instruments in the secondary market, where the assets have their well defined price. These markets are not affected by the trading activity of the FI. We also assume that the accounting information disclosed (i.e. the value of the portfolio under different accounting regimes) does not affect assets prices, because it does not introduce any new information in the market[9].

The previous analytical papers on FV accounting already show that the relevant differences between accounting regimes appear when imperfections exist. In particular, in our model we assume that asset prices are perfectly known at any moment, but that there exist frictions (transaction costs) when selling/buying the assets (especially the risky ones). This implies limitations to the possibilities for re-balancing the portfolio[10] Using the terminology of Plantin et al. (2007), we assume liquid and "hard" secondary markets. Without transaction costs, the differences between accounting regimes would become irrelevant, as one can liquidate the financial portfolio at the end of each period (registering the cash value, the same under any accounting regime) and then re-buying the desired financial portfolio at the beginning of the next period, and so on.

However, contrary to Plantin et al. (2007), in the present work the assets illiquidity (limited absorption for sales in the secondary markets leading to "beauty contests") is not the primary cause of distinct portfolio choices: we obtain distinct portfolio allocations under HC and FV even for "small" imperfections (transactions costs leading to limited rebalancing possibilities).

In Section 5 we introduce the case of illiquid risky assets (viewed as assets for which transaction costs are relevant) and we show how the results obtained for liquid risky assets are changing.

2.2 Assumptions about portfolio rebalancing

We analyze the behavior of a FI living for more than one year, taking into account some aspects of modern portfolio management, such as the possibility

[7] We don't question here the also strongly debated weak point of the FV accounting regime, i.e. that fair value of an instrument cannot be always reliably determined

[8] The same approach is followed by Campbell and Viceira (2002) when designing portfolio strategies

[9] In Plantin, Sapra and Shin (2007), and in Burkhardt and Strausz (2004) the accounting information disclosed affects the degree of liquidity and implicitly the price of the asset.

[10] We are inspired by the example of secondary markets for banks and insurance, where, due to information asymmetry, there exists restrictions when re-balancing a portfolio.

of rebalancing the portfolio in each period[11].

We do not quantify transactions costs explicitly in our model. We simply assume the existence of sufficiently high transaction costs such that the following rebalancing restrictions hold[12]:

1) there is a single possibility[13] for rebalancing the portfolio (and it appears in our model at $T = 1$[14], when new information arrives.)

2) it is not possible to hold cash from one period to the other; this means that, at any moment, the portfolio is composed only by risky and risk-free assets; in our setting it is not efficient to hold cash as long as risk-free assets are available in the economy (and they are better than cash), they are perfectly liquid at $T = 1, 2$ the only moments when claims for consumption can be made.

3) the rebalancing activity consists of selling assets, buying other assets and consuming part of the assets. Importantly, we asked to not re-buy the assets already sold.

4) short-selling of assets is not allowed. Pollack (1986) states that "on NYSE the total volume of short selling is around 8 percent of total volume", and only 1.5 percent is undertaken by non-members of the exchange, i.e. our FI (not active investors). Legal restrictions for the functioning of FIs whose stability is a social concern is another argument why short sales should not be available. Also Hull (2003) reminds us that "*regulators in the United States currently allow a stock to be shorted only on an uptick-that is, when the most recent movement in the price of the stock was an increase.*"

2.3 Decisions

According to the objective function and the restrictions about rebalancing portfolio provided above, the FI has to take 3 decisions in our model:

Decision 1: at $T = 0$ it selects a portfolio composed by α_1 risky assets and β_1 risk-free ones, denoted (α_1, β_1), using the entire endowment I_0, and the whole information available at $T = 0$. The prices at $T = 0$ (uniquely determined) of the two assets are respectively X_0 and 1, such that we have:

$$\alpha_1 X_0 + \beta_1 = I_0 \tag{2}$$

This portfolio is held until $T = 1$. Immediately before $T = 1$ (i.e. at the time called $T = 1_-$) this portfolio values:

[11] This is one of the reasons the Financial Institutions are created: "to rebalance portfolios on behalf of investors who find this task costly to execute" :Campbell and Viceira (2002)

[12] One can view our FI as a non active investor, due to transaction costs.

[13] This assumption is not very restrictive: the unique transaction can be viewed as being composed by different transactions, in the same accounting year, realized with a single (average or $T = 1$) price; the only requirement is not to sell and re-buy the same asset in the same accounting year.

[14] strictly speaking, it is the *new information* about the financial assets available in the interval $(T = 1_-, T = 1_+)$

$$\alpha_1 X_1 + \beta_1(1 + R_1^f) = W_1 \tag{3}$$

where R_1^f represents the risk-free interest rate corresponding to the first period.[15]

Decisions 2: at $T = 1$, according to the *new information* about the assets prices, FI will choose a level of consumption c_1 and consequently the level of re-investment Inv_1.

Decisions 3: at $T = 1$ FI will re-balance the Inv_1 amount into the new portfolio (α_2, β_2).

2.4 Assumptions about asset returns

We assume that the asset gross return $1 + R_1^x = \dfrac{X_1}{X_0}$ is log-normally distributed (where $R_1^x = \dfrac{X_1 - X_0}{X_0}$ represents the asset net return).

This is a common assumption when dealing with financial (liquid) assets and helps us to obtain closed form solutions when coupled with log-utility[16].

We consider the following "initial parameters", which determine the shapes of the expected utility curves: the risk-free interest rate for the first period R_1^f, known at $T = 0$; the expected value of the net return of the risky asset for the first period $E_0(R_1^x)$ and the variance of the natural logarithm of X_1, $\sigma_0^2 = Var_0(\log(X_1))$.

We denote by $r_1^x = \log(1 + R_1^x)$ and $r_1^f = \log(1 + R_1^f)$ the log-returns of the risky, respectively risk free assets, for the first period and $\mu = E_0(r_1^x)$. Values for the second period are defined in a a similar way: R_2^f, known at $T = 1$, $E_1(R_2^x)$ the expected return of the risky asset for the second period after learning the updated information at $T = 1$, $\sigma_1^2 = Var_1(\log(X_2))$, $r_2^x = \log(1 + R_2^x)$ and $r_2^f = \log(1 + R_2^f)$.

Finally, we consider the values $q_1 = \frac{E_0 r_1^x - r_1^f}{\sigma_0^2}$, $q_2 = \frac{E_1 r_2^x - r_2^f}{\sigma_1^2}$. They can be written in terms of our "initial parameters", as $q_1 = \frac{1}{\sigma_0^2} \log \frac{1+E_0 R_1^x}{1+R_1^f} - \frac{1}{2}$, and $q_2 = \frac{1}{\sigma_1^2} \log \frac{1+E_1 R_2^x}{1+R_2^f} - \frac{1}{2}$ (see Annex point 1). Importantly, we ask the following parametric restrictions: $q_1 \in (-\frac{1}{2}, \frac{1}{2})$, $q_2 \in (-\frac{1}{2}, \frac{1}{2})$.[17]

We describe at this point the approach we followed to quantify the expected returns of such portfolios and the expected consumption. Considering the portfolio composed by α_1 risky and β_1 risk-free assets, we denote by $\alpha_1^* = \frac{\alpha_1 X_0}{I_0}$ the

[15] At $T = 0$ we are not asking for a similar decision of splitting the initial resources I_0 between consumption and investment, but we invest all the resources I_0. It can be solved very easy the similar problem by going one-step back in our dynamic portfolio choice problem.

[16] The same approach is followed by Campbell and Viceira (2002)

[17] It will be clear in Section 3. This is equivalent with asking that the FB portfolio distribution to have an "interior solution", i.e. to contain a positive number of both risky and risk-free assets (as long as short-selling is not allowed in our model).

share of the initial endowment I_0 invested in the risky assets (and $\alpha_2^* = \frac{\alpha_2 X_1}{Inv_1}$ similarly for the second period).

The following formula for the gross returns of the portfolio can be derived[18]

$$R_1^p + 1 = \alpha_1^*(R_1^x + 1) + (1 - \alpha_1^*)(1 + R_1^f) \tag{4}$$

or equivalently in the log-form:

$$\log(R_1^p + 1) = \log[\alpha_1^*(R_1^x + 1) + (1 - \alpha_1^*)(1 + R_1^f)]$$

Following Campbell and Viceira (2002), we use discrete approximation of the gross returns. These authors points out that "*as the time interval shrinks , the non-lognormality of the portfolio return diminishes, and it disappears altogether in the limit of continuous time*", making the following approximation an exact equality:

$$\log(R_1^p + 1) = \alpha_1^* \log(R_1^x + 1) + (1 - \alpha_1^*) \log(1 + R_1^f) + \frac{1}{2}\alpha_1^*(1 - \alpha_1^*)\sigma_0^2$$

or, re-written with our notations:

$$r_1^p = r_1^f + \alpha_1^*(r_1^x - r_1^f) + \frac{1}{2}\alpha_1^*(1 - \alpha_1^*)\sigma_0^2 \tag{5}$$

and a similar formula for the second period:

$$r_2^p = r_2^f + \alpha_2^*(r_2^x - r_2^f) + \frac{1}{2}\alpha_2^*(1 - \alpha_2^*)\sigma_1^2 \tag{6}$$

Expression (6) and the fact that consumption at the end of the second period is equal to the terminal wealth, i.e. $c_2 = W_2$, allows us to use the following equivalent form of the initial objective function (1):

$$\delta E_0\{\log(c_1) + \delta \log(W_1 - c_1) + \delta(\frac{-\sigma_1^2}{2})\alpha_2^{*^2} + \delta(q_2 + \frac{1}{2})\sigma_1^2\alpha_2^* + \delta r_2^f\} \tag{7}$$

We will use this form for the rest of the analysis[19].

To simplify the evaluation of the expected utility at $T = 0$, we assume that the best estimators at $T = 0$ of the "initial parameters" for the second period are $E_0(R_2^x) = E_0(R_1^x)$, $E_0\left(\sigma_1^2\right) = \sigma_0^2$ and respectively $E_0\left(r_2^f\right) = r_1^f$. We also assume that: $E_0(q_2) = q_1$[20]

[18] See details of the computations for this section in Annex point 2

[19] Note that (7) is not an *exact* replacement of (1), because it depends on approximation (6).

[20] This is equivalent with saying that $E_0(\alpha_2^{*FB}) = \alpha_1^{*FB}$, the proportions of risky assets in the FB portfolio. For this expression the concept of best estimators of the *initial parameters* is not enough, because it is a complex function. Hence, we have to assume it as a block.

3 "First Best" without Accounting Restrictions

The FI is interested in maximizing its expected utility at $T = 0$:

Problem FB0

$$
\begin{aligned}
&\max E_0\{\delta \log(c_1) + \delta^2 \log(c_2)\} \\
&\text{s. t.} \alpha_1 \geq 0 \\
&\quad \beta_1 \geq 0 \\
&\quad c_1 \geq 0 \\
&\quad c_1 \leq W_1 \\
&\quad \alpha_2 \geq 0 \\
&\quad \beta_2 \geq 0
\end{aligned}
$$

Consistent with our comments from Section 2.4, we replace the objective function (1) by the approximated version (7), and instead of solving **Problem FB0** for $\alpha_1, \beta_1, c_1, \alpha_2$ and β_2, we equivalently express it in terms of α_1, β_1, c_1 and α_2^* - the proportion of the investment in the risky asset for the second period.

Accordingly, at $T = 0$ the FI solves the following problem:

Problem FB0'

$$
\begin{aligned}
&\max \delta E_0\{\log(c_1) + \delta \log(W_1 - c_1) + \delta(\frac{-\sigma_1^2}{2})\alpha_2^{*^2} + \delta(q_2 + \frac{1}{2})\sigma_1^2 \alpha_2^* + \delta r_2^f\} \\
&\text{s. t.} \alpha_1 \geq 0 \\
&\quad \beta_1 \geq 0 \\
&\quad c_1 \geq 0 \\
&\quad c_1 \leq W_1 \\
&\quad \alpha_2^* \geq 0 \\
&\quad \alpha_2^* \leq 1
\end{aligned}
$$

This is the main task of the present section. One can remark that another merit of the approximation (7) is to reduce the number of unknowns from 5 (like in **Problem FB0**) to 4, in **Problem FB0'**.

However, solving **Problem FB0'** in this form is not obvious: there are 4 variables (α_1, β_1, c_1 and α_2^*) corresponding to the decisions that the FI has to make at different time moments ($T = 0$ and 1). For this reason, we employ the technique of dynamic programming: backwards analysis. We fix the "trajectory" of decisions up to one point (i.e. we consider we already made the

Decision 1 of choosing a pair (α_1, β_1)) and then we solve for the optimal path starting with that point (i.e. starting at $T = 1$). Later we move one step back and so on. In our case, assuming we have fixed the initial decision (α_1, β_1) at $T = 0$, this leads to the following problem the FI solves at $T = 1$ for deciding c_1 and α_2^* (corresponding to the given (α_1, β_1) pair).[21]

ProblemFB1 *Suppose the FI has chosen a fixed arbitrary pair (α_1, β_1) of risky, respectively risk-free assets at $T = 0$ and it has to decide the consumption c_1 and the way to redistribute the assets for the second period (the proportion α_2^* or similarly the numbers of assets (α_2, β_2)). At $T = 1$ it is known the value X_1 (also W_1) and the FI solves the following maximization problem*:

$$\max \delta E_1\{\log(c_1) + \delta \log(W_1 - c_1) + \delta(\frac{-\sigma_1^2}{2})\alpha_2^{*^2} + \delta(q_2 + \frac{1}{2})\sigma_1^2\alpha_2^* + \delta r_2^f \mid_{(\alpha_1,\beta_1)}\}$$

s. t. $c_1 \geq 0$

$c_1 \leq W_1$

$\alpha_2^* \geq 0$

$\alpha_2^* \leq 1$

The solutions of this problem are: $\alpha_2^{*FB}(\alpha_1, \beta_1) = \frac{E_1 r_2^x - r_2^f}{\sigma_1^2} + \frac{1}{2} = q_2 + \frac{1}{2}$[22]
$c_1^{FB}(\alpha_1, \beta_1) = \frac{W_1}{1+\delta}$.

Proof: see Annex point 3

Now we move one step back to find the optimal starting pair (recall we obtained, by solving **Problem FB1,** the optimal path when given a fixed (α_1, β_1), hence we know how to optimally continue for any starting (α_1, β_1) chosen at $T = 0$.)

We are able to solve **Problem FB0'** (re-phrased as **Problem FB0"**) by replacing $c_1 = c_1^{FB}(\alpha_1, \beta_1)$ and $\alpha_2^* = \alpha_2^{*FB}(\alpha_1, \beta_1)$, the solutions of **Problem FB1**:

Problem FB0"

$$\max \delta E_0\{\log(c_1^{FB}) + \delta \log(W_1 - c_1^{FB}) + \delta(\frac{-\sigma_1^2}{2})\alpha_2^{*FB^2} + \delta(q_2 + \frac{1}{2})\sigma_1^2\alpha_2^{*FB} + \delta r_2^f\}$$

s. t. $\alpha_1 \geq 0$

$\beta_1 \geq 0$

[21] We also follow the dynamic programming approach when solving the cases with accounting restrictions

[22] We forced the parameters q_1 and q_2 such that the proportion $\alpha_2^*(\alpha_1, \beta_1) \in [0, 1]$ (one can see at this point why we asked $q_2 = \frac{E_1 r_2^x - r_2^f}{\sigma_1^2} \in (-\frac{1}{2}, \frac{1}{2})$

$q_2 = \frac{E_1 r_2^x - r_2^f}{\sigma_1^2} \in (-\frac{1}{2}, \frac{1}{2}) \Leftrightarrow \alpha_2^{*FB} = \frac{E_1 r_2^x - r_2^f}{\sigma_1^2} + \frac{1}{2} \in (0, 1))$

The result is $\alpha_1^{FB} = \frac{\alpha_1^{*FB} I_0}{X_0}$, $\beta_1^{FB} = (1 - \alpha_1^{*FB}) I_0$ where $\alpha_1^{*FB} = q_1 + \frac{1}{2} \in (0, 1)$.

Proof: Annex point 4

We conclude the analysis of the FB case in the next proposition.

Proposition 1 *a) Optimal Decisions*

A FI endowed with I_0 at $T = 0$ has to make the following optimal decisions in order to maximize the expected utility of consumption:

Decision 1

First, the FI chooses at $T = 0$, (α_1, β_1) as

$$\alpha_1^{FB} = \frac{\alpha_1^{*FB} I_0}{X_0} \tag{8}$$

$$\beta_1^{FB} = (1 - \alpha_1^{*FB}) I_0 \tag{9}$$

where

$$\alpha_1^{*FB} = \frac{E_0 r_1^x - r_1^f + \frac{\sigma_0^2}{2}}{\sigma_0^2} = q_1 + \frac{1}{2} \tag{10}$$

At $T = 1$ the optimal decisions 2 and 3 are:

Decision 2

Consumes:

$$c_1^{FB} = W_1 \frac{1}{1 + \delta} \tag{11}$$

Reinvests :

$$Inv_1^{FB} = W_1 \frac{\delta}{1 + \delta} \tag{12}$$

Decision 3

The reinvested quantity is optimally distributed as (α_2, β_2), where

$$\alpha_2^{FB} = \frac{\alpha_2^{*FB} Inv_1^{FB}}{X_1} \tag{13}$$

$$\beta_2^{FB} = \frac{(1 - \alpha_2^{*FB}) Inv_1^{FB}}{1 + R_1^f} \tag{14}$$

and

$$\alpha_2^{*FB} = \frac{E_1 r_2^x - r_2^f + \frac{\sigma_1^2}{2}}{\sigma_1^2} \tag{15}$$

b) Expected Utility

With the optimal decisions described at a), the ex-ante (at $T = 0$) expected utility of our FI is:

$$E_0^{FB}\{\delta \log(c_1) + \delta^2 \log(c_2)\} =$$
$$= (\delta+\delta^2)\left[\log(I_0) + \left(\frac{\delta}{1+\delta}\log(\delta) - \log(\delta+1)\right) + \frac{1+2\delta}{1+\delta}\left(r_1^f + \frac{1}{2}\sigma_0^2(q_1+\frac{1}{2})^2\right)\right] \quad (16)$$

Proposition 2 *a) Expected Consumption*

*A FI endowed with I_0 and following the optimal decisions described in **Proposition 1, a)** expects at $T = 0$ the following level of consumption c_1^{FB} for the moment $T = 1$:*

$$E_0(c_1^{FB}) = \frac{1}{1+\delta} I_0[(\frac{1}{2} - q_1)(1 + R_1^f) + (\frac{1}{2} + q_1)e^{\mu+\frac{\sigma_0^2}{2}}]; \quad (17)$$

b) Expected Number of Transacted Assets at $T = 1$

*A FI endowed with I_0 and following the optimal decisions described in **Proposition 1, a)** expects at $T = 0$ to transact at $T = 1$ the following number of risky assets:*

$$E_0(\alpha_2^{FB}) - \alpha_1^{FB} = (q_2+\frac{1}{2})\frac{I_0}{X_0}\{\frac{\delta}{1+\delta}[q_1+\frac{1}{2}+(1+R_1^f)(\frac{1}{2}-q_1)e^{-\mu+\frac{\sigma_0^2}{2}}]-1\} \quad (18)$$

and respectively risk-free assets:

$$E_0(\beta_2^{FB}) - \beta_1^{FB} = (\frac{1}{2} - q_2)\frac{\delta}{1+\delta} I_0[(\frac{1}{2} - q_1) + \frac{q_1+\frac{1}{2}}{1+R_1^f}e^{\mu+\frac{\sigma_0^2}{2}}] - I_0(\frac{1}{2} - q_1) \quad (19)$$

Proof: Annex point 5.

Up to here, we worked with the ideal case, where FIs optimally use their initial resources and the updated information at $T = 1$, in this inter-temporal consumption model, and they don't care about accounting restrictions.

In line with our "welfare" interest, we call this path of decisions a "First Best" and we define in Section 4 "Second Bests" to be compared with it.

4 Portfolio/Consumption Choice under different Accounting Regimes

Instead of freely selecting the portfolios, the FIs have to comply with the accounting rules. Like in the FB case, at $T = 0$ they select the first portfolio. At $T = 1$, they decide the consumption level and how to balance the portfolio.

The main change with the "First Best" case is: now the owners' consumption at $T = 1$ can be realized only through dividends (which can be distributed only when there is a positive profit corresponding to the first period $\Pi > 0$). The dividends are bounded above by the accounting profit the firm registers at $T = 1$, depending on the accounting regime.

We assume there are no retained profits from previous periods to be used as a reserve for $T = 1$[23]. At $T = 2$ the consumption is not influenced by the accounting regime in force, as the firm is liquidated and only the market prices (in our model they are equal with FV) count, in any accounting regime case[24].

We briefly introduce here the general assumptions we do about the accounting regimes role.

Under HC regime, we call “HC Good Time” the case when risky assets appreciate at $T = 1$ with respect to the initial moment, i.e. $X_1 > X_0$, and “HC Bad Time” the opposite case: $X_1 \leq X_0$. At $T = 1$ the company cannot recognize any profit, (but is not obliged to communicate any loss), unless it does not change (by selling some assets) its portfolio *before reporting* at $T = 1$, the end of the accounting year. Hence the unique way to report profit, and then to have the possibility to distribute dividends for consumption at $T = 1$, is to balance the portfolio at $T = 1_{-}$ (before reporting) through a net selling of assets that performed well during the first period. In line with this strategy, during “HC Good Time”, when risky assets appreciate with respect to $T = 0$, the firm can sell part of them or possibly risk free assets (which surely appreciate) and can recognize the profit. Importantly, even in “HC Bad Time” the firm can recognize profit by selling part of the risk-free assets.

Under FV regime, at $T = 1$, the company has to recognize the “fair value” of the portfolio. In particular we follow the “*FV option principle*”[25]: the difference between the FV of the portfolio at $T = 1$ and $T = 0$ (an “unrealized” profit or loss) is recognized into profit section. Accordingly, only if the portfolio value W_1 at $T = 1$ is greater than the initial investment I_0 (case called "FV Good Time") the company can distribute dividends. In the other case, when $W_1 < I_0$ ("FV Bad Time"), there is no such a possibility.

We present in details the FIs' utility maximization problem under the two accounting regimes.

4.1 HC Accounting

Under the HC accounting regime, the FIs' decisions are: at $T = 0$ they select a portfolio $(\alpha_1^{HC}, \beta_1^{HC})$ of risky, respectively risk-free assets, using the entire

[23] The motivation is similar with that for Assumption 2 from rebalancing restrictions: it is based on the existence of the risk-free security (investing in it is always better than keeping cash as a profit reserve.

[24] For this reason - to condition consumption on accounting regimes - we had to work with a multi-periodic (two-period) model, because in a single period model we could not offer any importance to the accounting reports, as FI would immediately liquidate and the accounting reports would become irrelevant.

[25] This is the dominant approach according to the new accounting standards

endowment I_0, and the whole information available at $T = 0$. This portfolio they hold for one period.

Hence:

$$\alpha_1^{HC} X_0 + \beta_1^{HC} = I_0 \tag{20}$$

at $T = 0$.

At $T = 1_-$, this portfolio values:

$$\alpha_1^{HC} X_1 + \beta_1^{HC}(1 + R_1^f) = W_1^{HC} \tag{21}$$

Around $T = 1$ new information arises about the future assets prices. (We assume this information appears at $T = 1_-$ and it remains the same until $T = 1_+$).

As we introduced (Section 2.2), we allow for a unique possibility of balancing the portfolio, around $T = 1$, when new information about asset prices appears. A priori there are two possibilities to balance the portfolio around $T = 1$: the FIs can choose to balance it either at $T = 1_-$ (before reporting) or at $T = 1_+$ (immediately after reporting). If balancing at $T = 1_+$ there is zero profit at $T = 1$, hence no consumption possible at $T = 1$, but the FIs can balance the wealth W_1 optimally in the second period, following a portfolio distribution rule similar to the FB case, for the second period. (In our case the FIs will never choose this option, as it is inefficient, when they use the information available at $T = 1$: it leads to $c_1^{HC} = 0$ which means bankruptcy considering our utility function).

On the other hand, if balancing *before reporting* (the case we consider for HC regime analysis in this work), then consumption different from zero in $T = 1$ is always possible, independently whether risky asset goes bad in the first period, but the "price" is that, when the FI decides to rebalance from $(\alpha_1^{HC}, \beta_1^{HC})$ into $(\alpha_2^{HC}, \beta_2^{HC})$ at $T = 1_-$, then this last portfolio should be kept up to $T = 2$, and it is possible (in the majority of cases it is sure) to be different from the optimal portfolio distribution the firm would choose if no accounting restriction were imposed.

Similarly to the FB case, we solve the investment-consumption allocation by backwards analysis. We assume the FI has chosen an arbitrary pair $(\alpha_1^{HC}, \beta_1^{HC})$ at $T = 0$ and it contemplates the ways this portfolio can be changed into the "targeted" portfolio for the second period $(\alpha_2^{HC}, \beta_2^{HC})$, according to the actualized information set. We discuss the possible strategies appearing at $T = 1_-$, conditioned by the value X_1 of the risky asset (at $T = 1_-$, X_1 is known, hence the state of the nature: "HC Good Time" or "HC Bad Time").

We are analyzing first the "HC Good Time" scenario (i.e. when $X_1 > X_0$). In this case, one can recognize profit by selling each of the assets: risky and risk-free. Hence, during "HC Good Time" there are feasible to be applied the following two strategies (4.1.1 and 4.1.2):

4.1.1 Strategy 1: $\alpha_2^{HC} < \alpha_1^{HC}$ and $\beta_2^{HC} > \beta_1^{HC}$ (sell risky, buy risk-free)

The firm sells (profitably) $\alpha_1^{HC} - \alpha_2^{HC}$ risky assets and it receives $(\alpha_1^{HC} - \alpha_2^{HC})X_1$ in cash.

With this transaction the firm recognizes a gain of:

$$\Pi_1^{HC} = (\alpha_1^{HC} - \alpha_2^{HC})(X_1 - X_0) \tag{22}$$

Hence the available consumption is bounded above by the profit value[26]:

$$c_1^{HC} \leq (\alpha_1^{HC} - \alpha_2^{HC})(X_1 - X_0) \tag{23}$$

On the other hand, the cash $(\alpha_1^{HC} - \alpha_2^{HC})X_1$ is divided between consumption c_1 and the rest for investment in the second period. (We are not obliging the whole profit to be consumed at $T = 1$. We allow for re-investing the profit obtained in the first period, in line with the optimization problem and we abstract for taxes).

As we do not allow for holding cash in our model, the firm has to acquire for the second period any of the financial assets available in the economy. In this case, we asked for buying only risk-free assets, as there is no economic meaning to re-buy the risky assets. Hence the cash available to buy risk-free assets at $T = 1_-$ is $(\alpha_1^{HC} - \alpha_2^{HC})X_1 - c_1^{HC}$.

It leads to

$$\beta_2^{HC} - \beta_1^{HC} = \frac{(\alpha_1^{HC} - \alpha_2^{HC})X_1 - c_1^{HC}}{1 + R_1^f}$$

or

$$\beta_2^{HC} = \beta_1^{HC} + \frac{(\alpha_1^{HC} - \alpha_2^{HC})X_1 - c_1^{HC}}{1 + R_1^f} \tag{24}$$

Taking into account the accounting restrictions presented above, given $(\alpha_1^{HC}, \beta_1^{HC})$ an arbitrary initial portfolio, at $T = 1_-$ the company following *Strategy 1* would choose in "HC Good Time" c_1^{HC}, α_2^{HC}, β_2^{HC} the results of the next problem, directly expressed in the equivalent form (7), like in the FB case.

Problem HC1Strategy1 *At $T = 1_-$, if $X_1 > X_0$, a FI wanting to rebalance the arbitrary portfolio $(\alpha_1^{HC}, \beta_1^{HC})$ into $(\alpha_2^{HC}, \beta_2^{HC})$ using Strategy 1 (i.e. selling risky assets and buying risk-free assets) solves the following maximization problem:*

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[26] We remark that $\Pi_1^{HC} < W_1$, because $\Pi_1^{HC} = (\alpha_1^{HC} - \alpha_2^{HC})(X_1 - X_0) < \alpha_1^{HC} X_1 \leq W_1 = \alpha_1^{HC} X_1 + \beta_1^{HC}(1 + R_1^f)$, hence by asking that $c_1^{HC} \leq (\alpha_1^{HC} - \alpha_2^{HC})(X_1 - X_0)$ we are sure that $c_1^{HC} < W_1$.

$$\max \delta E_1\{\log(c_1) + \delta \log(W_1 - c_1) + \delta(\frac{-\sigma_1^2}{2})\alpha_2^{*^2} + \delta(q_2 + \frac{1}{2})\sigma_1^2 \alpha_2^* + \delta r_2^f \mid_{(\alpha_1^{HC},\beta_1^{HC})}\}$$

subject to $\alpha_2^{HC} < \alpha_1^{HC}$ (25a)

$$\alpha_2^{HC} \geq 0 \quad (25b)$$

$$\beta_2^{HC} = \beta_1^{HC} + \frac{(\alpha_1^{HC} - \alpha_2^{HC})X_1 - c_1^{HC}}{1 + R_1^f} \quad (25c)$$

$$\beta_2^{HC} > \beta_1^{HC} \quad (25d)$$

$$\beta_2^{HC} > 0 \quad (25e)$$

$$c_1^{HC} \geq 0 \quad (25f)$$

$$c_1^{HC} \leq \Pi_1^{HC} \quad (25g)$$

The solution of **Problem HC1Strategy1** is the following: the FI makes the decisions according to the next algorithm, depending on the position of X_1:

1) If $X_1 > X_{FB}$ the FI chooses solutions of the FB type (i.e. $\alpha_2^{*HC} = q_2 + \frac{1}{2} = \alpha_2^{*FB}$, $c_1^{HC} = c_1^{FB}$).

2) If $X_1 \leq X_{FB}$ there exists the following candidates for global maxima and the FI should decide between them (*Corner* or *Interior Solutions*):

2.1) *Corner Solution*: (it depends on the X_1 value)
If $X_1 \geq CornerThreshold_{Str1}$ then the *Corner Solution* is $\alpha_2^{*HC} = 0$, $c_1^{HC} = c_1^{FB} = W_1 \frac{1}{1+\delta}$ (sells all the risky assets and consumes the same amount as in the FB case);
If $X_1 < CornerThreshold_{Str1}$ then the *Corner Solution* is $\alpha_2^{*HC} = 0$, $c_1^{HC} = \Pi_1^{HC} = \alpha_1(X_1 - X_0)$ (sells all the risky assets and consume all the available profit);

2.2) *Interior Solutions*
Finds the solutions Q' of the equation

$$Q^3 - (a+b)Q^2 + (ab + \frac{1}{\sigma_1^2})Q - (\frac{1+\delta}{\delta}\frac{1}{\sigma_1^2})b = 0 \quad (26)$$

satisfying $Q' > b$ and $Q' \leq a + (\frac{1}{2} + q_2)$ and then it distributes the portfolio as $\alpha_2^{*HC} = \frac{X_1}{X_1 - X_0} - Q'$ and $c_1^{HC} = W_1 - \frac{X_1(\frac{W_1}{X_1 - X_0} - \alpha_1^{HC})}{Q'}$

where

$X_{FB}^{aprox} = X_0 + \frac{[\alpha_1^{HC}X_0+\beta_1^{HC}(1+R_1^f)]+\beta_1^{HC}(1+R_1^f)\delta(q_2+\frac{1}{2})}{\alpha_1^{HC}\delta(\frac{1}{2}-q_2)}$ is an approximated value of X_{FB},

$CornerThreshold_{Str1} = (I_0 - \alpha_1^{HC}X_0)(1+R_1^f)\frac{1}{\alpha_1^{HC}\delta} + \frac{1+\delta}{\delta}X_0$,

$a = \frac{X_1}{X_1-X_0} - (\frac{1}{2}+q_2) > 0$ and $b = \frac{X_1}{X_1-X_0} - \frac{\alpha_1^{HC}X_1}{W_1} > 0$.

Proof: Annex point 6.

Remarks:
i) at $T=1$ we know the values of a, b;
ii) the condition $\beta_2^{HC} > 0$ can be eliminated;
iii) the values c_1^{HC}, α_2^{HC} and β_2^{HC} are functions of the initial portfolio $(\alpha_1^{HC}, \beta_1^{HC})$: $c_1^{HC} = c_1^{HC}(\alpha_1^{HC}, \beta_1^{HC})$, $\alpha_2^{HC} = \alpha_2^{HC}(\alpha_1^{HC}, \beta_1^{HC})$, $\beta_2^{HC} = \beta_2^{HC}(\alpha_1^{HC}, \beta_1^{HC})$;
iv) we have computed also the *exact* formula for the threshold X_{FB}. We describe the second possible strategy available in "HC Good Time" at $T = 1_-$.

4.1.2 Strategy 2: $\alpha_2^{HC} > \alpha_1^{HC}$ and $\beta_2^{HC} < \beta_1^{HC}$ (sell risk-free buy risky)

The firm sells $\beta_1^{HC} - \beta_2^{HC}$ risk-free assets and it receives $(\beta_1^{HC} - \beta_2^{HC})(1+R_1^f)$ as cash. There is a gain of $(\beta_1^{HC} - \beta_2^{HC})R_1^f$.

We make an important remark here: this strategy is always feasible, whenever there exists a positive number of risk-free assets in the portfolio; it works independently on the position of the risky asset ("HC Good Time" $X_1 > X_0$ or " HC Bad Time" $X_1 \leq X_0$).

Similarly with the first strategy, one obtains:

$$c_1^{HC} \leq (\beta_1^{HC} - \beta_2^{HC})R_1^f \tag{27}$$

The quantity $(\beta_1^{HC} - \beta_2^{HC})R_1^f$ is divided between consumption and reinvestment (in risky instruments this time) in the second period, s.t.

$$\alpha_2^{HC} - \alpha_1^{HC} = \frac{(\beta_1^{HC} - \beta_2^{HC})(1+R_1^f) - c_1^{HC}}{X_1}$$

or

$$\alpha_2^{HC} = \alpha_1^{HC} + \frac{(\beta_1^{HC} - \beta_2^{HC})(1+R_1^f) - c_1^{HC}}{X_1} \tag{28}$$

An equivalent problem with **Problem HC1Strategy1** leads to the solutions for optimally choosing α_2^{HC}, c_1^{HC} and then β_2^{HC}, at $T = 1_-$, when *Strategy 2* is chosen:

Problem HC1Strategy2 *At $T = 1_-$, a FI wanting to rebalance the arbitrary portfolio $(\alpha_1^{HC}, \beta_1^{HC})$ into $(\alpha_2^{HC}, \beta_2^{HC})$ using Strategy 2 (i.e. selling risk-free assets and buying risky assets) solves the following maximization problem:*

$$\max \delta E_1\{\log(c_1) + \delta\log(W_1 - c_1) + \delta(\frac{-\sigma_1^2}{2})\alpha_2^{*^2} + \delta(q_2 + \frac{1}{2})\sigma_1^2\alpha_2^* + \delta r_2^f \mid_{(\alpha_1^{HC}, \beta_1^{HC})}\}$$

subject to $\beta_2^{HC} < \beta_1^{HC}$ (29a)

$$\beta_2^{HC} \geq 0 \quad (29b)$$

$$\alpha_2^{HC} = \alpha_1^{HC} + \frac{(\beta_1^{HC} - \beta_2^{HC})(1 + R_1^f) - c_1}{X_1} \quad (29c)$$

$$\alpha_2^{HC} > \alpha_1^{HC} \quad (29d)$$

$$\alpha_2^{HC} > 0 \quad (29e)$$

$$c_1^{HC} \geq 0 \quad (29f)$$

$$c_1^{HC} \leq \Pi_1^{HC} \quad (29g)$$

The solution of **Problem HC1Strategy2** is the following:

The FI should decide between the following candidates for global maxima (*Corner* or *Interior Solutions*):

1) Corner Solution : $\alpha_2^{*HC} = 1$ and $c_1^{HC} = \beta_1 R_1^f = (I_0 - \alpha_1 X_1)R_1^f$ (i.e. sells all the risk-free assets and consumes all the available profit);

2) *Interior Solutions*: Finds the solutions R' of the equation

$$R^3 - R^2(c + d) + R(cd - \frac{1}{\sigma_1^2}) - \frac{(1+\delta)d}{\delta\sigma_1^2} = 0 \quad (30)$$

satisfying $R' \in (d, c + \frac{1}{2} - q_2]$ and then it distributes the portfolio as $\alpha_2^{*HC} = R' - \frac{1}{R_1^f}$ and $c_1^{HC} = W_1(1 - \frac{d}{R'})$,

where $c = \frac{1}{R_1^f} + (\frac{1}{2} + q_2) > 0$ and $d = \frac{1}{R_1^f} + \frac{\alpha_1 X_1}{W_1} > 0$.

Proof: Annex point 7.

Remarks:

i) at $T = 1$ we know the values of c and d;

ii) the condition $\beta_2^{HC} > 0$ can be eliminated;

iii) the values c_1^{HC}, α_2^{HC} and β_2^{HC} are functions of the initial portfolio $(\alpha_1^{HC}, \beta_1^{HC})$: $c_1^{HC} = c_1^{HC}(\alpha_1^{HC}, \beta_1^{HC})$, $\alpha_2^{HC} = \alpha_2^{HC}(\alpha_1^{HC}, \beta_1^{HC})$, $\beta_2^{HC} = \beta_2^{HC}(\alpha_1^{HC}, \beta_1^{HC})$.

Concluding, in "HC Good Time", at $T = 1_{-}$, depending on the value of X_1, the FI has to decide which strategy to use and inside of each strategy, how to make the decisions c_1^{HC}, α_2^{HC}, β_2^{HC}. This is the most difficult part of the present work. We have to solve separately the problems **Problem HC1Strategy1** and **Problem HC1Strategy2** and to compare their solutions.

Solving exactly our objective from this point is difficult: from one point of view, the solutions of the 3rd degree equations in **Problem HC1Strategy1** and **Problem HC1Strategy2** can be exactly found with the formula of Cardano, but they lead to complicated expressions. Also, the values of a, b and d, known at $T = 1$, are difficult to be estimated at $T = 0$.

To address these drawbacks, we show in **Proposition 3** that the previous strategies decisions can be re-written as approximations of their exact decision algorithms by neglecting the interior solutions:

Proposition 3 *1) At $T = 1_{-}$, if $X_1 > X_0$ (in "HC Good Time") , a FI wanting to rebalance the arbitrary portfolio $(\alpha_1^{HC}, \beta_1^{HC})$ into $(\alpha_2^{HC}, \beta_2^{HC})$ using Strategy 1 (i.e. selling risky assets and buying risk-free assets) makes the decisions according to the following algorithm, depending on the position of X_1:*

1.1) If $X_1 > X_{FB}$, then FI chooses solutions of the FB type (i.e. $\alpha_2^{*HC} = q_2 + \frac{1}{2} = \alpha_2^{*FB}$, $c_1^{HC} = c_1^{FB}$);

1.2) If $X_{FB} \geq X_1 \geq CornerThreshold_{Str1}$, then $\alpha_2^{*HC} = 0$, $c_1^{HC} = c_1^{FB} = W_1 \frac{1}{1+\delta}$ (it sells all the risky assets and consumes the same amount as in the FB case);

1.3) If $CornerThreshold_{Str1} > X_1$ then $\alpha_2^{*HC} = 0$, $c_1^{HC} = \Pi_1^{HC} = \alpha_1^{HC}(X_1 - X_0)$ (it sells all the risky assets and consumes all the available profit)

*where X_{FB}^{aprox} and $CornerThreshold_{Str1}$ are defined in **Problem HC1Strategy1**;*

2) At $T = 1_{-}$, a FI wanting to rebalance the arbitrary portfolio $(\alpha_1^{HC}, \beta_1^{HC})$ into $(\alpha_2^{HC}, \beta_2^{HC})$ using Strategy 2 (i.e. selling risk-free assets and buying risky assets) sells all the risk-free assets and consumes all the available profit: $\alpha_2^{*HC} = 1$ and $c_1^{HC} = \beta_1^{HC} R_1^f = (I_0 - \alpha_1^{HC} X_1) R_1^f$.

Proof: Annex point 8.

According to **Proposition 3**, during "HC Good Time" there exists 4 possible rebalancing decisions (1.1, 1.2, 1.3 and 1.4, coming from the two available Strategies) and the company has to decide between them. We have finished the description of the optimal strategies at $T = 1_{-}$ during "HC Good Time".

During "HC Bad Time" (i.e. when $X_1 \leq X_0$) the analysis is simpler: *Strategy 1* is useless (i.e. selling risky assets and buying risk-free ones) as it leads to

$c_1^{HC} = 0$. Hence, only *Strategy 2* is feasible, and the allocations corresponding to *Strategy 2* studied for the "HC Good Time" case apply; it is not necessary to make a separate analysis of the "HC Bad Time" case. Taking into account the findings of **Proposition 3** relative to Strategy 2, the conclusion is that during "HC Bad Time", at $T = 1_-$, the company always chooses α_2^{HC}, c_1^{HC} and β_2^{HC} by choosing the "corner solution" $\alpha_2^{*HC} = 1$ and $c_1^{HC} = \beta_1^{HC} R_1^f = (I_0 - \alpha_1^{HC} X_1) R_1^f$.

At this point we can put together the results of the two strategies analysis and to obtain the optimal path for the HC case at $T = 1_-$, given $(\alpha_1^{HC}, \beta_1^{HC})$ an arbitrary initial portfolio.

Proposition 4 *At $T = 1_-$, a FI wanting to rebalance the arbitrary portfolio $(\alpha_1^{HC}, \beta_1^{HC})$ into $(\alpha_2^{HC}, \beta_2^{HC})$ using any of the Strategies 1 or 2 makes the decisions according to the following algorithm, depending on the position of X_1:*

1) If $X_1 < ThreshStr$ (during "HC Bad Time" and the small values of X_1 from "HC Good Time") it uses Strategy 2 as: $\alpha_2^{*HC} = 1$, $c_1^{HC} = \beta_1^{HC} R_1^f = (I_0 - \alpha_1^{HC} X_0) R_1^f$ (it sells all the risk-free assets and consumes all the available profit); $ThreshStr = X_0 \left(1 + \frac{(1-\alpha_1^*) R_1^f}{\alpha_1^*} e^{\delta q_2 \sigma_1^2}\right)$;

For the rest of the cases it uses Strategy 1 as follows:

2) If $X_1 \in [ThreshStr, CornerThrStr1)$
$\alpha_2^{*HC} = 0$, $c_1^{HC} = \alpha_1^{HC}(X_1 - X_0)$ (it sells all the risky assets and consumes all the available profit);

3) If $X_1 \in [CornerThrStr1, X_{FB}^{aprox})$
$\alpha_2^{*HC} = 0$, $c_1^{HC} = c_1^{FB} = \frac{1}{1+\delta} W_1$ (it sells all the risky assets and consumes the same amount as in the FB case);

4) If $X_1 \geq X_{FB}^{aprox}$
$\alpha_2^{*HC} = \alpha_2^{*FB} = q_2 + \frac{1}{2}$, $c_1^{HC} = c_1^{FB} = \frac{1}{1+\delta} W_1$ (it chooses allocations of the FB type).

Proof: Annex point 9

We make the last step of our backwards analysis. In the following proposition, with the aid of **Proposition 4,** we estimate an approximate value of our expected utility function (7) at $T = 0$ and we decide the (approximated) optimal strategy the FI has to follow under the HC accounting regime. We denote by $E_0 f(X_1) \mid_{(\alpha_1^{HC}, \beta_1^{HC})}$ the expected utility of the objective function (7) when starting with an arbitrary pair $(\alpha_1^{HC}, \beta_1^{HC})$ and applying at $T = 1_-$ the optimal decisions described in **Proposition 4**.

Proposition 5 *a) Expected Utility*

For an arbitrary starting portfolio $(\alpha_1^{HC}, \beta_1^{HC})$, *the FI ex-ante utility* $E_0 f(X_1) \mid_{(\alpha_1^{HC}, \beta_1^{HC})}$ *can be approximated as* $Int_1 + Int_2 + Int_3 + Int_4$, *where:*

$$Int_1 = \delta\{k_1^{HC}[\mu\Phi(N) - \sigma_0\rho(N)] + k_2^{HC}\Phi(N)\}$$

$$Int_3 = \delta\{k_5^{HC}\{\mu[\Phi(P) - \Phi(M)] + \sigma_0[\rho(M) - \rho(P)]\} + k_6^{HC}[\Phi(P) - \Phi(M)]\}$$

$$Int_4 = \delta\{k_7^{HC}\{\mu[1 - \Phi(P)] + \sigma_0\rho(P)\} + k_8^{HC}[1 - \Phi(P)]\}$$

$$Int_2 = \begin{cases} \delta\{k_3^{HC}[\Phi(N) - \Phi(M)] + k_4^{HC}[\Phi(M) - \Phi(N)]\} & \text{if } CornerThr1/X_0 < 3 \\ \begin{array}{c}\delta\{k_3^{HC}\{\Phi(N) - \Phi(V) + \mu[\Phi(M) - \Phi(V)] + \\ +\sigma_0[\rho(V) - \rho(M)]\} + k_4^{HC}[\Phi(M) - \Phi(N)]\}\end{array} & \begin{array}{c}\text{if } ThreshStr/X_0 \le 3 \le \\ \le CornerThr1/X_0\end{array} \\ \begin{array}{c}\delta\{k_3^{HC}\{\mu[\Phi(M) - \Phi(N)] + \sigma_0[\rho(N) - \rho(M)]\} + \\ +k_4^{HC}[\Phi(M) - \Phi(N)]\end{array} & \text{if } 3 < ThreshStr/X_0 \end{cases}$$

where $\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, $\Phi(x)$ *is the cumulative distribution function of the standard normal, and the parameters are:*

$k_1^{HC} = \delta \frac{\alpha_1^*(1+R_1^f)}{1+\alpha_1^* R_1^f}$;

$k_2^{HC} = \delta[\log I_0 + \log(1+\alpha_1^* R_1^f) - r_1^f \frac{\alpha_1^*(1+R_1^f)}{1+\alpha_1^* R_1^f} + \frac{1}{2}\frac{\alpha_1^*(1+R_1^f)}{1+\alpha_1^* R_1^f}\frac{1-\alpha_1^*}{1+\alpha_1^* R_1^f}\sigma_0^2] + \log I_0 +$
$\log(1-\alpha_1^*) + \log R_1^f + \delta(-\frac{\sigma_1^2}{2}) + \delta(q_2 + \frac{1}{2})\sigma_1^2 + \delta r_1^f$;

$k_3^{HC} = 1$; $k_4^{HC} = \log \alpha_1^* + (1+\delta)\log I_0 + \delta \log(1 + R_1^f - \alpha_1^* R_1^f) + \delta r_2^f$;

$k_5^{HC} = (1+\delta)\alpha_1^*$;
$k_6^{HC} = (1+\delta)[\log I_0 + (1-\alpha_1^*) r_1^f + \frac{1}{2}\alpha_1^*(1-\alpha_1^*)\sigma_0^2] + \log(\frac{1}{\delta+1}) + \delta \log(\frac{\delta}{\delta+1}) + \delta r_2^f$;
$k_7^{HC} = k_5^{HC} = (1+\delta)\alpha_1^*$; $k_8^{HC} = k_6^{HC} + \delta\frac{\sigma_0^2}{2}(q_2 + \frac{1}{2})^2$;

$N = \frac{\ln(T_{11} + \frac{1}{\alpha_1^*}T_{12}) - \mu}{\sigma_0}$; $M = \frac{\ln(T_{21} + \frac{1}{\alpha_1^*}T_{22}) - \mu}{\sigma_0}$; $P = \frac{\ln(T_{31} + \frac{1}{\alpha_1^*}T_{32}) - \mu}{\sigma_0}$;

$V = \frac{\ln(3/X_0) - \mu}{\sigma_0}$;

$T_{11} = 1 - R_1^f e^{\delta\sigma_1^2 q_2}$; $T_{12} = R_1^f e^{\delta\sigma_1^2 q_2}$;
$T_{21} = 1 - \frac{R_1^f}{\delta}$; $T_{22} = \frac{1+R_1^f}{\delta}$;
$T_{31} = 1 - \frac{R_1^f}{\delta(\frac{1}{2} - q_2)} - \frac{(1+R_1^f)\delta(q_2 + \frac{1}{2})}{\delta(\frac{1}{2} - q_2)}$; $T_{32} = \frac{(1+R_1^f)}{\delta(\frac{1}{2} - q_2)} + \frac{(1+R_1^f)\delta(q_2 + \frac{1}{2})}{\delta(\frac{1}{2} - q_2)}$;

b) Optimal Decisions

Under HC regime, the FI optimal decision at $T = 0$ *is to choose the initial portfolio* $(\alpha_1^{HC}, \beta_1^{HC})$ *that maximizes the expected utility* $E_0 f(X_1) \mid_{(\alpha_1^{HC}, \beta_1^{HC})}$

described at a). This leads to choosing the following proportion of risky assets at $T = 0$:

$$\alpha_1^{*HC} = \frac{1}{1 + \delta R_1^f \left(1 - \alpha_1^{*FB}\right) + \left\{1 + \delta\left(\alpha_1^{*FB} - 1\right)\left[\mu + R_1^f - r_1^f + \frac{\sigma_0^2\left(1 - 2\alpha_1^{*FB}\right)}{2(1+R_1^f)}\right]\right\}\frac{cons_1}{cons_2}} \tag{31}$$

where $cons_1 = \left(\frac{1}{2} + \frac{\rho(X_N) + \frac{1}{\sqrt{2\pi}}}{2} X_N\right)$ *and* $cons_2 = (X_M - X_N)\left(\frac{\rho(X_M)+\rho(X_N)}{2}\right)$;

$X_N = \frac{\ln(T_{11} + \frac{1}{\alpha_1^{*FB}} T_{12}) - \mu}{\sigma_0}$; $X_M = \frac{\ln(T_{21} + \frac{1}{\alpha_1^{*FB}} T_{22}) - \mu}{\sigma_0}$; $X_P = \frac{\ln(T_{31} + \frac{1}{\alpha_1^{*FB}} T_{32}) - \mu}{\sigma_0}$;

*At $T = 1_-$, the FI follows the decision rules described in **Proposition 4**.*

Proof: Annex point 10

Proposition 6 *a) Expected Consumption*

*Under HC regime, a FI endowed with I_0 and following the optimal decisions described in **Proposition 5, b)** expects at $T = 0$ the following level of consumption c_1^{HC} for the moment $T = 1$:*

$E_0(c_1^{HC}) = I_0 R_1^f (1 - \alpha_1^{*HC})\Phi(N) + \alpha_1^{*HC} I_0 \{e^{\mu + \frac{\sigma_0^2}{2}}[\Phi(M - \sigma_0) - \Phi(N - \sigma_0)] - \Phi(M) + \Phi(N)\} + \frac{1}{1+\delta}\{\alpha_1^{*HC} I_0 e^{\mu + \frac{\sigma_0^2}{2}}[1 - \Phi(M - \sigma_0)] + I_0(1 + R_1^f)(1 - \alpha_1^{*HC})[1 - \Phi(M)]\}$;

b) Expected Number of Transacted Assets at $T = 1$

*Under HC regime, a FI endowed with I_0 and following the optimal decisions described in **Proposition 5, b)** expects at $T = 0$ to transact at $T = 1$ the following number of risky assets:*

$E_0(\alpha_2^{HC}) - \alpha_1^{HC} = \alpha_1^{*HC}\frac{I_0}{X_0}\{\Phi(N) - 1 + (q_2 + \frac{1}{2})\frac{\delta}{\delta+1}[1 - \Phi(P)]\} + \frac{I_0(1-\alpha_1^{*HC})}{X_0} e^{-\mu + \frac{\sigma_0^2}{2}}\{\Phi(N + \sigma_0) + (q_2 + \frac{1}{2})\frac{\delta}{\delta+1}(1 + R_1^f)[1 - \Phi(P + \sigma_0)]\}$;

and respectively risk-free assets:

$E_0(\beta_2^{HC}) - \beta_1^{HC} = \frac{1}{1+R_1^f}\alpha_1^{*HC} I_0\{\Phi(M) - \Phi(N) + \frac{\delta}{1+\delta} e^{\mu + \frac{\sigma_0^2}{2}}[\Phi(P - \sigma_0) - \Phi(M - \sigma_0) + (\frac{1}{2} - q_2)(1 - \Phi(P - \sigma_0))]\} + I_0(1 - \alpha_1^{*HC})\{\Phi(M) - \Phi(N) + \frac{\delta}{1+\delta}[\Phi(P) - \Phi(M) + (\frac{1}{2} - q2)(1 - \Phi(P))]\} - I_0(1 - \alpha_1^{*HC})$

Proof: Annex point 11.

We completed the description of the optimal path a FI should follow under the HC regime in our model. However, we make the following comments on the limitations and implications of the strategies analyzed for the HC case. First, a third strategy can be considered for our analysis: $\alpha_2^{HC} < \alpha_1^{HC}$ and $\beta_2^{HC} < \beta_1^{HC}$. It means selling both types of assets, consuming part of the profit and then reinvesting. We disregarded this strategy, as it obliges to re-buy the sold assets, or to hold cash the second period, situations we already rejected for present analysis.

The second comment is the following: the model with two assets - one risky and the other risk-free - is clearly a simplification of reality. A step further would be allowing for holding cash from one period to the other (i.e. to be added a third security - money, bearing the interest rate risk). From one point of view without cash the problem is simpler, but on the other hand it implies the risk of truncating reality when designing our strategies in the HC case. Importantly, in terms of consequences, not allowing for holding cash is not affecting essentially the results: we show that, even working with this simplified hypothesis, the HC is better in terms of consumption smoothing than FV, hence improving the analyzed HC strategies would increase the efficiency of the HC regime (and maybe it will enlarge the set of the points where HC coincides with the FB), but will not change the order of preferences between regimes.

4.2 FV Accounting (with "Fair Value Option" principle)

Consistent with the previously analyzed frameworks (FB and HC accounting), we describe by backward analysis the optimal decisions and we compute the ex-ante utility of a FI facing the FV accounting regime.

The FI starts with an arbitrary portfolio $(\alpha_1^{FV}, \beta_1^{FV})$ at $T = 0$. At $T = 1$ it owns the portfolio $(\alpha_1^{FV}, \beta_1^{FV})$ and the endowment $W_1^{FV} = \alpha_1^{FV} X_1 + \beta_1^{FV}(1 + R_1^f)$.

There exists two possible scenarios at $T = 1$:

1. If $W_1^{FV} \leq I_0$ ("FV Bad Time") the FI cannot register any (unrealized) profit ($\Pi_1^{FV} \leq 0$); it implies $c_1^{FV} = 0$ and the FI goes bankrupt, according to our utility function.

2. If $W_1^{FV} > I_0$ ("FV Good Time") the FI registers the profit $\Pi_1^{FV} = W_1^{FV} - I_0$.

We compute the profit value

$$\Pi_1^{FV} = \alpha_1^{FV} X_1 + \beta_1^{FV}(1+R_1^f) - \alpha_1^{FV} X_0 + \beta_1^{FV} = \alpha_1^{FV}(X_1 - X_0) + \beta_1^{FV} R_1^f \quad (32)$$

and the consumption should satisfy the following restriction:

$$c_1^{FV} \in [0, \Pi_1^{FV}] \tag{33}$$

Problem FV1 *At $T = 1$, a FI wanting to rebalance the arbitrary portfolio $(\alpha_1^{FV}, \beta_1^{FV})$ into $(\alpha_2^{FV}, \beta_2^{FV})$ makes the decisions according to the following algorithm, depending on the position of W_1^{FV} (or equivalently X_1):*

1. *In case of $W_1^{FV} \le I_0$ the FI consumes $c_1^{FV} = 0$ and it goes bankrupt.*

2. *In case of $W_1^{FV} > I_0$ the FI solves the following maximization problem :*

$$\max \delta E_1\{\log(c_1) + \delta \log(W_1 - c_1) + \delta(\frac{-\sigma_1^2}{2})\alpha_2^{*^2} + \delta(q_2 + \frac{1}{2})\sigma_1^2\alpha_2^* + \delta r_2^f \mid_{(\alpha_1^{FV}, \beta_1^{FV})}\}$$

$$\begin{aligned}
\text{s. t.} \; & c_1^{FV} \ge 0 \\
& c_1^{FV} \le \Pi_1^{FV} = \alpha_1^{FV}(X_1 - X_0) + \beta_1^{FV} R_1^f \\
& \alpha_2^{*FV} \ge 0 \\
& \alpha_2^{*FV} \le 1
\end{aligned}$$

The solutions of **Problem FV1** are:

1. *If $X_1 \le Threshold_{FV}$, then $c_1^{FV} = 0$ and the FI enters into bankruptcy.*

2.1 If $Threshold_{FV} < X_1 < CornerThreshold_{FV}$, then the FI consumes all the profit: $c_1^{FV} = \alpha_1^{FV}(X_1 - X_0) + \beta_1^{FV} R_1^f = \alpha_1^{FV}(X_1 - X_0) + (I_0 - \alpha_1^{FV} X_0) R_1^f$ and $\alpha_2^{*FV} = \alpha_2^{*FB}$;

2.2 If $CornerThreshold_{FV} \le X_1$, then the FI chooses the FB allocation: $c_1^{FV} = \frac{W_1}{1+\delta} = c_1^{FB}$ and $\alpha_2^{*FV} = q_2 + \frac{1}{2} = \alpha_2^{*FB}$

where

$Threshold_{FV} = X_0(1 + R_1^f) - \frac{I_0 R_1^f}{\alpha_1^{FV}}$ is the threshold that assures $\Pi_1^{FV} > 0$ and

$CornerThreshold_{FV} = X_0(1 + R_1^f) + \frac{I_0(1-\delta R_1^f)}{\delta \alpha_1^{FV}}$ is the threshold that distinguishes between the two types of solutions.

Proof: Annex point 12

Remarks:
i) $CornerThreshold_{FV} > X_0 > Threshold_{FV}$.

ii) The thresholds and the solutions c_1^{FV} and α_2^{*FV} are functions of α_1^{FV} (or equivalently of α_1^{*FV}).

However, one can note that if $Threshold_{FV} \geq 0$, there exists ex-ante *a positive probability* for the FI to default at $T = 1$ (i.e. the case when X_1 would belong to the interval $(0, Threshold_{FV}]$), hence the FI is obliged to choose the initial proportion of risky assets α_1^* such that to have no risk of default[27]. This is equivalent with choosing α_1^* such that $Threshold_{FV} \leq 0$. We prove (Annex point 12) this implies the FI should start with an initial portfolio $(\alpha_1^{FV}, \beta_1^{FV})$ satisfying $\alpha_1^{*FV} \in \left(0, \frac{R_1^f}{1+R_1^f}\right]$. We make now the last step of the backwards analysis of our portfolio decision rules.

Similarly to the HC case, we denote by $E_0 f(X_1) \mid_{(\alpha_1^{FV}, \beta_1^{FV})}$the expected utility of the objective function (7) when starting with an arbitrary pair $(\alpha_1^{FV}, \beta_1^{FV})$ (satisfying $\alpha_1^{*FV} \in \left(0, \frac{R_1^f}{1+R_1^f}\right]$) and applying at $T = 1$ the optimal decisions described in the solution of **Problem FV1**.

Proposition 7 *a) Expected Utility*

For an arbitrary starting portfolio $(\alpha_1^{FV}, \beta_1^{FV})$ *satisfying* $\alpha_1^{*FV} \in \left(0, \frac{R_1^f}{1+R_1^f}\right]$, *the FI ex-ante utility* $E_0 f(X_1) \mid_{(\alpha_1^{FV}, \beta_1^{FV})}$ *can be approximated as:*

$$E_0 f(X_1) \mid_{(\alpha_1^{FV}, \beta_1^{FV})} = \delta \left(I_1^{FV} + I_2^{FV}\right) + \delta^2 \left(\frac{\sigma_1^2}{2}(q_2 + \frac{1}{2})^2 + r_2^f\right) \qquad (34)$$

where

$$I_1^{FV} = k_1^{FV}[\mu\Phi(S) - \sigma_0\rho(S)] + k_2^{FV}\Phi(S)$$

$$I_2^{FV} = k_3^{FV}\{\mu[1 - \Phi(S)] + \sigma_0\rho(S)\} + k_4^{FV}[1 - \Phi(S)]$$

with Φ *the cumulative distribution function of the standard normal, and the parameters:*

$S = \frac{\ln(T_{41} + \frac{1}{\alpha_1^*}T_{42}) - \mu}{\sigma_0}$; $k_1^{FV} = \alpha_1^* \frac{1+R_1^f}{R_1^f}$;

$k_2^{FV} = -\frac{1}{2}\alpha_1^{*2}(\frac{1+R_1^f}{R_1^f})^2\sigma_0^2 + \alpha_1^*(\frac{1+R_1^f}{R_1^f}\frac{\sigma_0^2}{2} - \frac{1+R_1^f}{R_1^f}r_1^f) + (\delta+1)\log(I_0) + \log(R_1^f)$;

$k_3^{FV} = (1+\delta)\alpha_1^*$;

$k_4^{FV} = -\frac{1}{2}\alpha_1^{*2}(1+\delta)\sigma_0^2 + \alpha_1^*(1+\delta)(\frac{\sigma_0^2}{2} - r_1^f) + (1+\delta)(\log(I_0) + r_1^f) + \delta\log(\delta) - (\delta+1)\log(\delta+1)$;

[27] The explanation is the following: as long as there exists an ex-ante positive probability to default (i.e. such that $u(c_1) = -\infty$), it implies the expected utility $E_0\{\delta u(c_1) + \delta^2 u(c_2)\} = -\infty$, and this is worse than any finite value of the expected utility obtained when there is not such a positive probability (in particular, by starting with $\alpha_1^{*FV} \in \left(0, \frac{R_1^f}{1+R_1^f}\right]$).

$T_{41} = 1 + R_1^f; T_{42} = \frac{1}{\delta} - R_1^f;$

b) Optimal Decisions

Under FV regime, the FI optimal decision at $T = 0$ is to choose the initial portfolio $(\alpha_1^{FV}, \beta_1^{FV})$ that maximizes the expected utility $E_0 f(X_1) \mid_{(\alpha_1^{FV}, \beta_1^{FV})}$ described at a). An approximate expression for the proportion of risky assets at $T = 0$ is $\alpha_1^{*FV} = \frac{R_1^f}{2(1+R_1^f)}$, taking into account there is a very small region where feasible α_1^{*FV} lay[28].

*At $T = 1$, the FI follows the decision rules described in **Problem FV1**.*

Proof: Annex point 13

Proposition 8 *a) Expected Consumption*

*Under FV regime, a FI endowed with I_0 and following the optimal decisions described in **Proposition 7, b)** expects at $T = 0$ the following level of consumption c_1^{FV} for the moment $T = 1$:*

$$E_0(c_1^{FV}) = \frac{1}{1+\delta}\{I_0\alpha_1^{*FV}e^{\mu+\frac{\sigma_0^2}{2}}[1+\delta\Phi(S-\sigma_0)]+I_0(1-\alpha_1^{*FV})R_1^f[1+\delta\Phi(S)] - \Phi(S)I_0(\alpha_1^{*FV}\delta+1)+I_0(1-\alpha_1^{*FV})\};$$

b) Expected Number of Transacted Assets at $T = 1$

*Under FV regime, a FI endowed with I_0 and following the optimal decisions described in **Proposition 7, b)** expects at $T = 0$ to transact at $T = 1$ the following number of risky assets:*

$$E_0(\alpha_2^{FV})-\alpha_1^{FV} = \frac{I_0}{X_0}\{(q_2+\frac{1}{2})\{e^{-\mu+\frac{\sigma_0^2}{2}}[\Phi(S+\sigma_0)+\frac{\delta}{1+\delta}(1+R_1^f)(1-\alpha_1^{*FV})(1-\Phi(S+\sigma_0))]+\frac{\delta}{1+\delta}\alpha_1^{*FV}(1-\Phi(S))\} - \alpha_1^{*FV}\};$$

and respectively risk-free assets:

$$E_0(\beta_2^{FV})-\beta_1^{FV} = \frac{1-\alpha_2^{*FV}}{1+R_1^f}I_0\Phi(S)+\frac{1-\alpha_2^{*FV}}{1+R_1^f}\frac{\delta}{1+\delta}\alpha_1^{*FV}I_0e^{\mu+\frac{\sigma_0^2}{2}}[1-\Phi(S-\sigma_0)]+\frac{1-\alpha_2^{*FV}}{1+R_1^f}\frac{\delta}{1+\delta}I_0(1-\alpha_1^{*FV})(1+R_1^f)[1-\Phi(S)] - I_0(1-\alpha_1^{*FV}).$$

Proof: Annex point 14.

We completed the description of the optimal path a FI should follow under the FV regime in our model.

[28] For a more rigorous proof, see Annex point 13

5 Accounting Regimes Comparison

Proposition 1, a), **Proposition 5, b)** and **Proposition 7, b)** tells that a FI interested in consumption smoothing makes different decisions in case of no accounting restrictions (FB), historic cost accounting (HC) regime and Fair Value accounting (FV) regime.

We compare first the portfolio structure in the first (or non-terminal) period, for the three cases; the non-terminal period decisions are important as they can be considered representative for the analyzed frameworks, while the second (or terminal) period decisions are not directly influenced by the accounting regimes because the consumption c_2 at $T = 2$ is always given by the market value of the liquidated portfolio.

We plot in *Figure 1* the proportion of risky assets in the first period portfolio for the three cases: FB, HC and FV accounting. The parameters used to plot the figure are $I_0 = 100$, $R_1^f = 0.04$, $E_0 R_1^x = 0.2$ and σ_0^2 is chosen such that the proportion of risky assets in the FB portfolio for the first period is 0.25[29]. Similar patterns are obtained when $E_0 R_1^x \in (0.05, 1.5)$ and σ_0^2 corresponds to other proportions of risky assets (of 0, 0.5, 0, 75, 1).

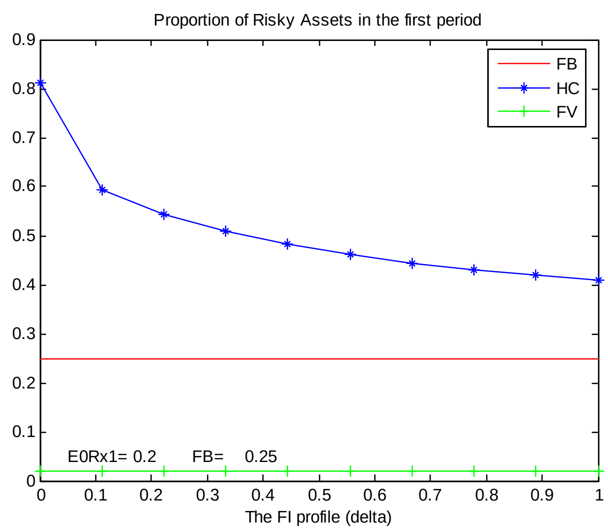

Figure 1: Proportion of risky assets in the portfolio at T=0

One can note from *Figure 1* that the proportion of risky assets in the HC case is higher than under FV (in some cases, depending on the risky asset expected mean and variance, this proportion is also higher than the optimal

[29] we are not expressing the variance σ_0^2 in absolute value, but in relative terms; for any expected net return $E_0 R_1^x$ we can identify the equivalence pairs $(\sigma_0^2, \alpha_1^{*FB})$ of the variance level leading to an optimal proportion α_1^{*FB} of risky assets in the FB portfolio. See (10) and Annex point 1: $\alpha_1^{*FB} = q_1 + \frac{1}{2} = \frac{1}{\sigma_0^2} \log \frac{1+E_0 R_1^x}{1+R_1^f}$ or $\sigma_0^2 = \frac{1}{\alpha_1^{*FB}} \log \frac{1+E_0 R_1^x}{1+R_1^f}$

proportion of risky assets from the FB case, like in the present figure); the FV regime shows a very conservative behavior (a very low number of risky assets in the portfolio), a consequence of the lack of protection in bad times under the FV regime, as we saw in Section 4. The high proportion of risky assets at $T = 0$ under the HC regime has two explanations: first, it is a consequence of the protection (insurance) against the possible low outcomes of the risky asset at $T = 1$ this regime offers during bad time, when the losses are not recognized and positive consumption is always possible. On the other hand, during HC Good Time, the FI is incentivated to sell risky assets at $T = 1$ in order to be able to recognize a positive profit (an "windows dressing" activity) at this moment and to remain in the same time with a level of risky assets still close to the FB proportion. Hence, by anticipating the "windows dressing" transactions activity, the FI has to carry a sufficient level of risky assets in the non-terminal portfolio. One can also note, in case of HC accounting, the proportion of risky assets decreases when moving from a small δ to one closer to 1 (i.e. moving from FIs with short-term horizons to those with long-term consumption smoothing concerns): the "windows dressing" activity is not so important at $T = 1$ when early consumption is not a priority.

We compare the expected utility of consumption $E_0\{\delta u(c_1) + \delta^2 u(c_2)\}$ (our main objective rephrased as (7)) for a FI under no accounting restrictions, HC accounting and FV accounting regimes. To plot the expected utility for the three cases, we use **Proposition 1, b)** (for the FB case), **Proposition 5, a)** (for the HC case), respectively **Proposition 7, a)** (for the FV case) applied to the optimal proportions α_1^* specific to each regime and plotted in *Figure 1*. The parameters used are the same as those used for *Figure 1*.

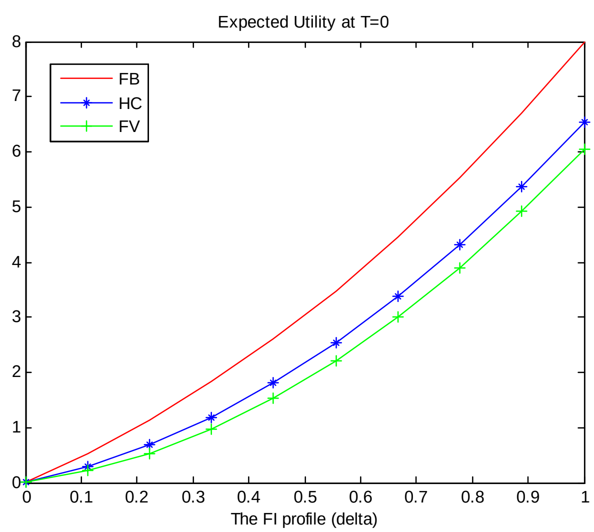

Figure 2: Expected utility comparison

Figure 2 shows the expected utility of future consumption the FI can predict at $T = 0$. The HC regime is superior in terms of consumption smoothing to the FV regime, and the difference between the two expected outcomes increases when moving from FI with short-term horizons to those with long-term horizons. The welfare loss (the difference between the FB expected utility and the expected utility under a given accounting regime) is higher for the FV regime and it is increasing in δ, the FI horizon. Our findings support the idea that the most affected by the FV introduction are the FIs having an interest in smoothing their owners' long-term consumption.

The next analysis is about the consumption level for the non-terminal period. The expected consumption at $T = 1$ is computed using the results of **Proposition 2, a)**, **Proposition 6, a)**, respectively **Proposition 8, a)**. *Figure 3* compares the three consumption profiles (with the same parameters as in the previous figures).

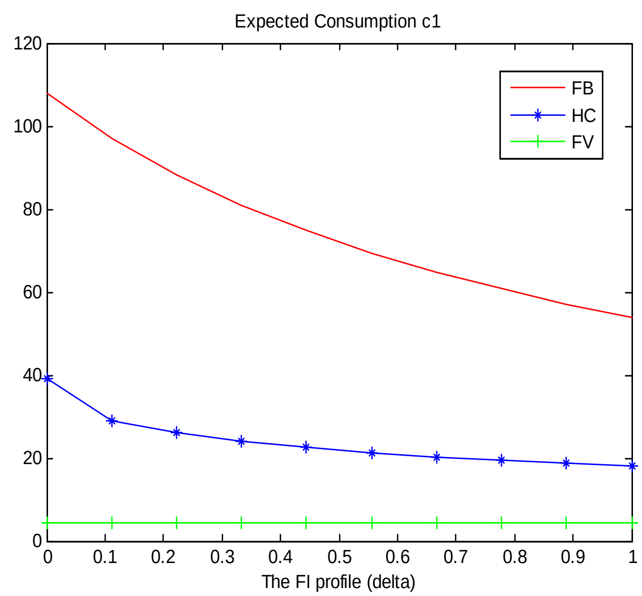

Figure 3: Expected consumption c1 comparison

One can note the two accounting regimes imply suboptimal consumption (or equivalently, dividend distribution) levels and the difference is more severe for the FV case.

Taking into account the results of our propositions, one can also analyze the expected "agressivity" of the FI to transact risky and risk-free assets. This is interesting to be studied, as Beatty, Chamberlain and Magliolo (1996) finds "*banks that more frequently traded their investments, with longer maturing investments, and that are more fully hedged against interest rate changes, were the most negatively impacted by the standard* [SFAS 115]"[30]. We count the expected

[30] a "fair value" standard

total number of risky assets transacted at $T = 0$, 1 and 2. At $T = 0$ the FI has to acquire α_1 risky assets (we obtain these values by multiplying α_1^* plotted in *Figure 1* with I_0/X_0). At $T = 1$, the expected number of risky assets transacted is $|E_0\alpha_2 - \alpha_1|$ and it is obtained from **Proposition 2, b)**, **Proposition 6, b)**, respectively **Proposition 8, b)** for the three cases: FB, HC and FV. Finally, at $T = 2$, the FI has to liquidate the portfolio (it lives for two periods) and it has to sell the $E_0\alpha_2$ risky assets. The $E_0\alpha_2$ values are obtained by summing $\alpha_1 + (E_0\alpha_2 - \alpha_1)$ we already know. Hence the expected number of risky assets transacted during the FI life is $\alpha_1 + |E_0\alpha_2 - \alpha_1| + E_0\alpha_2$.

We plot in *Figure 4* this expected number for the following parameters: $I_0 = 100$, $R_1^f = 0.04$, $E_0R_1^x = 0.2$ and $\sigma_0^2 = 0.25$ (i.e. the same parameters as before, in the left side) and $\sigma_0^2 = 0.75$ (an "improved" risky asset, less volatile, in the right side). As a difference with the previous graphs, one can note the risky asset transaction activity is influenced by the risky asset expected mean and variance; this is the reason of plotting the two figures together. However, the transacted volume in the FV case is quite linked to the FB number (as in the terminal period the portfolio structure is the same), but the HC transacted volume is very sensitive to the risky asset patterns. The "windows dressing" activity is becoming less active when the risky asset improves (when it is more predictable).

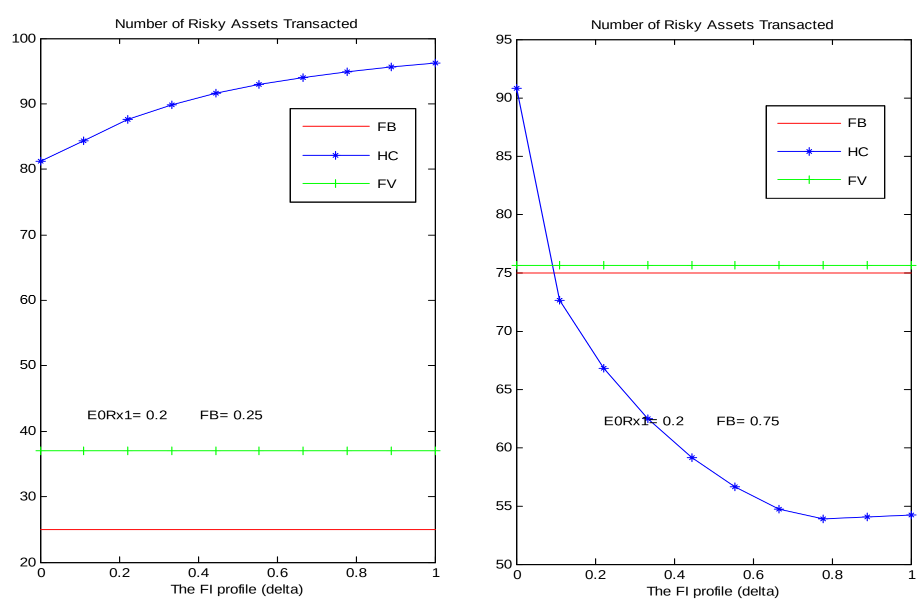

Figure 4: Number of risky assets transacted

One can conclude the ex-ante optimal decisions for the two accounting regimes imply transacting different volumes of risky assets and the HC volume in particular is very sensitive to the risky asset patterns (expected mean and variance).

Our concern for the number of risky assets transacted is linked with the problem of costs with transacting (especially) these assets. We solved in the present work for the optimal decisions paths under the FB case, HC, respectively FV accounting by imposing only rebalancing restrictions and without counting the costs with transacting the risky assets. When these costs are relevant (e.g. when the risky assets are not liquid) the FI has to take them into account[31].

Assuming the costs are proportional with the volume of risky assets transacted, one can infer from *Figure 4* that, in the left side the transaction costs are greater in case of HC regime than for the FV one, while in the right side (when the risky asset improves) the FV regime decisions imply less transaction costs than the HC regime (for a δ greater than 0.1). Consequently, with relevant transaction costs, the FI has to re-analyze the optimal decisions paths.

Solving our consumption smoothing problem by introducing explicitly the transaction costs is a very complicated task and we leave it for a further research. However, in order to introduce the comparison between the accounting regimes when transaction costs are relevant we propose a simplified approach: we assume the FI applies the same decisions as discussed in this work for the three analyzed cases but it has to make some re-adjustments.

First, we identify the costs (assumed to be proportional with the number of risky assets transacted and equal to s for buying or selling one unit of risky asset). According to the number of risky assets transacted, the costs at $T = 0$, 1 and 2 are respectively $Cost_0 = s\alpha_1$, $Cost_1 = s\left|E_0\alpha_2 - \alpha_1\right|$ and $Cost_2 = sE_0\alpha_2$.

We show (annex point 15) these additional costs imply (an approximated solution) solving equivalently our initial problem with an investment of $I_0 - s(\alpha_1 + \frac{1}{1+R_1^f}\left|E_0\alpha_2 - \alpha_1\right| + \frac{1}{(1+R_1^f)(1+R_1^f)}E_0\alpha_2)$ instead of I_0.

In *Figure 5* we plot the expected utility curves when the transaction costs are taken into account (or when the risky assets are illiquid). The parameters are those used in *Figure 4* and $s = 0.5$. Importantly, one note in the left figure the order of preference between accounting regimes we obtained for the liquid assets reverts: the HC regime is worse than the FV one. The explanation is the higher agressivity for transacting risky assets (as we saw in *Figure 4,* left side) can be considered an inefficiency of the HC regime, coming from the necessity of "windows dressing". When transaction costs are relevant, this inefficiency penalizes ex-ante the HC regime and it makes preferable the FV one. In the right side of the *Figure 5* the results are similar with those obtained for liquid assets: the HC regime is superior to the FV one in terms of consumption smoothing. This is consistent with the reduced number of risky assets the FI expects to transact in the HC case (*Figure 4,* right side) according to the risky asset patterns (expected mean and variance).

[31] Plantin, Sapra and Shin (2004) finds, in a different setting, the illiquidity of the asset accentuates the superiority of the HC regime.

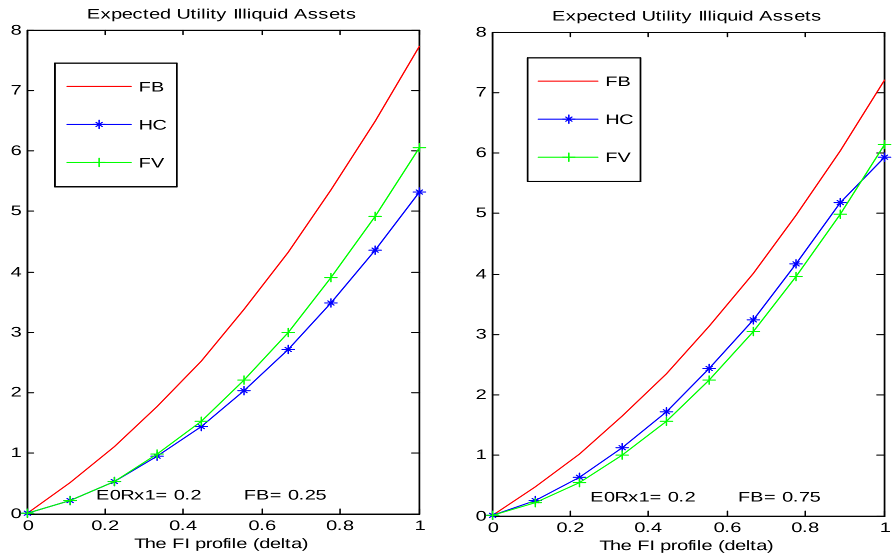

Figure 5: Expected utility comparison (illiquid assets)

We conclude that with illiquid assets or relevant transaction costs for the risky assets the order of preference between accounting regimes sensibly depends on the patterns of the risky asset and the amount of transaction costs. For the same expected return of the asset, the FV regime is better in terms of consumption smoothing for high variances of the asset (or equivalently for small proportion of risky assets in the FB portfolio) while the HC is better for small variances (when the risky asset improves and the proportion of risky assets in the portfolio is sufficiently high). The cut-off point and the rigorous ex-ante optimal decisions for the two accounting regimes remain open questions and they will be the subject of a separate work.

6 Conclusion

The present work analyzes in a dynamic setting the consequences of using different accounting regimes (HC vs. FV) for the optimal choice of a financial portfolio, when the owner is interested in consumption for two periods, c_1 and c_2, and two types of liquid assets are available in the economy: one risky and one risk-free. We assume that dividend distribution, hence consumption c_1, is conditioned by the existence of a positive profit at $T = 1$. By ex-ante comparisons with the theoretical optimal portfolio decisions (First Best), we find that both regimes are inefficient, but FV is worse than HC in terms of consumption smoothing and the welfare loss is higher for the companies concerned with long-term business than for those with short-term horizons. In absolute terms, the consumption level (dividend distribution) for the non-terminal period is suboptimal for both accounting regimes, but the FV consumption is less than the HC one.

The three portfolio choices (FB, HC and FV) show the following characteristics:

1. the FB is the optimal path when no accounting restrictions are taken into account;

2. the HC regime offers an insurance to the company, by not recognizing the unrealized losses of the portfolio in bad times and assuring a minimal positive consumption c_1 independently on how bad the risky asset behaves, each time when the portfolio is not fully composed of risky assets. With this protection, in bad time the HC accounting regime is ex-ante superior to the FV regime, where no minimal consumption c_1 is guaranteed by a portfolio containing risky assets. However, in good time the HC is inefficient: the unrealized gains are neither recognized, and in order to be able to consume at $T = 1$ an amount comparable with the optimal level c_1^{FB}, the company has to sacrifice the optimal theoretical policy to a strategy that allows for positive HC profit to be recognized at $T = 1$. Hence, particularly in good time, the HC portfolio choice strategy is not identical with the optimal theoretical strategy, but is concerned with "windows dressing" of the $T = 1$ profit, depending on the relative importance of the early consumption.

3. the FV regime recognizes both the unrealized gains and losses, hence no protection is guaranteed in bad time; in the same time, there are no incentives to portfolio rebalancing strategies for "windows dressing" purposes, as they are inactive - the FV profit value is independent on the rebalancing strategy. FV accounting is efficient in good time, when the company rebalancing decisions are identical with the FB ones, but it is inefficient in bad time, when $c_1 = 0$ (or bankruptcy) due to negative profits, is induced, unless the portfolio is not containing a sufficiently high proportion of risk-free assets. To address the ex-ante bankruptcy risk the FI should maintain a very conservative portfolio.

Regarding the portfolio structure under HC and FV, we proved that they differ from the optimal theoretical one (FB), due to the inefficiencies encountered in good, respectively bad time. Consistent with the previous remarks, under HC accounting the portfolio contains a high level of risky assets (the insurance effect and the necessity of the "windows dressing" activity) and in some cases this proportion is higher than the optimal one. On the other hand, under FV the portfolio is very conservative, with a very low level of risky assets in the portfolio (no protection guaranteed).

Finally, when the risky asset is illiquid and/or costs associated with transacting it are relevant to be taken into account, the ex-ante consumption smoothing superiority of the HC regime to the FV one is not always true but depends on the risky asset patterns (expected return and variance) and the transaction costs amount.

References

[1] Beatty, A.; Chamberlain, S.; Magliolo, J. (1996): "*An empirical analysis of the economic implications of fair value accounting for investment securities*", Journal of Accounting and Economics, 22, 43-77

[2] Burkhardt, K.; Strausz, R. (2004): "*The Effect of Fair vs. Book Value Accounting on the Liquidity and Investment Behavior of Banks*", Working Paper

[3] Campbell, J.Y., Viceira, L.M. (2002): "*Strategic Asset Allocation*", Oxford University Press

[4] European Central Bank (2004): "*Fair Value Accounting and Financial Stability*"

[5] Freixas, X.; Tsomocos, D. P. (2004): "*Books vs. Fair Value Accounting in Banking, and Intertemporal Smoothing*", Working Paper

[6] Geneva Association (2004): "*Impact of a Fair Value Financial Reporting System on Insurance Companies*", Geneva Papers on Risk and Insurance: Issues and Practice, 29, 540-581

[7] Hull, J. C. (2003), "*Options, Futures, and Other Derivatives*", Prentice Hall

[8] O'Hara, M. (1993): "*Real Bills Revisited: Market Value Accounting and Loan Maturity*", Journal of Financial Intermediation 3, 51-76

[9] Plantin, G.; Sapra, H., Shin, H.S. (2007): "*Marking-to-Market: Panacea or Pandora's Box ?*", forthcoming in Journal of Accounting Research

[10] Pliska,S (1997): "*Introduction to Mathematical Finance: Discrete Time Models*", Blackwell Publishers, Oxford

[11] Pollack, I.M. (1986): "*Short-Sale Regulation of NASDAQ Securities*", Washington, D.C.: National Association of Securities Dealers, Inc.

Annex (proofs)

1. Formula for q_1

$1+R_1^x = \dfrac{X_1}{X_0}$ is log-normally distributed $\Leftrightarrow \log(\frac{X_1}{X_0}) \sim N(\mu_0, \sigma_0^2) \Leftrightarrow \log(X_1) \sim N(\mu_0 + \log(X_0), \sigma_0^2)$. We can apply the following general equality, true for any log-normal variable Y (taking into account that $\frac{X_1}{X_0} = 1 + R_1^x$ is log-normally distributed): $\log E_0(Y) = E_0 \log(Y) + \frac{1}{2}Var_0(\log(Y)) \Rightarrow \log E_0(1 + R_1^x) =$

$E_0 \log(1+R_1^x) + \frac{1}{2} Var_0(\log(1+R_1^x)) \Leftrightarrow E_0 r_1^x = \log E_0(1+R_1^x) - \frac{1}{2} Var_0(\log(1+R_1^x))$. But $Var_0(\log(1+R_1^x)) = Var_0(\log(\frac{X_1}{X_0})) = Var_0[(\log(X_1)) - \log(X_0)] = Var_0(\log(X_1)) = \sigma_0^2$. Then $E_0 r_1^x = \log E_0(1+R_1^x) - \frac{1}{2}\sigma_0^2 = \log(1+E_0 R_1^x) - \frac{1}{2}\sigma_0^2 \Rightarrow E_0(r_1^x - r_1^f) = \log(1+E_0 R_1^x) - \frac{1}{2}\sigma_0^2 - \log(1+R_1^f) \Rightarrow q_1 \overset{def}{=} \frac{E_0(r_1^x - r_1^f)}{\sigma_0^2} = \frac{1}{\sigma_0^2} \log \frac{1+E_0 R_1^x}{1+R_1^f} - \frac{1}{2}$.

2. $R_1^p = \frac{W_1 - I_0}{I_0} = \frac{\alpha_1(X_1 - X_0) + \beta_1 R_1^f}{I_0} = \frac{\alpha_1 X_0}{I_0}\frac{(X_1 - X_0)}{X_0} + \frac{(I_0 - \alpha_1 X_0) R_1^f}{I_0} = \alpha_1^* R_1^x + (1-\alpha_1^*) R_1^f \Leftrightarrow R_1^p + 1 = \alpha_1^*(R_1^x + 1) + (1-\alpha_1^*)(1+R_1^f) \Rightarrow \log(R_1^p + 1) = \log[\alpha_1^*(R_1^x + 1) + (1-\alpha_1^*)(1+R_1^f)]$. An important approximation, due to Campbell and Viceira (2002) is: $\log(R_1^p + 1) = \alpha_1^* \log(R_1^x + 1) + (1-\alpha_1^*)\log(1+R_1^f) + \frac{1}{2}\alpha_1^*(1-\alpha_1^*)\sigma_0^2 \Leftrightarrow r_1^p = \alpha_1^* r_1^x + (1-\alpha_1^*) r_1^f + \frac{1}{2}\alpha_1^*(1-\alpha_1^*)\sigma_0^2 = r_1^f + \alpha_1^*(r_1^x - r_1^f) + \frac{1}{2}\alpha_1^*(1-\alpha_1^*)\sigma_0^2$.

The consequences for the portfolio distributions are: $E_0 r_1^p = r_1^f + \alpha_1^* E_0(r_1^x - r_1^f) + \frac{1}{2}\alpha_1^*(1-\alpha_1^*)\sigma_0^2$ and similarly for $E_1 r_2^p : E_1 r_2^p = r_2^f + \alpha_2^* E_1(r_2^x - r_2^f) + \frac{1}{2}\alpha_2^*(1-\alpha_2^*)\sigma_1^2$.

To simplify the computations, we assume $E_0(R_2^x) = E_0(R_1^x)$, $E_0\left(\sigma_1^2\right) = \sigma_0^2$ and respectively $E_0\left(r_2^f\right) = r_1^f$.

Also we ask that :

1) $E_0(q_2) = q_1$(this is equivalent with saying that $E_0(\alpha_2^{*FB}) = \alpha_1^{*FB}$, the proportions of risky assets in the FB portfolio) and

2) $E_0 r_2^p = E_0\left(r_2^f + \alpha_2^* E_1(r_2^x - r_2^f) + \frac{1}{2}\alpha_2^*(1-\alpha_2^*)\sigma_1^2\right) = r_1^f + \alpha_2^* E_0(r_1^x - r_1^f) + \frac{1}{2}\alpha_2^*(1-\alpha_2^*)\sigma_0^2$, for any proportion of risky assets α_2^* in the second period portfolio.

Taking into account these consequences, we are interested in computing: (when working with log-utility and log-normally distributed assets) max $E_0\{\delta\log(c_1) + \delta^2 \log(c_2)\}$. But $\delta \log(c_1) + \delta^2\log(c_2) = \delta\{\log(c_1) + \delta\log(W_1 - c_1) + \delta r_2^p\} \simeq \delta\{\log(c_1) + \delta\log(W_1 - c_1) + \delta(\frac{-\sigma_1^2}{2})\alpha_2^{*^2} + \delta(q_2 + \frac{1}{2})\sigma_1^2\alpha_2^* + \delta r_2^f\}$. Hence the objective function (approximated with the Campbell and Viceira formula) to be maximized is: $\delta E_0\{\log(c_1) + \delta\log(W_1 - c_1) + \delta(\frac{-\sigma_1^2}{2})\alpha_2^{*^2} + \delta(q_2 + \frac{1}{2})\sigma_1^2\alpha_2^* + \delta r_2^f\}$.

3. Problem FB1

Solving with Kuhn-Tucker:

$$\begin{cases} 1) & \frac{\partial}{\partial c_1} = \delta(\frac{1}{c_1} - \frac{\delta}{W_1 - c_1}) = 0 & \Leftrightarrow c_1^{FB} = \frac{W_1}{1+\delta} \\ 2) & \frac{\partial}{\partial \alpha_2^*} = -\delta\sigma_1^2\alpha_2^* + \delta(q_2 + \frac{1}{2})\sigma_1^2 = 0 & \Leftrightarrow \alpha_2^{*FB} = q_2 + \frac{1}{2} \end{cases}$$

Remark: We forced the parameters such that the proportion $\alpha_2^{*FB} \in (0,1)$ (we asked $q_2 = \frac{E_1 r_2^x - r_2^f}{\sigma_1^2} \in (-\frac{1}{2}, \frac{1}{2}) \Leftrightarrow \alpha_2^{*FB} = \frac{E_1 r_2^x - r_2^f}{\sigma_1^2} + \frac{1}{2} \in (0,1)$).

4. Problem FB0"

The objective function to be maximized (according to (7)) is: $max \delta E_0\{\log(c_1) + \delta\log(W_1 - c_1) + \delta(\frac{-\sigma_1^2}{2})\alpha_2^{*^2} + \delta(q_2 + \frac{1}{2})\sigma_1^2\alpha_2^* + \delta r_2^f\} = \delta E_0\{\log(c_1) + \delta\log(W_1 -$

$c_1) + \delta r_2^{p,FB}\} = \delta E_0\{\log(\frac{W_1}{1+\delta}) + \delta\log(\frac{\delta W_1}{1+\delta}) + \delta r_2^{p,FB}\} = \delta E_0\{(1+\delta)\log(W_1) + \log(\frac{1}{1+\delta}) + \delta\log(\frac{\delta}{1+\delta}) + \delta r_2^{p,FB}\}$.

This is equivalent with solving $maxE_0\{\log(W_1)\}$ (it is the characteristic of log-utility function in a multi-periodic model -"myopia": it leads to concerns for only one period). But $\log(W_1) = \log I_0(1+R_1^p) = \log I_0 + r_1^p$. Using $E_0 r_1^p = r_1^f + \alpha_1^* E_0(r_1^x - r_1^f) + \frac{1}{2}\alpha_1^*(1-\alpha_1^*)\sigma_0^2$, for finding the optimal Decision 1 at $T=0$ one solves: $maxE_0\{\log(W_1)\} = \log I_0 + \max E_0 r_1^p = \log I_0 + \max\{r_1^f + \alpha_1^* E_0(r_1^x - r_1^f) + \frac{1}{2}\alpha_1^*(1-\alpha_1^*)\sigma_0^2\}$ with respect to $\alpha_1^* \in (0,1)$. In a similar manner with the second period, when computing the derivative w.r.t. α_1^* one obtains the optimal distribution for the first period given by $\alpha_1^* = \alpha_1^{*FB} = q_1 + \frac{1}{2}$. $(\frac{\partial}{\partial\alpha_1^*} = E_0(r_1^x - r_1^f) - \alpha_1^*\sigma_0^2 + \frac{1}{2}\sigma_0^2 = 0 \Leftrightarrow \alpha_1^* = \frac{E_0 r_1^x - r_1^f}{\sigma_0^2} + \frac{1}{2} = q_1 + \frac{1}{2})$. It is clear the requirement $q_1 \in (-\frac{1}{2}, \frac{1}{2}) \Rightarrow \alpha_1^{*FB} = q_1 + \frac{1}{2}$ for having an internal solution.

To compute the value of the objective function we first do (taking into account that $E_0 r_1^x - r_1^f = q_1\sigma_0^2$): $E_0 r_1^{p,FB} = r_1^f + \alpha_1^{*FB} E_0(r_1^x - r_1^f) + \frac{1}{2}\alpha_1^{*FB}(1-\alpha_1^{*FB})\sigma_0^2 = r_1^f + q_1\sigma_0^2(q_1+\frac{1}{2}) + \frac{1}{2}\sigma_0^2(q_1+\frac{1}{2})(\frac{1}{2}-q_1) = r_1^f + \sigma_0^2(q_1+\frac{1}{2})(q_1+\frac{1}{4}-\frac{q_1}{2}) = r_1^f + \sigma_0^2(q_1+\frac{1}{2})(\frac{q_1}{2}+\frac{1}{4}) = r_1^f + \frac{1}{2}\sigma_0^2(q_1+\frac{1}{2})^2$. Hence $E_0 r_2^{p,FB} = E_0 r_1^{p,FB} = r_1^f + \frac{1}{2}\sigma_0^2(q_1+\frac{1}{2})^2$.

Then,

$max\delta E_0\{\delta\log(c_1) + \delta^2\log(c_2)\} = max\delta E_0\{\log(c_1) + \delta\log(W_1 - c_1) + \delta(\frac{-\sigma_1^2}{2})\alpha_2^{*^2} + \delta(q_2+\frac{1}{2})\sigma_1^2\alpha_2^* + \delta r_2^f\} = \max\delta E_0\{(1+\delta)\log(W_1) + \log(\frac{1}{1+\delta}) + \delta\log(\frac{\delta}{1+\delta}) + \delta r_2^{p,FB}\} = \delta E_0\{(1+\delta)(\log I_0 + r_1^{p,FB}) + \log(\frac{1}{1+\delta}) + \delta\log(\frac{\delta}{1+\delta}) + \delta r_2^{p,FB}\} = \delta\{(1+\delta)\log I_0 + (1+2\delta)[r_1^f + \frac{1}{2}\sigma_0^2(q_1+\frac{1}{2})^2] - (1+\delta)\log(1+\delta) + \delta\log\delta\} = (\delta+\delta^2)\left\{\log(I_0) + \left[\frac{\delta}{1+\delta}\log(\delta) - \log(\delta+1)\right] + \frac{1+2\delta}{1+\delta}\left[r_1^f + \frac{1}{2}\sigma_0^2(q_1+\frac{1}{2})^2\right]\right\}$.

5. Proof of Proposition 2

We introduce the following formula for the moments of a log-normal variable:

$$\int_z^\infty y^q \frac{1}{y}\frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{(\ln y-\mu)^2}{2\sigma_0^2}} dy = e^{q\mu + \frac{q^2\sigma_0^2}{2}}[1 - \Phi(\frac{\ln(z) - (\mu + q\sigma_0^2)}{\sigma_0})] = e^{q\mu+\frac{q^2\sigma_0^2}{2}}\Phi(-\frac{\ln(z)-(\mu+q\sigma_0^2)}{\sigma_0})$$

.

We remark that (with $\rho_{\frac{X_1}{X_0}}(y) = \frac{1}{y\sigma_0\sqrt{2\pi}} e^{-\frac{(\ln y-\mu)^2}{2\sigma_0^2}}$): $\int_a^b \rho_{X_1}(x_1)dx_1 = \int_{a/X_0}^{b/X_0}\rho_{\frac{X_1}{X_0}}(y)dy =$

$\int_{a/X_0}^{b/X_0}\frac{1}{y\sigma_0\sqrt{2\pi}} e^{-\frac{(\ln y-\mu)^2}{2\sigma_0^2}} dy = \int_{\ln(a/X_0)}^{\ln(b/X_0)}\frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{(z-\mu)^2}{2\sigma_0^2}} dz = \Phi(\frac{\ln(b/X_0)-\mu}{\sigma_0}) - \Phi(\frac{\ln(a/X_0)-\mu}{\sigma_0})$ and applying the formula for the moments for the particular cases of $q=1$ respectively $q=-1$, one obtains:

$\int_a^b \frac{X_1}{X_0}dX_1 = \int_a^b \frac{X_1}{X_0}\rho_{X_1}(x_1)dx_1 = e^{\mu+\frac{\sigma_0^2}{2}}[\Phi(\frac{\ln(b/X_0)-\mu}{\sigma_0} - \sigma_0) - \Phi(\frac{\ln(a/X_0)-\mu}{\sigma_0} - \sigma_0)]$ and

$\int_a^b \left(\frac{X_1}{X_0}\right)^{-1} dX_1 = \int_a^b \left(\frac{X_1}{X_0}\right)^{-1}\rho_{X_1}(x_1)dx_1 = e^{-\mu+\frac{\sigma_0^2}{2}}[\Phi(\frac{\ln(b/X_0)-\mu}{\sigma_0} + \sigma_0) - \Phi(\frac{\ln(a/X_0)-\mu}{\sigma_0} + \sigma_0)]$.

With these formulas we start the analysis for the FB case.

$E_0(c_1^{FB}) = E_0(W_1 \frac{1}{1+\delta}) = \frac{1}{1+\delta} E_0(W_1);$
$E_0(\alpha_2^{FB}) = E_0\left(\frac{\alpha_2^{*FB} Inv_1^{FB}}{X_1}\right) = \alpha_2^{*FB} E_0\left(\frac{Inv_1^{FB}}{X_1}\right) = \alpha_2^{*FB} \frac{\delta}{1+\delta} E_0\left(\frac{W_1}{X_1}\right);$
$E_0(\beta_2^{FB}) = E_0\left(\frac{(1-\alpha_2^{*FB}) Inv_1^{FB}}{1+R_1^f}\right) = \frac{1-\alpha_2^{*FB}}{1+R_1^f} E_0\left(Inv_1^{FB}\right) = \frac{1-\alpha_2^{*FB}}{1+R_1^f} \frac{\delta}{1+\delta} E_0(W_1);$
$E_0(W_1) = I_0(1+R_1^f)(1-\alpha_1^{*FB}) + I_0 \alpha_1^{*FB} E_0(\frac{X_1}{X_0}) = I_0(1+R_1^f)(\frac{1}{2}-q_1) + I_0(q_1+\frac{1}{2}) e^{\mu+\frac{\sigma_0^2}{2}};$
$E_0\left(\frac{W_1}{X_1}\right) = I_0 \alpha_1^{*FB} \frac{1}{X_0} + \frac{I_0(1+R_1^f)(1-\alpha_1^{*FB})}{X_0} E_0(\frac{X_0}{X_1}) =$
$= I_0(q_1+\frac{1}{2}) \frac{1}{X_0} + \frac{I_0(1+R_1^f)(\frac{1}{2}-q_1)}{X_0} e^{-\mu+\frac{\sigma_0^2}{2}};$
Then:
a) $E_0(c_1^{FB}) = \frac{1}{1+\delta} I_0[(\frac{1}{2}-q_1)(1+R_1^f) + (\frac{1}{2}+q_1) e^{\mu+\frac{\sigma_0^2}{2}}];$
b) $E_0(\alpha_2^{FB}) - \alpha_1^{FB} = (q_2+\frac{1}{2}) \frac{I_0}{X_0} \{\frac{\delta}{1+\delta}[q_1+\frac{1}{2}+(1+R_1^f)(\frac{1}{2}-q_1) e^{-\mu+\frac{\sigma_0^2}{2}}]-1\};$
$E_0(\beta_2^{FB}) - \beta_1^{FB} = \frac{\frac{1}{2}-q_2}{1+R_1^f} \frac{\delta}{1+\delta} I_0[(1+R_1^f)(\frac{1}{2}-q_1) + (\frac{1}{2}+q_1) e^{\mu+\frac{\sigma_0^2}{2}}] - I_0(1-\alpha_1^{*FB}) = (\frac{1}{2}-q_2) \frac{\delta}{1+\delta} I_0[(\frac{1}{2}-q_1) + \frac{q_1+\frac{1}{2}}{1+R_1^f} e^{\mu+\frac{\sigma_0^2}{2}}] - I_0(\frac{1}{2}-q_1);$

6. Problem HC1Strategy 1

Solving the problem in the initial form is a complicated task. For this reason we prove first it can be reduced to the simpler problem:

Lemma 1 *The system from Problem HC1Strategy1 can be equivalently written as*

$$\max \delta E_1\{\log(c_1) + \delta \log(W_1 - c_1) + \delta(\frac{-\sigma_1^2}{2}) \alpha_2^{*^2} + \delta(q_2 + \frac{1}{2}) \sigma_1^2 \alpha_2^* + \delta r_2^f \mid_{(\alpha_1, \beta_1)}\}$$

s. t. $c_1 > 0$

$c_1 < W_1$

$\alpha_2^{*HC} \geq 0$

$\alpha_2^{*HC} < 1$

$$-c_1 \alpha_2^*(X_1 - X_0) + c_1 X_1 + \alpha_2^*(X_1 - X_0) W_1 \leq \alpha_1(X_1 - X_0) X_1 \qquad \text{(R1)}$$

Proof of Lemma 1:

We compute the equivalent form of the restrictions of the system in **Problem HC1Strategy1.**

Taking into account that $\alpha_2^{*HC} = \frac{\alpha_2^{HC} X_1}{Inv_1} \Rightarrow \alpha_2^{HC} = \frac{\alpha_2^{*HC} Inv_1}{X_1} = \frac{\alpha_2^{*HC}(W_1 - c_1)}{X_1}$ and $Inv_1^{HC} = \alpha_2^{HC} X_1 + \beta_2^{HC}(1+R_1^f) = W_1 - c_1^{HC} \Rightarrow \beta_2^{HC} = \frac{Inv_1^{HC} - \alpha_2^{HC} X_1}{1+R_1^f} = \frac{Inv_1^{HC}}{1+R_1^f}\left(1 - \frac{\alpha_2^{HC} X_1}{Inv_1^{HC}}\right) = \frac{Inv_1^{HC}}{1+R_1^f}\left(1 - \alpha_2^{*HC}\right)$, we have:

(25a) $\Leftrightarrow \alpha_2^{*HC}(W_1 - c_1^{HC}) < \alpha_1^{HC} X_1 \Leftrightarrow -c_1^{HC}\alpha_2^{*HC} + W_1\alpha_2^{*HC} < \alpha_1^{HC} X_1$;
(25b) $\Leftrightarrow \alpha_2^{*HC} \geq 0$;
(25c) tautology;
(25d) $\Leftrightarrow \left(1 - \alpha_2^{*HC}\right)(W_1 - c_1^{HC}) > \beta_1^{HC}(1 + R_1^f) \Leftrightarrow c_1^{HC}\alpha_2^{*HC} - c_1^{HC} - \alpha_2^{*HC} W_1 > \beta_1^{HC}(1 + R_1^f) - W_1 \Leftrightarrow -c_1^{HC}\alpha_2^{*HC} + c_1^{HC} + \alpha_2^{*HC} W_1 < \alpha_1^{HC} X_1 \Rightarrow$ (25a) is redundant; we only keep condition (25d); hence (25e) is also redundant;

From $c_1^{HC} < W_1$ (see footnote 24) we have $Inv_1^{HC} = W_1 - c_1^{HC} > 0$.

However (25e) $\Rightarrow \alpha_2^{*HC} < 1$ (because $\alpha_2^{*HC} < 1 \Leftrightarrow \frac{\alpha_2^{HC} X_1}{Inv_1} < 1 \Leftrightarrow \alpha_2^{HC} X_1 < Inv_1 = \alpha_2^{HC} X_1 + \beta_2^{HC}(1 + R_1^f)$ which is true for (25e) $\beta_2^{HC} > 0$).

(25f) we can assume $c_1^{HC} > 0$ (we look only for non-negative consumption because any feasible solution of the initial problem with $c_1^{HC} > 0$ is superior to any allocation with $c_1^{HC} = 0$);

(25g) $\Leftrightarrow c_1^{HC} \leq (\alpha_1^{HC} - \alpha_2^{HC})(X_1 - X_0) \Leftrightarrow c_1^{HC} \leq (\alpha_1^{HC} - \frac{\alpha_2^{*HC}(W_1 - c_1^{HC})}{X_1})(X_1 - X_0) \Leftrightarrow c_1^{HC} X_1 \leq \left(\alpha_1^{HC} X_1 - \alpha_2^{*HC}(W_1 - c_1^{HC})\right)(X_1 - X_0) \Leftrightarrow c_1^{HC}(X_1 - \alpha_2^{*HC}(X_1 - X_0)) \leq (\alpha_1^{HC} X_1 - \alpha_2^{*HC} W_1)(X_1 - X_0) \Leftrightarrow -c_1^{HC}\alpha_2^{*HC}(X_1 - X_0) + c_1^{HC} X_1 + \alpha_2^{*HC} W_1(X_1 - X_0) \leq (X_1 - X_0)\alpha_1^{HC} X_1$.

Now if we multiply the equivalent form of (25d) with $(X_1 - X_0) > 0$ we obtain: $-c_1^{HC}\alpha_2^{*HC}(X_1 - X_0) + c_1^{HC}(X_1 - X_0) + \alpha_2^{*HC} W_1(X_1 - X_0) < \alpha_1^{HC} X_1(X_1 - X_0)$.

We remark that (25g) $\Rightarrow$ (25d), hence (25d) is redundant, and we only keep (25g). Hence the important relation is (25g) completed with $\alpha_2^{*HC} \geq 0$, $c_1^{HC} > 0$, $\alpha_2^{*HC} < 1$ and $c_1^{HC} < W_1$. **q.e.d.**

We make the following change of variables: $P = W_1 - c_1$, $Q = \frac{X_1}{X_1 - X_0} - \alpha_2^*$; we denote by $ct_1 = X_1(\frac{W_1}{X_1 - X_0} - \alpha_1)$, $A = -\frac{\delta\sigma_1^2}{2} < 0$, $B = \delta\sigma_1^2(\frac{X_1}{X_1 - X_0} - (q_2 + \frac{1}{2})) > 0$, $C = \delta(-\frac{\sigma_1^2}{2}(\frac{X_1}{X_1 - X_0})^2 + (q_2 + \frac{1}{2})\sigma_1^2\frac{X_1}{X_1 - X_0} + r_2^f)$ and $f_1(P, Q) = \log(W_1 - P) + \delta\log(P) + AQ^2 + BQ + C$.

Lemma 2 *By rewriting the objective function from* ***Problem HC1Strategy1*** *in terms of the new variables P and Q, at $T = 1_-$, the maximization problem from* ***Lemma 1*** *is equivalent with:*

$$\max \delta E_1 f_1(P, Q)$$
$$\text{s. t.} PQ \geq ct_1$$
$$0 < P < W_1$$
$$\frac{X_0}{X_1 - X_0} < Q \leq \frac{X_1}{X_1 - X_0}$$

Proof. ■

With $P = W_1 - c_1$, $Q = \frac{X_1}{X_1 - X_0} - \alpha_2^*$, and $ct_1 = X_1(\frac{W_1}{X_1 - X_0} - \alpha_1)$,
we show first the condition (R1) can be rewritten as $PQ \geq ct_1$.

This is true because (R1) says: $-c_1\alpha_2^*(X_1 - X_0) + c_1 X_1 + \alpha_2^*(X_1 - X_0)W_1 \leq \alpha_1(X_1 - X_0)X_1 \Leftrightarrow$ (taking into account that $\frac{1}{X_1 - X_0} > 0$)$\Leftrightarrow -c_1\alpha_2^* + c_1\frac{X_1}{X_1 - X_0} +$

$\alpha_2^* W_1 \leq \alpha_1 X_1 \Leftrightarrow -(c_1 - W_1)\left(\alpha_2^* - \frac{X_1}{X_1 - X_0}\right) + \frac{W_1 X_1}{X_1 - X_0} - \alpha_1 X_1 \leq 0 \Leftrightarrow PQ \geq ct_1$.
Hence (R1) is equivalent with $PQ \geq ct_1$.

On the other hand we have $0 < P < W_1$ because $0 < c_1 < W_1$.

Also $\frac{X_0}{X_1 - X_0} < Q \leq \frac{X_1}{X_1 - X_0}$ because $\alpha_2^* \in [0, 1)$ and $\frac{X_1}{X_1 - X_0} > 1$.

Considering our objective function to be maximized, $\max \delta E_1\{\log(c_1) + \delta \log(W_1 - c_1) + \delta(\frac{-\sigma_1^2}{2}){\alpha_2^*}^2 + \delta(q_2 + \frac{1}{2})\sigma_1^2 \alpha_2^* + \delta r_2^f\} = max E_1\{\log(W_1 - P) + \delta \log(P) + \delta(\frac{-\sigma_1^2}{2})\left(\frac{X_1}{X_1 - X_0} - Q\right)^2 + \delta(q_2 + \frac{1}{2})\sigma_1^2\left(\frac{X_1}{X_1 - X_0} - Q\right) + \delta r_2^f\}$.

But $\log(W_1 - P) + \delta \log(P) + \delta(\frac{-\sigma_1^2}{2})[(\frac{X_1}{X_1 - X_0})^2 - \frac{2X_1}{X_1 - X_0}Q + Q^2] + [\delta(q_2 + \frac{1}{2})\sigma_1^2 \frac{X_1}{X_1 - X_0} - \delta(q_2 + \frac{1}{2})\sigma_1^2 Q + \delta r_2^f] = \log(W_1 - P) + \delta \log(P) + \delta(\frac{-\sigma_1^2}{2})Q^2 + (\delta\sigma_1^2 \frac{X_1}{X_1 - X_0} - \delta(q_2 + \frac{1}{2})\sigma_1^2)Q + \delta(\frac{-\sigma_1^2}{2}(\frac{X_1}{X_1 - X_0})^2 + (q_2 + \frac{1}{2})\sigma_1^2 \frac{X_1}{X_1 - X_0} + r_2^f) = \log(W_1 - P) + \delta \log(P) + AQ^2 + BQ + C$.

q.e.d.

We have to solve this problem by Kuhn-Tucker method: (we only consider the restriction (R1), then the domains restrictions for P and Q).

First we compute the derivatives with respect to P and Q:

$$\begin{cases} 1) & \frac{\partial f_1}{\partial P} = -\frac{1}{W_1 - P} + \frac{\delta}{P} \\ 2) & \frac{\partial f_1}{\partial Q} = 2AQ + B \end{cases}$$

$R_1 : -PQ \leq X_1\left(\alpha_1 - \frac{W_1}{X_1 - X_0}\right)$

$$\begin{cases} 1) & \frac{\partial R_1}{\partial P} = -Q \\ 2) & \frac{\partial R_1}{\partial Q} = -P \end{cases}$$

Applying Kuhn-Tucker:

$$\begin{cases} 1) & -\frac{1}{W_1 - P} + \frac{\delta}{P} = \lambda_1(-Q) \\ 2) & 2AQ + B = \lambda_1(-P) \\ 3) & \lambda_1\left(-PQ - X_1\left(\alpha_1 - \frac{W_1}{X_1 - X_0}\right)\right) = 0 \Leftrightarrow \quad \lambda_1\left(PQ + X_1\left(\alpha_1 - \frac{W_1}{X_1 - X_0}\right)\right) = 0 \end{cases}$$

with $\lambda_1 \geq 0$.

We are analyzing separately the following cases:

Case 1: $\lambda_1 = 0 \Rightarrow \frac{\delta}{P} = \frac{1}{W_1 - P} \Leftrightarrow P^{HC} = \frac{\delta W_1}{1 + \delta} = P^{FB}$.

$2AQ + B = 0 \Leftrightarrow Q^{HC} = -\frac{B}{2A} = \frac{X_1}{X_1 - X_0} - (q_2 + \frac{1}{2}) \Leftrightarrow$

$\Leftrightarrow {\alpha_2^*}^{HC} = q_2 + \frac{1}{2} = {\alpha_2^*}^{FB}$ (these are solutions of the FB type).

We have to check when this solution is feasible: (i.e. when $P = \frac{\delta}{\delta + 1}W_1$ and $Q = \frac{X_1}{X_1 - X_0} - (q_2 + \frac{1}{2})$ satisfy the restriction (R1)). Hence it has to: $-PQ \leq X_1\left(\alpha_1 - \frac{W_1}{X_1 - X_0}\right) \Leftrightarrow -\frac{\delta}{\delta + 1}W_1\left(\frac{X_1}{X_1 - X_0} - (q_2 + \frac{1}{2})\right) \leq X_1\left(\alpha_1 - \frac{W_1}{X_1 - X_0}\right) \Leftrightarrow W_1 X_1 + \delta W_1(X_1 - X_0)(q_2 + \frac{1}{2}) \leq (\delta + 1)X_1\alpha_1(X_1 - X_0) \Leftrightarrow X_1^2\alpha_1\delta(q_2 - \frac{1}{2}) + X_1\left(\alpha_1 X_0(1 + \delta(\frac{1}{2} - q_2)) + \beta_1 S_1(1 + \delta(q_2 + \frac{1}{2}))\right) - \beta_1 S_1 X_0 \delta(q_2 + \frac{1}{2}) \leq 0$

This leads to the inequality (in order to have FB): $M_\alpha X_1^2 + N_\alpha X_1 + P_\alpha \le 0$, where $M_\alpha = \alpha_1\delta(q_2 - \frac{1}{2}) < 0$, $N_\alpha = \alpha_1 X_0(1+\delta(\frac{1}{2}-q_2)) + \beta_1(1+R_1^f)(1+\delta(q_2+\frac{1}{2})) > 0$ (as a linear combination of positive terms and $P_\alpha = -\beta_1(1+R_1^f)X_0\delta(q_2+\frac{1}{2}) < 0$.

We compute $\Delta_\alpha = N_\alpha^2 - 2\alpha_1 X_0\beta_1(1+R_1^f)(\frac{\delta^2}{2} - 2\delta^2 q_2^2) < N_\alpha^2 = \Delta_\alpha^{aprox}$.

We show first $\Delta_\alpha > 0$, $\forall X_1 > X_0$.

$\Delta_\alpha = N_\alpha^2 - 4M_\alpha P_\alpha = \alpha_1^2 X_0^2(1+\delta(\frac{1}{2}-q_2))^2 + \beta_1^2(1+R_1^f)^2(1+\delta(q_2+\frac{1}{2}))^2 + 2\alpha_1 X_0\beta_1(1+R_1^f)(1+\delta(\frac{1}{2}-q_2))(1+\delta(q_2+\frac{1}{2})) + 4\alpha_1\delta(q_2-\frac{1}{2})\beta_1(1+R_1^f)X_0\delta(q_2+\frac{1}{2}) = \alpha_1^2X_0^2(1+\delta(\frac{1}{2}-q_2))^2 + \beta_1^2(1+R_1^f)^2(1+\delta(q_2+\frac{1}{2}))^2 + 2\alpha_1 X_0\beta_1(1+R_1^f)(1+\delta(q_2+\frac{1}{2}) + \delta(\frac{1}{2}-q_2)) + 2\alpha_1 X_0\beta_1(1+R_1^f)\delta^2(q_2^2 - \frac{1}{4}) > 0$.

We compute $M_\alpha X_0^2 + N_\alpha X_0 + P_\alpha = \alpha_1 X_0^2 + \beta_1(1+R_1^f)X_0 > 0$. Taking into account $M_\alpha < 0$, it comes $X_0 \in (x_1, x_2)$, where x_1, x_2 are the solutions of the second degree equation. In order to have FB solution, one has to have $X_1 > X_0$ and $X_1 \notin (x_1, x_2)$, hence $X_1 \ge x_2 \overset{def}{=} X_{FB} = -\frac{N_\alpha}{2M_\alpha} - \frac{\sqrt{\Delta_\alpha}}{2M_\alpha}$.

We consider the following approximated threshold $X_{FB}^{aprox} = x_2^{aprox} = -\frac{N_\alpha}{2M_\alpha} - \frac{N_\alpha}{2M_\alpha} = -\frac{N_\alpha}{M_\alpha} = X_0 + \frac{[\alpha_1 X_0 + \beta_1(1+R_1^f)] + \beta_1(1+R_1^f)\delta(q_2+\frac{1}{2})}{\alpha_1\delta(\frac{1}{2}-q_2)} > X_0$.

Case 2: $\lambda_1 \neq 0 \Rightarrow \frac{-\frac{1}{W_1-P} + \frac{\delta}{P}}{2AQ+B} = \frac{Q}{P}$ and $PQ = ct_1$

$\frac{-P+\delta(W_1-P)}{W_1-P} = 2AQ^2 + BQ \Leftrightarrow \frac{-P}{W_1-P} = 2AQ^2 + BQ - \delta \Leftrightarrow$

$-P = (W_1-P)\left(2AQ^2 + BQ - \delta\right) \Leftrightarrow \frac{-ct_1}{Q} = 2AW_1Q^2 + BW_1Q - \delta W_1 - 2AQct_1 - Bct_1 + \frac{\delta ct_1}{Q} \Leftrightarrow -2AW_1Q^3 + (2Act_1 - BW_1)Q^2 + (\delta W_1 + Bct_1)Q - ct_1(1+\delta) = 0$.

In particular, we are looking for the solutions Q' of the previous equation that satisfies $Q' \in (\frac{X_0}{X_1-X_0}, \frac{X_1}{X_1-X_0})$ (Interior Solutions) and to lead to $P' = \frac{ct1}{Q'} = \frac{X_1(\frac{W_1}{X_1-X_0} - \alpha_1)}{Q'} < W_1$ (we know that $P' > 0$).

It results: $P' < W_1 \Leftrightarrow \frac{ct1}{Q'} < W_1 \Leftrightarrow Q' > \frac{ct1}{W_1}$.

We use the following notations: $a = \frac{X_1}{X_1-X_0} - (\frac{1}{2}+q_2) > 0$ and $b = \frac{X_1}{X_1-X_0} - \frac{\alpha_1 X_1}{W_1} > 0$.Then $ct_1 = W_1 b$. This leads to $Q' > \frac{ct1}{W_1} \Leftrightarrow \frac{X_1}{X_1-X_0} > Q' > b$. We rewrite the equation in terms of a and $b:$ $Q^3 - (a+b)Q^2 + (ab + \frac{1}{\sigma_1^2})Q - (\frac{1+\delta}{\delta}\frac{1}{\sigma_1^2})b = 0$ and we want the solutions satisfying $Q' > b$ and $Q' \le \frac{X_1}{X_1-X_0} = a + (\frac{1}{2}+q_2)$. From here one can deduce $\alpha_2^{*HC} = \frac{X_1}{X_1-X_0} - Q'$ and $c_1^{HC} = W_1 - P = W_1 - \frac{ct_1}{Q'} = W_1 - \frac{W_1 b}{Q'} = W_1(1 - \frac{b}{Q'})$.

Case 3: Corner Solutions, when Kuhn-Tucker cannot be applied:

Are solutions having $\alpha_2^* = 0$ ($\Leftrightarrow Q = \frac{X_1}{X_1-X_0}$).

In terms of the initial notation, we have to solve at $T = 1$ (for $\alpha_2^* = 0$):

$\max\delta E_1\{\log(c_1) + \delta\log(W_1 - c_1) + \delta(\frac{-\sigma_1^2}{2}){\alpha_2^*}^2 + \delta(q_2+\frac{1}{2})\sigma_1^2\alpha_2^* + \delta r_2^f\} = \max\delta\{\log(c_1) + \delta\log(W_1-c_1) + \delta r_2^f\}$ (it is not necessary E_1 as we only have risk-free terms). Also, (R1) reduces in this case ($\alpha_2^* = 0 \;\; \Leftrightarrow \alpha_2^{HC} = 0$) to: $c_1 \le \Pi_1^{HC} = \alpha_1^{HC}(X_1 - X_0)$.

Considering the function $f_c(c_1) = \log(c_1) + \delta \log(W_1 - c_1) + \delta r_2^f$, it has the global maximum $c_1^{FB} = \frac{1}{\delta+1} W_1$ (because $\frac{\partial f_c}{\partial c_1} = \frac{1}{c_1} - \frac{\delta}{W_1 - c_1} = 0 \Leftrightarrow c_1^{FB} = \frac{1}{\delta+1} W_1$). Also, $f_c(c_1)$ is increasing on $(0, c_1^{FB})$ and decreasing on (c_1^{FB}, W_1). Hence, the maximum of the function $f_c(c_1)$ on the restricted domain $(0, \alpha_1^{HC}(X_1 - X_0)]$ is:

$$\begin{cases} \text{If } c_1^{FB} \leq \alpha_1(X_1 - X_0) \Rightarrow \text{we choose } c_1 = c_1^{FB}; \\ \text{If } c_1^{FB} > \alpha_1(X_1 - X_0) \Rightarrow \text{we choose } c_1 = \alpha_1(X_1 - X_0). \end{cases}$$

They are the two possible corner solutions. We have to decide the threshold point at which the corner solutions are changing:

$c_1^{FB} \leq \alpha_1(X_1 - X_0) \Leftrightarrow \frac{1}{\delta+1} W_1 \leq \alpha_1(X_1 - X_0) \Leftrightarrow \alpha_1 X_1 \geq (I_0 - \alpha_1 X_0)(1 + R_1^f)\frac{1}{\delta} + \frac{1+\delta}{\delta}\alpha_1 X_0 \Leftrightarrow X_1 \geq (I_0 - \alpha_1 X_0)(1 + R_1^f)\frac{1}{\alpha_1 \delta} + \frac{1+\delta}{\delta} X_0 \stackrel{def}{=} CornerThreshold_1$

If $X_1 \geq CornerThreshold_1$, $\alpha_2^{*HC} = 0$, $c_1^{HC} = c_1^{FB} = \frac{1}{\delta+1} W_1 \Leftrightarrow Q = \frac{X_1}{X_1 - X_0}$, $P = \frac{\delta}{\delta+1} W_1$;

If $X_1 < CornerThreshold_1$, $\alpha_2^{*HC} = 0 \Leftrightarrow \alpha_2^{HC} = 0$, $c_1^{HC} = \Pi_1^{HC} = \alpha_1(X_1 - X_0) \Leftrightarrow Q = \frac{X_1}{X_1 - X_0}$, $P = W_1 - \alpha_1(X_1 - X_0)$;

One can note $CornerThreshold_1 = X_0 + \frac{[\alpha_1 X_0 + \beta_1(1+R_1^f)](\frac{1}{2} - q_2)}{\alpha_1 \delta(\frac{1}{2} - q_2)} = X_0 + \frac{[\alpha_1 X_0 + \beta_1(1+R_1^f)]}{\alpha_1 \delta} > X_0$ and $CornerThreshold_1 < X_{FB}^{aprox}$. To prove the last inequality we note it can be equivalently written as $(I_0 - \alpha_1 X_0)(1 + R_1^f)\frac{1}{\alpha_1 \delta} + \frac{1}{\delta} X_0 < \frac{[\alpha_1 X_0 + \beta_1(1+R_1^f)] + \beta_1(1+R_1^f)\delta(q_2 + \frac{1}{2})}{\alpha_1 \delta(\frac{1}{2} - q_2)} \Leftrightarrow \frac{[\alpha_1 X_0 + \beta_1(1+R_1^f)](\frac{1}{2} - q_2)}{\alpha_1 \delta(\frac{1}{2} - q_2)} < \frac{[\alpha_1 X_0 + \beta_1(1+R_1^f)] + \beta_1(1+R_1^f)\delta(q_2 + \frac{1}{2})}{\alpha_1 \delta(\frac{1}{2} - q_2)} \Leftrightarrow$
$[\alpha_1 X_0 + \beta_1(1 + R_1^f)](\frac{1}{2} - q_2) < [\alpha_1 X_0 + \beta_1(1 + R_1^f)] + \beta_1(1 + R_1^f)\delta(q_2 + \frac{1}{2}) \Leftrightarrow$
$\alpha_1 X_0 + \beta_1(1 + R_1^f) + \beta_1(1 + R_1^f)\delta > 0$ (always true).

7. Problem HC1Strategy2

Lemma 3 *The system from **Problem HC1Strategy2** can be written as:*

$$\begin{aligned} &\max \delta E_1\{\log(c_1) + \delta \log(W_1 - c_1) + \delta(\frac{-\sigma_1^2}{2})\alpha_2^{*^2} + \delta(q_2 + \frac{1}{2})\sigma_1^2 \alpha_2^* + \delta r_2^f \mid_{(\alpha_1, \beta_1)}\} \\ &\text{s. t.} c_1 > 0 \\ &\quad c_1 < W_1 \\ &\quad \alpha_2^{*HC} > 0 \\ &\quad \alpha_2^{*HC} \leq 1 \\ &\quad c_1 \alpha_2^* - \alpha_2^* W_1 + c_1 \frac{1}{R_1^f} \leq -\alpha_1 X_1 \qquad \text{(R2)} \end{aligned}$$

Proof :

We take into account that $\alpha_2^{*HC} = \frac{\alpha_2^{HC} X_1}{Inv_1} \Rightarrow \alpha_2^{HC} = \frac{\alpha_2^{*HC} Inv_1}{X_1} = \frac{\alpha_2^{*HC}(W_1 - c_1)}{X_1}$,

$Inv_1^{HC} = \alpha_2^{HC} X_1 + \beta_2^{HC}(1 + R_1^f) \Rightarrow \beta_2^{HC} = \frac{Inv_1^{HC} - \alpha_2^{HC} X_1}{1 + R_1^f} = \frac{Inv_1^{HC}}{1 + R_1^f}\left(1 - \frac{\alpha_2^{HC} X_1}{Inv_1^{HC}}\right) = \frac{Inv_1^{HC}}{1 + R_1^f}\left(1 - \alpha_2^{*HC}\right)$,

$\alpha_1^{HC}X_0+\beta_1^{HC}=I_0\Rightarrow\beta_1^{HC}=I_0-\alpha_1^{HC}X_0$ and $\Pi_1^{HC}=(\beta_1^{HC}-\beta_2^{HC})R_1^f$.

We have the following equivalent restrictions for those of the system from **Problem HC1Strategy2**:

(29a) $\Leftrightarrow$ $\left(1-\alpha_2^{*HC}\right)(W_1-c_1^{HC})<\beta_1^{HC}(1+R_1^f)\Leftrightarrow c_1^{HC}\alpha_2^{*HC}-c_1^{HC}-\alpha_2^{*HC}W_1<\beta_1^{HC}(1+R_1^f)-W_1\Leftrightarrow c_1^{HC}\alpha_2^{*HC}-c_1^{HC}-\alpha_2^{*HC}W_1<-\alpha_1^{HC}X_1\Leftrightarrow$
$-c_1^{HC}\alpha_2^{*HC}+c_1^{HC}+\alpha_2^{*HC}W_1>\alpha_1^{HC}X_1$

(29b) $\Leftrightarrow\alpha_2^{*HC}\leq 1$ (assuming $c_1^{HC}<W_1$, because $c_1^{HC}\leq\Pi_1^{HC}=(\beta_1^{HC}-\beta_2^{HC})R_1^f\leq\beta_1^{HC}R_1^f<W_1=\alpha_1^{HC}X_1+\beta_1^{HC}(1+R_1^f)$);

(29c) tautology

(29d) $\Leftrightarrow\alpha_2^{*HC}(W_1-c_1^{HC})>\alpha_1^{HC}X_1\Leftrightarrow -c_1^{HC}\alpha_2^{*HC}+W_1\alpha_2^{*HC}>\alpha_1^{HC}X_1\Rightarrow$ (29a) is redundant; we only keep condition (29d); (29e) is also redundant

(29e) $\Leftrightarrow\alpha_2^{*HC}>0$

(29f) similarly with (25f), we can assume $c_1^{HC}>0$ (we look only for non-negative consumption because any feasible solution of the initial problem with $c_1^{HC}>0$ is superior to any allocation with $c_1^{HC}=0$);

(29g) $\Leftrightarrow c_1^{HC}\leq(\beta_1^{HC}-\beta_2^{HC})R_1^f\Leftrightarrow c_1^{HC}\leq(\beta_1^{HC}-\frac{Inv_1^{HC}}{1+R_1^f}\left(1-\alpha_2^{*HC}\right))R_1^f\Leftrightarrow$
$c_1^{HC}(1+R_1^f)\leq\left(\beta_1^{HC}(1+R_1^f)-(W_1-c_1^{HC})\left(1-\alpha_2^{*HC}\right)\right)R_1^f\Leftrightarrow$

$c_1^{HC}(1+R_1^f-\left(1-\alpha_2^{*HC}\right)R_1^f)\leq\left(\beta_1^{HC}(1+R_1^f)-W_1\left(1-\alpha_2^{*HC}\right)\right)R_1^f\Leftrightarrow$

$\Leftrightarrow -c_1^{HC}\alpha_2^{*HC}R_1^f-c_1^{HC}+W_1\alpha_2^{*HC}R_1^f\geq\alpha_1^{HC}X_1R_1^f\Leftrightarrow c_1\alpha_2^*-\alpha_2^*W_1+c_1\frac{1}{R_1^f}\leq -\alpha_1X_1$

Now if we multiply the equivalent form of (29d) with $R_1^f>0$ we obtain: $-c_1^{HC}\alpha_2^{*HC}R_1^f+W_1\alpha_2^{*HC}R_1^f>\alpha_1^{HC}X_1R_1^f$.

We remark that (29g) $\Rightarrow$ (29d), hence (29d) is redundant, and we only keep (29g). **q.e.d.**

We make the change of variables $P=W_1-c_1$, $R=\frac{1}{R_1^f}+\alpha_2^*$, and $ct_2=\alpha_1X_1+W_1\frac{1}{R_1^f}>0$, $A_2=-\frac{\delta\sigma_1^2}{2}<0$, $B_2=\delta\sigma_1^2\left(\frac{1}{R_1^f}+(q_2+\frac{1}{2})\right)>0$, $C_2=\delta\left(-\frac{\sigma_1^2}{2}\left(\frac{1}{R_1^f}\right)^2-(q_2+\frac{1}{2})\sigma_1^2\frac{1}{R_1^f}+r_2^f\right)$, $f_2(P,R)=(\log(W_1-P)+\delta\log(P)+A_2R^2+B_2R+C_2)$.

Lemma 4 *By rewriting the objective function from **Problem HC1Strategy2** in terms of the new variables P and R, at T = 1_ , the maximization problem from **Lemma 3** is equivalent with:*

$$
\begin{aligned}
&\max \delta E_1 f_2(P,R)\\
&\text{s. t.} -PR\leq -ct_2\\
&\quad 0<P<W_1\\
&\quad \frac{1}{R_1^f}<R\leq 1+\frac{1}{R_1^f}
\end{aligned}
$$

Proof :

Using the notations $P = W_1 - c_1$, $R = \frac{1}{R_1^f} + \alpha_2^*$, and $ct_2 = \alpha_1 X_1 + W_1 \frac{1}{R_1^f} > 0$ we show (R2) can be rewritten as $PR \geq ct_2$. The condition (R2) is $c_1\alpha_2^* - \alpha_2^* W_1 + c_1 \frac{1}{R_1^f} \leq -\alpha_1 X_1 \Leftrightarrow (c_1 - W_1)\left(\alpha_2^* + \frac{1}{R_1^f}\right) \leq -\left(\alpha_1 X_1 + W_1 \frac{1}{R_1^f}\right) \Leftrightarrow -PR \leq -ct_2$ q.e.d. $P = W_1 - c_1 \Leftrightarrow c_1 = W_1 - P \Rightarrow 0 < P \leq W_1$;

$R = \frac{1}{R_1^f} + \alpha_2^* \Leftrightarrow \alpha_2^* = R - \frac{1}{R_1^f} \Rightarrow \frac{1}{R_1^f} < R \leq \frac{1}{R_1^f} + 1$.

With notations P, R and ct_2 the objective function from **Problem HC1Strategy2** becomes: $\max \delta E_1\{\log(c_1) + \delta \log(W_1 - c_1) + \delta(\frac{-\sigma_1^2}{2})\alpha_2^{*^2} + \delta(q_2 + \frac{1}{2})\sigma_1^2\alpha_2^* + \delta r_2^f\} = maxE_1\{\log(W_1 - P) + \delta \log(P) + \delta(\frac{-\sigma_1^2}{2})\left(R - \frac{1}{R_1^f}\right)^2 + \delta(q_2 + \frac{1}{2})\sigma_1^2\left(R - \frac{1}{R_1^f}\right) + \delta r_2^f\}$.

But $\log(W_1 - P) + \delta \log(P) + \delta(\frac{-\sigma_1^2}{2})[R^2 - 2R\frac{1}{R_1^f} + (\frac{1}{R_1^f})^2] + [\delta(q_2 + \frac{1}{2})\sigma_1^2 R - \delta(q_2 + \frac{1}{2})\sigma_1^2\frac{1}{R_1^f} + \delta r_2^f] = \log(W_1 - P) + \delta \log(P) + \delta(\frac{-\sigma_1^2}{2})R^2 + R[2\frac{1}{R_1^f}\delta\frac{\sigma_1^2}{2} + \delta\sigma_1^2(q_2 + \frac{1}{2})] + \delta(\frac{-\sigma_1^2}{2})(\frac{1}{R_1^f})^2 - \delta(q_2 + \frac{1}{2})\sigma_1^2\frac{1}{R_1^f} + \delta r_2^f = \log(W_1 - P) + \delta \log(P) + A_2R^2 + B_2R + C_2$;

We have to solve this problem by Kuhn-Tucker method: (we only consider the restriction (R2) then the domains restrictions for P and R).

First we compute the derivatives with respect to P and R:

$$\begin{cases} 1) & \frac{\partial f_2}{\partial P} = -\frac{1}{W_1 - P} + \frac{\delta}{P} \\ 2) & \frac{\partial f_2}{\partial R} = 2A_2R + B_2 \end{cases}$$

$R_2 : -PR \leq -\left(\alpha_1 X_1 + W_1 \frac{1}{R_1^f}\right)$

$$\begin{cases} 1) & \frac{\partial R_2}{\partial P} = -R \\ 2) & \frac{\partial R_2}{\partial R} = -P \end{cases}$$

Applying Kuhn-Tucker:

$$\begin{cases} 1) & -\frac{1}{W_1 - P} + \frac{\delta}{P} = \lambda_2(-R) \\ 2) & 2A_2R + B_2 = \lambda_2(-P) \\ 3) & \lambda_2\left(-PR + \alpha_1 X_1 + W_1\frac{1}{R_1^f}\right) = 0 \end{cases}$$

with $\lambda_2 \geq 0$.

Case 1: $\lambda_2 = 0 \Rightarrow \frac{\delta}{P} = \frac{1}{W_1 - P} \Leftrightarrow P^{HC} = \frac{\delta W_1}{1+\delta} = P^{FB}$;

$2A_2R + B_2 = 0 \Leftrightarrow R^{HC} = -\frac{B_2}{2A_2} = \frac{1}{R_1^f} + \left(q_2 + \frac{1}{2}\right) \Leftrightarrow \alpha_2^{*HC} = q_2 + \frac{1}{2} = \alpha_2^{*FB}$

(these are solutions of the FB type)

We prove with Strategy 2 we cannot obtain the FB path (assuming $R_1^f < 1$). In order to be a feasible solution of the system, $P^{HC} = \frac{\delta W_1}{1+\delta}$ and $R^{HC} = \frac{1}{R_1^f} +$

$\left(q_2+\frac{1}{2}\right)$ should fulfill $PR \geq ct_2 \Leftrightarrow \frac{\delta W_1}{1+\delta}\left(\frac{1}{R_1^f}+\left(q_2+\frac{1}{2}\right)\right) \geq \alpha_1 X_1 + W_1\frac{1}{R_1^f} \Leftrightarrow \frac{W_1}{\delta+1}\frac{R_1^f\delta\left(q_2+\frac{1}{2}\right)-1}{R_1^f} - \alpha_1 X_1 \geq 0$.

Assuming that $R_1^f < 1$ it comes that $R_1^f\delta\left(q_2+\frac{1}{2}\right)-1<0$, then the LHS of the inequality is negative, hence the required condition is never fulfilled.

We conclude with Strategy 2 will never obtain the FB path.

Case 2: $\lambda_2 \neq 0 \Rightarrow \frac{-\frac{1}{W_1-P}+\frac{\delta}{P}}{2A_2R+B_2} = \frac{R}{P}$ and $PR = \alpha_1 X_1 + W_1\frac{1}{R_1^f} \Leftrightarrow (PR = ct_2)$

$\frac{-P+\delta(W_1-P)}{W_1-P} = 2A_2R^2 + B_2R \Leftrightarrow \frac{-P}{W_1-P} = 2A_2R^2 + B_2R - \delta \Leftrightarrow$

$\frac{-ct_2}{R} = 2A_2W_1R^2 + B_2W_1R - \delta W_1 - 2A_2Rct_2 - B_2ct_2 + \frac{\delta ct_2}{R} \Leftrightarrow$

$-2A_2W_1R^3 + (2A_2ct_2 - B_2W_1)R^2 + (\delta W_1 + B_2ct_2)R - ct_2(1+\delta) = 0$ (Interior solutions)

We denote $c = \frac{1}{R_1^f} + (\frac{1}{2}+q_2) > 0$ and $d = \frac{1}{R_1^f} + \frac{\alpha_1 X_1}{W_1} > 0$.

We are looking for the solutions R' of the previous equation that satisfies $R' \in (\frac{1}{R_1^f} + \frac{\alpha_1 X_1}{W_1}, 1 + \frac{1}{R_1^f}) = (d, c + \frac{1}{2} - q_2)$ and to lead to $P' = \frac{ct_2}{R'} = \frac{W_1 d}{R'} > 0$ and $P' < W_1 \Leftrightarrow \frac{d}{R'} < 1 \Leftrightarrow R' > d$.

The equation becomes, similarly to the proof from **annex point 6,** $R^3 - R^2(c+d) + R(cd + \frac{1}{\sigma_1^2}) - \frac{(1+\delta)d}{\delta\sigma_1^2} = 0$

Case 3: Corner Solutions, when Kuhn-Tucker cannot be applied:

Are solutions having $\alpha_2^* = 1$ ($\Leftrightarrow \beta_2 = 0 \Leftrightarrow R = 1 + \frac{1}{R_1^f}$)

In terms of the initial notation, we have to solve at $T = 1$:

$\max \delta E_1\{\log(c_1) + \delta\log(W_1 - c_1) + \delta(\frac{-\sigma_1^2}{2}) + \delta(q_2+\frac{1}{2})\sigma_1^2 + \delta r_2^f\}$ (for $\alpha_2^* = 1$)

subject to $c_1 \leq \Pi_2^{HC} = (\beta_1^{HC} - \beta_2^{HC})R_1^f = \beta_1^{HC}R_1^f$.

Considering the function $f_{c,2}(c_1) = \log(c_1) + \delta\log(W_1 - c_1) + \delta(\frac{-\sigma_1^2}{2}) + \delta(q_2+\frac{1}{2})\sigma_1^2 + \delta r_2^f$, it has the global maximum $c_1^{FB} = \frac{1}{\delta+1}W_1$(because $\frac{\partial f_{c,2}}{\partial c_1} = \frac{1}{c_1} - \frac{\delta}{W_1 - c_1} = 0 \Leftrightarrow c_1^{FB} = \frac{1}{\delta+1}W_1$).

Also, $f_{c,2}(c_1)$ is increasing on $(0, c_1^{FB})$ and decreasing on (c_1^{FB}, W_1). Hence, the maximum of the function $f_{c,2}(c_1)$ on the restricted domain $(0, \beta_1^{HC}R_1^f]$ is:

If $c_1^{FB} \leq \beta_1^{HC}R_1^f \Rightarrow$ we choose $c_1 = c_1^{FB}$

If $c_1^{FB} > \beta_1^{HC}R_1^f \Rightarrow$ we choose $c_1 = \beta_1^{HC}R_1^f$ (i.e. all the profit)

These are the two available corner solutions. We have to decide the threshold point at which the corner solutions are changing.

We show we can never reach the case $c_1^{FB} \leq \beta_1^{HC}R_1^f$.

To prove this, we assume that $c_1^{FB} \leq \beta_1^{HC}R_1^f \Leftrightarrow \frac{1}{\delta+1}W_1 \leq \beta_1^{HC}R_1^f \Leftrightarrow$

$W_1 \leq (\delta+1)\beta_1^{HC}R_1^f \Leftrightarrow \alpha_1^{HC}X_1 + \beta_1^{HC}\left(1+R_1^f\right) \leq (\delta+1)\beta_1^{HC}R_1^f \Leftrightarrow$

$\alpha_1^{HC}X_1 \leq \beta_1^{HC}\left[(\delta+1)R_1^f - \left(1+R_1^f\right)\right] = \beta_1^{HC}(\delta R_1^f - 1) < 0$: false, assuming $R_1^f < 1$ and $\delta < 1$. Hence, the only available corner solution is of the form: $\alpha_2^* = 1$ and $c_1^{HC} = \beta_1 R_1^f = (I_0 - \alpha_1 X_1)R_1^f \Leftrightarrow R' = 1 + \frac{1}{R_1^f}$, $P' = W_1 - c_1 = W_1 - \beta_1 R_1^f = I_0 + \alpha_1(X_1 - X_0)$;

8.Proof of Proposition 3:

1) For Strategy 1. It is based on the fact that $Q = \frac{X_1}{X_1 - X_0} - \alpha_2^*$, $a = \frac{X_1}{X_1-X_0} - (\frac{1}{2} + q_2)$ and $b = \frac{X_1}{X_1-X_0} - \frac{\alpha_1 X_1}{W_1}$ are approximatively equal (with $\frac{X_1}{X_1-X_0} - \alpha_2^{*HC}$).

We approximate the term $-(\frac{1+\delta}{\delta}\frac{1}{\sigma_1^2})b \simeq -(\frac{1+\delta}{\delta}\frac{1}{\sigma_1^2})Q$ in the 3rd degree equation giving the interior solutions, such that it becomes: $Q^3 - (a+b)Q^2 + (ab + \frac{1}{\sigma_1^2})Q - (\frac{1+\delta}{\delta}\frac{1}{\sigma_1^2})Q = 0 \Leftrightarrow Q^2 - (a+b)Q + (ab + \frac{1}{\sigma_1^2} - \frac{1+\delta}{\delta}\frac{1}{\sigma_1^2}) = 0$.

We approximated the last term (that of degree 0) in order to minimize the error committed (however, δ and σ_1^2 have an important role here for the accuracy of this approximation).

This equation has always two real solutions, because $\Delta = (a+b)^2 - 4\left(ab - \frac{1}{\delta}\frac{1}{\sigma_1^2}\right) = (a-b)^2 + 4\frac{1}{\delta}\frac{1}{\sigma_1^2} > 0$. Hence $Q_{1,2} = \frac{a+b\pm\sqrt{\Delta}}{2} = \frac{a+b}{2} \pm \frac{\sqrt{(a-b)^2 + 4\frac{1}{\delta}\frac{1}{\sigma_1^2}}}{2}$.

However, we have to check whether the solutions $Q_{1,2}$ satisfy both $Q > b$ and $Q \leq a + (\frac{1}{2} + q_2)$. We start by checking first whether $Q_{1,2} > b$.

If $a > b$: $Q_1 = \frac{a+b}{2} - \frac{\sqrt{\Delta}}{2} > b \Leftrightarrow a - \sqrt{\Delta} > b \Leftrightarrow a - b > \sqrt{\Delta} \Leftrightarrow (a-b)^2 > (a-b)^2 + 4\frac{1}{\delta}\frac{1}{\sigma_1^2}$ False. Hence $Q_1 < b$;

We analyze now Q_2 : $Q_2 = \frac{a+b}{2} + \frac{\sqrt{\Delta}}{2} > b \Leftrightarrow a + \sqrt{\Delta} > b \Leftrightarrow a - b > -\sqrt{\Delta}$ True. Hence $Q_2 > b$;.

If $a \leq b$: $Q_1 = \frac{a+b}{2} - \frac{\sqrt{\Delta}}{2} > b \Leftrightarrow a - \sqrt{\Delta} > b \Leftrightarrow a - b > \sqrt{\Delta}$ False. Hence $Q_1 < b$;

$Q_2 = \frac{a+b}{2} + \frac{\sqrt{\Delta}}{2} > b \Leftrightarrow a + \sqrt{\Delta} > b \Leftrightarrow a - b > -\sqrt{\Delta} \Leftrightarrow b - a < \sqrt{\Delta} \Leftrightarrow (a-b)^2 < (a-b)^2 + 4\frac{1}{\delta}\frac{1}{\sigma_1^2}$ True. Hence $Q_2 > b$;

We conclude we always choose $Q_2 = \frac{a+b}{2} + \frac{\sqrt{\Delta}}{2} > b$.

The only thing to be tested is whether $Q_2 \leq a + (\frac{1}{2} + q_2) \Leftrightarrow$

$\frac{a+b}{2} + \frac{\sqrt{\Delta}}{2} \leq a + (\frac{1}{2} + q_2) \Leftrightarrow b + \sqrt{\Delta} \leq a + 2(\frac{1}{2} + q_2) \Leftrightarrow$

$\sqrt{\Delta} \leq a - b + 2(\frac{1}{2} + q_2)$. (For this condition to be satisfied, we have to ask additionally that $a - b \geq -2(\frac{1}{2} + q_2)$).

$(a-b)^2 + 4\frac{1}{\delta}\frac{1}{\sigma_1^2} \leq (a-b)^2 + 4(\frac{1}{2} + q_2)^2 + 4(a-b)(\frac{1}{2} + q_2) \Leftrightarrow$

$\frac{1}{\delta}\frac{1}{\sigma_1^2} \leq (\frac{1}{2} + q_2)^2 + (a-b)(\frac{1}{2} + q_2) \Leftrightarrow (a-b)(\frac{1}{2} + q_2) \geq \frac{1}{\delta}\frac{1}{\sigma_1^2} - (\frac{1}{2} + q_2)^2 \Leftrightarrow$

$a - b \geq -(\frac{1}{2} + q_2) + \frac{1}{\delta}\frac{1}{\sigma_1^2(\frac{1}{2} + q_2)}$ (we have to ask only for this restriction, $a - b \geq -2(\frac{1}{2} + q_2)$ is redundant) $\Leftrightarrow$ (taking into account the definitions of a and b) $-(\frac{1}{2} + q_2) + \frac{\alpha_1 X_1}{W_1} \geq \frac{1}{\delta}\frac{1}{\sigma_1^2(\frac{1}{2} + q_2)} - (\frac{1}{2} + q_2) \Leftrightarrow$

$\frac{\alpha_1 X_1}{\alpha_1 X_1 + (I_0 - \alpha_1 X_0)(1 + R_1^f)} \geq \frac{1}{\delta}\frac{1}{\sigma_1^2(\frac{1}{2} + q_2)} \Leftrightarrow \alpha_1 X_1 \left(\delta\sigma_1^2(\frac{1}{2} + q_2) - 1\right) \geq (I_0 - \alpha_1 X_0)(1 + R_1^f)$.

We assume the parameters satisfy $\delta\sigma_1^2(\frac{1}{2}+q_2)-1 \le 0$. Then $X_1 \le \frac{(I_0-\alpha_1 X_0)(1+R_1^f)}{\alpha_1\left(\delta\sigma_1^2(\frac{1}{2}+q_2)-1\right)} < 0$. But this is impossible. Hence, the approximated 3rd order equation does not have convenient solutions. We renounce to the interior solutions.

2) For Strategy 2**.** We approximate the term $-(\frac{1+\delta}{\delta}\frac{1}{\sigma_1^2})d \simeq -(\frac{1+\delta}{\delta}\frac{1}{\sigma_1^2})R$ in the equation giving the interior solutions, such that it becomes:

$R^3-(c+d)R^2+(cd+\frac{1}{\sigma_1^2})R-(\frac{1+\delta}{\delta}\frac{1}{\sigma_1^2})R=0 \Leftrightarrow$

$R^2-(c+d)R+(cd+\frac{1}{\sigma_1^2}-\frac{1+\delta}{\delta}\frac{1}{\sigma_1^2})=0$. This equation has always two real solutions, because $\Delta=(c+d)^2-4\left(cd-\frac{1}{\delta}\frac{1}{\sigma_1^2}\right)=(c-d)^2+4\frac{1}{\delta}\frac{1}{\sigma_1^2}>0$. Hence $R_{1,2}=\frac{c+d\pm\sqrt{\Delta}}{2}=\frac{c+d}{2}\pm\frac{\sqrt{(c-d)^2+4\frac{1}{\delta}\frac{1}{\sigma_1^2}}}{2}$.

However, we have to check whether the solutions $R_{1,2}$ satisfy both $R>d$ and $R \le c+(\frac{1}{2}-q_2)$. We start by checking whether $R_{1,2}>d$.

If $c>d$: $R_1=\frac{c+d}{2}-\frac{\sqrt{\Delta}}{2}>d \Leftrightarrow c-\sqrt{\Delta}>d \Leftrightarrow c-d>\sqrt{\Delta} \Leftrightarrow (c-d)^2 > (c-d)^2+4\frac{1}{\delta}\frac{1}{\sigma_1^2}$ False. Hence $R_1<d$;

We try also R_2 : $R_2=\frac{c+d}{2}+\frac{\sqrt{\Delta}}{2}>d \Leftrightarrow c+\sqrt{\Delta}>d \Leftrightarrow c-d>-\sqrt{\Delta}$ True. Hence $R_2>d$;

If $c \le d$: $R_1=\frac{c+d}{2}-\frac{\sqrt{\Delta}}{2}>d \Leftrightarrow c-\sqrt{\Delta}>d \Leftrightarrow c-d>\sqrt{\Delta}$ False. Hence

$R_1<d$;

$R_2=\frac{c+d}{2}+\frac{\sqrt{\Delta}}{2}>d \Leftrightarrow c+\sqrt{\Delta}>d \Leftrightarrow c-d>-\sqrt{\Delta} \Leftrightarrow d-c<\sqrt{\Delta} \Leftrightarrow (c-d)^2<(c-d)^2+4\frac{1}{\delta}\frac{1}{\sigma_1^2}$ True. Hence $R_2>d$;

We conclude we always choose $R_2=\frac{c+d}{2}+\frac{\sqrt{\Delta}}{2}>d$.

The only thing to be tested is whether $R_2 \le c+(\frac{1}{2}-q_2) \Leftrightarrow$

$\frac{c+d}{2}+\frac{\sqrt{\Delta}}{2} \le c+(\frac{1}{2}-q_2) \Leftrightarrow c+d+\sqrt{\Delta} \le 2c+2(\frac{1}{2}-q_2) \Leftrightarrow$

$\sqrt{\Delta} \le c-d+2(\frac{1}{2}-q_2)$ (we have to ask additionally $c-d \ge 2(q_2-\frac{1}{2})$)

$(c-d)^2+4\frac{1}{\delta}\frac{1}{\sigma_1^2} \le (c-d)^2+4(\frac{1}{2}-q_2)^2+4(c-d)(\frac{1}{2}-q_2) \Leftrightarrow$

$c-d \ge -(\frac{1}{2}-q_2)+\frac{1}{\delta}\frac{1}{\sigma_1^2(\frac{1}{2}-q_2)}$ (we have to ask only this, $c-d \ge 2(q_2-\frac{1}{2})$ is redundant)

$\Leftrightarrow$ (taking into account the definitions of c and d)

$(\frac{1}{2}+q_2)-\frac{\alpha_1 X_1}{W_1} \ge \frac{1}{\delta}\frac{1}{\sigma_1^2(\frac{1}{2}-q_2)}-(\frac{1}{2}-q_2) \Leftrightarrow$

$\frac{\alpha_1 X_1}{\alpha_1 X_1+(I_0-\alpha_1 X_0)(1+R_1^f)} \le \frac{1}{\delta}\frac{\delta\sigma_1^2(\frac{1}{2}-q_2)-1}{\sigma_1^2(\frac{1}{2}-q_2)} \Leftrightarrow$

$\alpha_1 X_1 \le (I_0-\alpha_1 X_0)(1+R_1^f)\left(\delta\sigma_1^2(\frac{1}{2}-q_2)-1\right) \Leftrightarrow$

$X_1 \le \frac{(I_0-\alpha_1 X_0)(1+R_1^f)}{\alpha_1}\left(\delta\sigma_1^2(\frac{1}{2}+q_2)-1\right)$. But this never happens, under the assumption $\delta\sigma_1^2(\frac{1}{2}+q_2)-1 \le 0$.

Hence, again, we renounce to the interior solutions.

9. Proof of Proposition 4

We introduce first the following lemma:

Lemma 5 *The following approximation holds:* $\log(\alpha_1 X_1 + m) = \log(\alpha_1 X_0 + \frac{m}{1+R_1^f}) + r_1^f + \alpha_1^{*p}(r_1^x - r_1^f) + \frac{1}{2}\alpha_1^{*p}(1-\alpha_1^{*p})\sigma_0^2$, *whenever* $\alpha_1^{*p} = \frac{1}{1+\frac{m}{(1+R_1^f)\alpha_1 X_0}} \in (0,1)$.

Proof: Consider the portfolio composed by $\alpha_1$risky assets and $\beta_1 = \frac{m}{1+R_1^f}$ risk-free assets. The invested value is $Inv_0 = \alpha_1 X_0 + \frac{m}{1+R_1^f}$. This portfolio gives after 1 period (at $T=1$): $\alpha_1 X_1 + m$. The proportion invested in the risky assets for this portfolio is $\alpha_1^{*p} = \frac{\alpha_1 X_0}{\alpha_1 X_0 + \frac{m}{1+R_1^f}} = \frac{1}{1+\frac{m}{(1+R_1^f)\alpha_1 X_0}}$. Then $\log(\alpha_1 X_1 + m) = \log(Inv_0) + \alpha_1^{*p}\log(R_1^x+1) + (1-\alpha_1^{*p})\log(R_1^f+1) + \frac{1}{2}\alpha_1^{*p}(1-\alpha_1^{*p})\sigma_0^2 = \log(Inv_0) + \alpha_1^{*p} r_1^x + (1-\alpha_1^{*p}) r_1^f + \frac{1}{2}\alpha_1^{*p}(1-\alpha_1^{*p})\sigma_0^2 = \log(Inv_0) + r_1^f + \alpha_1^{*p}(r_1^x - r_1^f) + \frac{1}{2}\alpha_1^{*p}(1-\alpha_1^{*p})\sigma_0^2$, whenever $\alpha_1^{*p} \in (0,1)$.**q.e.d.**

We compute the objective function for the 4 decision paths in **Proposition 4**:

1) $f_1 = \delta\{\log(I_0 - \alpha_1 X_0)R_1^f + \delta\log[W_1 - (I_0-\alpha_1 X_0)R_1^f] + \delta(-\frac{\sigma_1^2}{2}) + \delta(q_2 + \frac{1}{2})\sigma_1^2 + \delta r_2^f) = \delta\{\log I_0 + \log(1-\alpha_1^*) + \log R_1^f + \delta\log(\alpha_1 X_1 + \beta_1) + \delta(-\frac{\sigma_1^2}{2}) + \delta(q_2+\frac{1}{2})\sigma_1^2 + \delta r_2^f\}$;

We used:

$\log(I_0 - \alpha_1 X_0)R_1^f = \log I_0 + \log(1-\alpha_1^*) + \log R_1^f$ and $\log[W_1 - (I_0 - \alpha_1 X_0)R_1^f)] = \log(\alpha_1 X_1 + \beta_1)$.

2) $f_2 = \delta\{\log c_1 + \delta\log(W_1 - c_1) + \delta r_2^f\} = \delta\{\log[\alpha_1(X_1 - X_0)] + \delta\log[W_1 - \alpha_1 X_1 + \alpha_1 X_0] + \delta r_2^f\} = \delta\{\log\alpha_1^* + (1+\delta)\log I_0 + \log(\frac{X_1}{X_0} - 1) + \delta\log[1 + R_1^f - \alpha_1^* R_1^f] + \delta r_2^f\}$;

$W_1 - \alpha_1 X_1 + \alpha_1 X_0 = I_0(1 + R_1^f - \alpha_1^* R_1^f)$ and $\log[\alpha_1(X_1 - X_0)] = \log\alpha_1^* + \log I_0 + \log(\frac{X_1}{X_0} - 1)$.

3) $f_3 = \delta\{\log(c_1) + \delta\log(W_1 - c_1) + \delta(\frac{-\sigma_1^2}{2})\alpha_2^{*^2} + \delta(q_2 + \frac{1}{2})\sigma_1^2\alpha_2^* + \delta r_2^f\} = \delta\{(1+\delta)\log W_1 + \log(\frac{1}{\delta+1}) + \delta\log(\frac{\delta}{\delta+1}) + \delta r_2^f\}$;

4) $f_4 = \delta\{\log(\frac{1}{\delta+1}W_1) + \delta\log(\frac{\delta}{\delta+1}W_1) + \delta[r_2^f + \frac{\sigma_1^2}{2}(q_2+\frac{1}{2})^2]\} = \delta\{(1+\delta)\log W_1 + \log(\frac{1}{\delta+1}) + \delta\log(\frac{\delta}{\delta+1}) + \delta[r_2^f + \frac{\sigma_1^2}{2}(q_2+\frac{1}{2})^2]\}$;

During "HC Bad Time" it is clear the FI uses Strategy 2, hence the objective function is given by $f1$. Similarly, for values of $X_1 \geq X_{FB}^{aprox}$, the FI uses Strategy 1 and the FB allocation and there is no need to check any allocation with Strategy 2, because we proved it is always inferior to the FB allocation. The only cases to discuss are when $X_1 \in (X_0, X_{FB}^{aprox})$ and both Strategies are feasible.

We show $f3 - f1 \geq 0$ always. $f_3 = \delta\{(1+\delta)\log W_1 + \log(\frac{1}{\delta+1}) + \delta\log(\frac{\delta}{\delta+1}) + \delta r_2^f\}$;

$f_1 = \delta\{\log I_0 + \log(1-\alpha_1^*) + \log R_1^f + \delta\log(\alpha_1 X_1 + \beta_1) + \delta(-\frac{\sigma_1^2}{2}) + \delta(q_2 + \frac{1}{2})\sigma_1^2 + \delta r_2^f\}$;

Applying **Lemma 5**, and considering

$\log W_1 = \log I_0 + \alpha_1^* r_1^x + (1-\alpha_1^*)r_1^f + \frac{1}{2}\alpha_1^*(1-\alpha_1^*)\sigma_0^2$

$\log(\alpha_1 X_1+\beta_1) = \log I_0 + \log(1+\alpha_1^* R_1^f) - \log(1+R_1^f) + r_1^f + (r_1^x - r_1^f)\frac{\alpha_1^*(1+R_1^f)}{1+\alpha_1^* R_1^f} + \frac{1}{2}\left(\frac{\alpha_1^*(1+R_1^f)}{1+\alpha_1^* R_1^f}\frac{1-\alpha_1^*}{1+\alpha_1^* R_1^f}\right)\sigma_0^2$;

we have: $f3 - f1 \geq 0 \Leftrightarrow$

$\alpha_1^*(r_2^x - r_2^f)\left(1+\delta-\delta\frac{1+R_1^f}{1+\alpha_1^* R_1^f}\right) + \frac{\sigma_0^2}{2}\alpha_1^*(1-\alpha_1^*)(1+\delta-\delta\frac{1+R_1^f}{(1+\alpha_1^* R_1^f)^2}) - \delta\log(1+\alpha_1^* R_1^f) - \log(1-\alpha_1^*) \geq \log R_1^f + \delta q_2\sigma_1^2 - r_2^f - \log(\frac{1}{\delta+1}) - \delta\log(\frac{\delta}{\delta+1})$

From $\frac{1+R_1^f}{1+\alpha_1^* R_1^f} < \frac{1+R_1^f}{1} = 1 + R_1^f \Rightarrow 1 - \frac{1+R_1^f}{1+\alpha_1^* R_1^f} > 1-(1+R_1^f) = -R_1^f \Rightarrow 1+\delta\left(1-\frac{1+R_1^f}{1+\alpha_1^* R_1^f}\right) > 1-\delta R_1^f > 0 \Rightarrow 1+\delta-\delta\frac{1+R_1^f}{1+\alpha_1^* R_1^f} = 1+\delta\frac{R_1^f(\alpha_1^*-1)}{1+\alpha_1^* R_1^f} > 0.$

Also, from $\frac{1+R_1^f}{(1+\alpha_1^* R_1^f)^2} < \frac{1+R_1^f}{1} = 1+R_1^f \Rightarrow 1 - \frac{1+R_1^f}{(1+\alpha_1^* R_1^f)^2} > 1-(1+R_1^f) = -R_1^f$

$\Rightarrow 1+\delta-\delta\frac{1+R_1^f}{(1+\alpha_1^* R_1^f)^2} = 1+\delta\left(1-\frac{1+R_1^f}{(1+\alpha_1^* R_1^f)^2}\right) > 1-\delta R_1^f > 0$. It is sufficient to show that $-\delta\log(1+\alpha_1^* R_1^f) - \log(1-\alpha_1^*) \geq \log R_1^f + \delta q_2\sigma_1^2 - r_2^f - \log(\frac{1}{\delta+1}) - \delta\log(\frac{\delta}{\delta+1}), \forall\ \alpha_1^* \in [0,1]$.

For $\alpha_1^* = 0$ this is equivalent to show that $0 \geq \log R_1^f + \delta q_2\sigma_1^2 - r_2^f - \log(\frac{1}{\delta+1}) - \delta\log(\frac{\delta}{\delta+1})$.

But $\log R_1^f + \delta q_2\sigma_1^2 - r_2^f - \log(\frac{1}{\delta+1}) - \delta\log(\frac{\delta}{\delta+1}) = \log R_1^f + \delta q_2\sigma_1^2 - r_2^f + \log(\delta+1) - \delta\log\delta + \delta\log(\delta+1) < 0$ for $\delta > \varepsilon$. Also, the LHS is increasing in α_1^*.

$\frac{\partial}{\partial\alpha_1^*}\left[-\delta\log(1+\alpha_1^* R_1^f) - \log(1-\alpha_1^*)\right] = -\frac{\delta R_1^f}{1+\alpha_1^* R_1^f} + \frac{1}{1-\alpha_1^*} = \frac{-\delta R_1^f(1-\alpha_1^*)+1+\alpha_1^* R_1^f}{\left(1+\alpha_1^* R_1^f\right)\left(1-\alpha_1^*\right)} = \frac{\alpha_1^* R_1^f(1+\delta)+1-\delta R_1^f}{\left(1+\alpha_1^* R_1^f\right)\left(1-\alpha_1^*\right)} > 0$

q.e.d.

Hence, for $X_1 \in [CornerThrStr_1, X_{FB}^{aprox})$ the FI always chooses Strategy 1 and the objective function is $f3$.

We are looking for $ThreshStr \in (X_0, CornerThrStr_1)$ the point where the FI has to decide between the two available strategies.

$f2 - f1 \geq 0 \Leftrightarrow \log(\frac{X_1}{X_0} - 1) - \delta\log(\alpha_1 X_1 + \beta_1) + \delta\log I_0 + \log\alpha_1^* - \log(1-\alpha_1^*) - \log R_1^f + \delta\log(1+R_1^f - \alpha_1^* R_1^f) - \delta q_2\sigma_1^2 \geq 0$

We employ the two following approximations: $\log(\alpha_1 X_1 + \beta_1) = \log(\alpha_1 X_0 + \beta_1) = \log I_0$ and $\log(1+R_1^f - \alpha_1^* R_1^f) = R_1^f - \alpha_1^* R_1^f = R_1^f(1-\alpha_1^*)$.

Hence $f2 - f1 \geq 0 \Leftrightarrow \log(\frac{X_1}{X_0} - 1) + \log\frac{\alpha_1^*}{(1-\alpha_1^*)R_1^f} + \delta R_1^f(1-\alpha_1^*) - \delta q_2\sigma_1^2 \geq 0 \Leftrightarrow$

$\log[(\frac{X_1}{X_0}-1)\frac{\alpha_1^*}{(1-\alpha_1^*)R_1^f}] \geq \delta q_2\sigma_1^2 - \delta R_1^f(1-\alpha_1^*) \Leftrightarrow X_1 \geq X_0[1+\frac{(1-\alpha_1^*)R_1^f}{\alpha_1^*}e^{\delta q_2\sigma_1^2-\delta R_1^f(1-\alpha_1^*)}]$.

We can approximate $e^{-\delta R_1^f(1-\alpha_1^*)} = 1$ because $\delta R_1^f(1-\alpha_1^*) \in (0, R_1^f)$, then $e^{-\delta R_1^f(1-\alpha_1^*)} \in (e^{-R_1^f}, 1)$. Hence an approximated value for the threshold between the two strategies is $ThreshStr = X_0\left(1+\frac{(1-\alpha_1^*)R_1^f}{\alpha_1^*}e^{\delta q_2\sigma_1^2}\right)$. For $X_1 \in (X_0, ThreshStr)$, Strategy 2 is superior and the objective function is given by $f1$. Similarly, for $X_1 \in [ThreshStr, CornerThreshold_1)$, the FI chooses Strategy 1 and the objective function is given by $f2$.

We discuss now the position of this approximated $ThreshStr$ with respect to $CornerThreshold_1$. We affirm $ThreshStr \leq CornerThreshold_1 \Leftrightarrow \frac{ThreshStr}{X_0} \leq \frac{CornerThreshold_1}{X_0} \Leftrightarrow 1+\frac{(1-\alpha_1^*)R_1^f}{\alpha_1^*}e^{\delta q_2\sigma_1^2} \leq 1+\frac{1}{\alpha_1^*}\left(\frac{1+R_1^f}{\delta}-\frac{\alpha_1^* R_1^f}{\delta}\right) \Leftrightarrow$

$(1-\alpha_1^*)R_1^f\left(\delta e^{\delta q_2\sigma_1^2}-1\right) \leq 1$

We assume $\delta e^{\delta q_2\sigma_1^2}-1 \leq 0 \Rightarrow (1-\alpha_1^*)R_1^f\left(\delta e^{\delta q_2\sigma_1^2}-1\right) \leq 0$ $,\forall \alpha_1^*$. True.

10. Proof of Proposition 5

Considering f_1, f_2, f_3 and f_4 as in Annex point 9, we denote:

$Int_1 = \int_0^{ThreshStr} f_1\, dX_1$, $Int_2 = \int_{ThreshStr}^{CornerThr1} f_2\, dX_1$, $Int_3 = \int_{CornerThr1}^{X_{FB}^{aprox}} f_3\, dX_1$ and $Int_4 = \int_{X_{FB}^{aprox}}^{\infty} f_4 dX_1$.

We are replacing first the functions (Int_1, Int_3, Int_4) with approximations given by *Lemma 5*, for the following cases:

i) $\log(W_1) = \log I_0 + \alpha_1^* r_1^x + (1-\alpha_1^*)r_1^f + \frac{1}{2}\alpha_1^*(1-\alpha_1^*)\sigma_0^2$; for $m = \beta_1(1+R_1^f)$ and $Inv_0 = I_0$.

ii) $\log(\alpha_1 X_1+\beta_1) = \log I_0 + \log(1+\alpha_1^* R_1^f) - \log(1+R_1^f) + r_1^f + (r_1^x - r_1^f)\frac{\alpha_1^*(1+R_1^f)}{1+\alpha_1^* R_1^f} + \frac{1}{2}\left(\frac{\alpha_1^*(1+R_1^f)}{1+\alpha_1^* R_1^f}\frac{1-\alpha_1^*}{1+\alpha_1^* R_1^f}\right)\sigma_0^2$;

for $m = \beta_1$, $Inv_0 = \alpha_1 X_0 + \frac{\beta_1}{1+R_1^f} = \frac{I_0+\alpha_1 X_0 R_1^f}{1+R_1^f}$ and $\alpha_1^{*p} = \frac{1}{1+\frac{\beta_1}{(1+R_1^f)\alpha_1 X_0}} = \frac{\alpha_1^*(1+R_1^f)}{1+\alpha_1^* R_1^f}$.

Now we can re-write the functions:

1) $f_1 = \delta\{\log I_0 + \log(1-\alpha_1^*) + \log R_1^f + \delta\log(\alpha_1 X_1+\beta_1) +$
$+\delta(-\frac{\sigma_1^2}{2}) + \delta(q_2+\frac{1}{2})\sigma_1^2 + \delta r_2^f\} = \delta\left(k_1 r_1^x + k_2\right)$ with $k_1 = \delta\frac{\alpha_1^*(1+R_1^f)}{1+\alpha_1^* R_1^f}$ and $k_2 = \delta[\log I_0 + \log(1+\alpha_1^* R_1^f) - r_1^f\frac{\alpha_1^*(1+R_1^f)}{1+\alpha_1^* R_1^f} + \frac{1}{2}\frac{\alpha_1^*(1+R_1^f)}{1+\alpha_1^* R_1^f}\frac{1-\alpha_1^*}{1+\alpha_1^* R_1^f}\sigma_0^2] + \log I_0 + \log(1-\alpha_1^*) + \log R_1^f + \delta(-\frac{\sigma_1^2}{2}) + \delta(q_2+\frac{1}{2})\sigma_1^2 + \delta r_2^f$;

2) $f_2 = \delta\{\log\alpha_1^* + (1+\delta)\log I_0 + \log(\frac{X_1}{X_0}-1) + \delta\log(1+R_1^f-\alpha_1^* R_1^f) + \delta r_2^f\}$

$f_2 = \delta[k_3\log(\frac{X_1}{X_0}-1) + k_4]$;

$k_3 = 1$; $k_4 = \log\alpha_1^* + (1+\delta)\log I_0 + \delta\log(1+R_1^f-\alpha_1^* R_1^f) + \delta r_2^f$;

3) $f_3 = \delta\{(1+\delta)\log W_1 + \log(\frac{1}{\delta+1}) + \delta\log(\frac{\delta}{\delta+1}) + \delta r_2^f\} = \delta\left(k_5 r_1^x + k_6\right)$;

$k_5 = (1+\delta)\alpha_1^*$; $k_6 = (1+\delta)[\log I_0 + (1-\alpha_1^*)r_1^f + \frac{1}{2}\alpha_1^*(1-\alpha_1^*)\sigma_0^2] + \log(\frac{1}{\delta+1}) + \delta\log(\frac{\delta}{\delta+1}) + \delta r_2^f$;

4) $f_4 = \delta\{(1+\delta)\log W_1 + \log(\frac{1}{\delta+1}) + \delta\log(\frac{\delta}{\delta+1}) + \delta[r_2^f + \frac{\sigma_0^2}{2}(q_2+\frac{1}{2})^2]\} = \delta\left(k_7 r_1^x + k_8\right)$;

$k_7 = k_5 = (1+\delta)\alpha_1^*$; $k_8 = k_6 + \delta\frac{\sigma_0^2}{2}(q_2+\frac{1}{2})^2$;

We have to evaluate $\int_a^b \rho(x_1)dx_1$, $\int_a^b r_1^x\,\rho(x_1)dx_1$ and $\int_a^b \log(\frac{X_1}{X_0}-1)\rho(x_1)dx_1$. The first two integrals are equal to:

$\int_a^b \rho_{X_1}(x_1)dx_1 = \Phi\left(\frac{\ln(b/X_0)-\mu}{\sigma_0}\right) - \Phi\left(\frac{\ln(a/X_0)-\mu}{\sigma_0}\right)$;

$\int_a^b r_1^x\,\rho_{X_1}(x_1)dx_1 = \int_a^b \ln(\frac{X_1}{X_0})\,\rho_{X_1}(x_1)dx_1 = \int_{a/X_0}^{b/X_0}\ln(y)\,\rho_{\frac{X_1}{X_0}}(y)dy =$
$= \int_{a/X_0}^{b/X_0}\ln(y)\frac{1}{y\sigma_0\sqrt{2\pi}}e^{-\frac{(\ln y-\mu)^2}{2\sigma_0^2}}\,dy = \int_{a/X_0}^{b/X_0}\ln(y)\frac{1}{\sigma_0\sqrt{2\pi}}e^{-\frac{(\ln y-\mu)^2}{2\sigma_0^2}}\,(\ln(y))'dy =$
$\int_{\ln(a/X_0)}^{\ln(b/X_0)} z\frac{1}{\sigma_0\sqrt{2\pi}}e^{-\frac{(z-\mu)^2}{2\sigma_0^2}}\,dz$

but this is a truncated mean of a normal variable, hence:

$\int_a^b r_1^x\,\rho_{X_1}(x_1)dx_1 = \mu[\Phi(\frac{\ln(b/X_0)-\mu}{\sigma_0}) - \Phi(\frac{\ln(a/X_0)-\mu}{\sigma_0})] +$
$+\sigma_0[\rho(\frac{\ln(a/X_0)-\mu}{\sigma_0}) - \rho(\frac{\ln(b/X_0)-\mu}{\sigma_0})]$;

The thresholds needed for the integrals are: $ThreshStr$, $CornerThrStr_1$, X_{FB}^{aprox}.

$\frac{ThreshStr}{X_0} = T_{11} + \frac{1}{\alpha_1^*}T_{12}$ with $T_{11} = 1 - R_1^f e^{\delta\sigma_1^2 q_2}$ and $T_{12} = R_1^f e^{\delta\sigma_1^2 q_2}$.
$\frac{CornerThrStr_1}{X_0} = T_{21} + \frac{1}{\alpha_1^*}T_{22}$; $T_{21} = 1 - \frac{R_1^f}{\delta}$; $T_{22} = \frac{1+R_1^f}{\delta}$;
$\frac{X_{FB}^{aprox}}{X_0} = T_{31} + \frac{1}{\alpha_1^*}T_{32}$; $T_{31} = 1 - \frac{R_1^f}{\delta(\frac{1}{2}-q_2)} - \frac{(1+R_1^f)\delta(q_2+\frac{1}{2})}{\delta(\frac{1}{2}-q_2)}$; $T_{32} = \frac{(1+R_1^f)}{\delta(\frac{1}{2}-q_2)} + \frac{(1+R_1^f)\delta(q_2+\frac{1}{2})}{\delta(\frac{1}{2}-q_2)}$;

With these formulas we can re-write (exactly) the integrals: Int_1, Int_3, Int_4.

$Int_1 = \int_0^{ThreshStr} f_1\,dX_1 = \int_0^{ThreshStr}\delta\left(k_1 r_1^x + k_2\right)dX_1 =$
$= \delta\{k_1\int_0^{ThreshStr} r_1^x dX_1 + k_2\int_0^{ThreshStr} dX_1\} = \{\delta k_1[\mu\Phi\left(N\right) - \sigma_0\rho\left(N\right)] + k_2\Phi(N)\}$;

For $Int_3 = \int_{CornerThr1}^{X_{FB}^{aprox}} f_3\,dX_1 = \int_{CornerThr1}^{X_{FB}^{aprox}}\delta\left(k_5 r_1^x + k_6\right)dX_1 = \delta\{k_5\int_{CornerThr1}^{X_{FB}^{aprox}} r_1^x dX_1 + k_6\int_{CornerThr1}^{X_{FB}^{aprox}} dX_1\} = \delta\{k_5\{\mu[\Phi\left(P\right) - \Phi\left(M\right)] + \sigma_0[\rho\left(M\right) - \rho\left(P\right)]\} + k_6[\Phi(P) - \Phi(M)]\}$;

For $Int_4 = \int_{X_{FB}^{aprox}}^{\infty} f_4\,dX_1 = \int_{X_{FB}^{aprox}}^{\infty}\delta\left(k_7 r_1^x + k_8\right)\,dX_1 =$
$= \delta\{k_7\int_{X_{FB}^{aprox}}^{\infty} r_1^x dX_1 + k_8\int_{X_{FB}^{aprox}}^{\infty} dX_1\} = \delta\{k_7[\mu\left(1-\Phi\left(P\right)\right) + \sigma_0\rho\left(P\right)] +$

$+k_8[1-\Phi(P)]\};$

Now we discuss the second integral.
$Int_2 = \int_{ThreshStr}^{CornerThr1} f_2\, dX_1 = \int_{ThreshStr}^{CornerThr1} \delta\{k_3 \log(\frac{X_1}{X_0}-1)+k_4\}dX_1 =$
$= \delta\{k_3 \int_{ThreshStr}^{CornerThr1} \log(\frac{X_1}{X_0}-1)dX_1 + k_4 \int_{ThreshStr}^{CornerThr1} dX_1\}$

$\int_{ThreshStr}^{CornerThr1} dX_1 = \Phi(M)-\Phi(N)$; Unfortunatelly, $\int_{ThreshStr}^{CornerThr1} \log(\frac{X_1}{X_0} - 1)dX_1$ cannot be writtten as before.

Hence $Int_2 = \delta\{k_3 \int_{ThreshStr}^{CornerThr1} \log(\frac{X_1}{X_0}-1)dX_1 + k_4[\Phi(M)-\Phi(N)]\}$; We denote $\frac{\ln(3/X_0)-\mu}{\sigma_0} = V$;

However, $\int_{ThreshStr}^{CornerThr1} \log(\frac{X_1}{X_0}-1)dX_1$ can be approximated in a similar manner with $\int_{ThreshStr}^{CornerThr1} \log(\frac{X_1}{X_0})dX_1$, by taking into account the position of the thresholds values:

1) if $CornerThr1/X_0 < 3$, then $\int_{ThreshStr}^{CornerThr1} \log(\frac{X_1}{X_0}-1)dX_1 \simeq \Phi(N)-\Phi(M)$;

2) if $ThreshStr/X_0 \leq 3 \leq CornerThr1/X_0$, then $\int_{ThreshStr}^{CornerThr1} \log(\frac{X_1}{X_0} - 1)dX_1 \simeq \Phi(N)-\Phi(V)+\mu[\Phi(M)-\Phi(V)]+\sigma_0[\rho(V)-\rho(M)]$;

3) if $3 < ThreshStr/X_0$, then $\int_{ThreshStr}^{CornerThr1} \log(\frac{X_1}{X_0}-1)dX_1 \simeq \mu[\Phi(M) - \Phi(N)] + \sigma_0[\rho(N)-\rho(M)]$;

Finally we write a simplified version of the sum of the 4 integrals, which alllows for an analytical result for α_1^*.

We approximate $\Phi(M)-\Phi(N) \simeq (M-N)\left(\frac{\rho(M)+\rho(N)}{2}\right)$ with the trapezoidal rule and $\rho(N)-\rho(M) \simeq 0$.

Moreover,
$N = \frac{\ln(T_{11}+\frac{1}{\alpha_1^*}T_{12})-\mu}{\sigma_0} \simeq X_N = \frac{\ln(T_{11}+\frac{1}{\alpha_1^{*FB}}T_{12})-\mu}{\sigma_0}$;
$M = \frac{\ln(T_{21}+\frac{1}{\alpha_1^*}T_{22})-\mu}{\sigma_0} \simeq X_M = \frac{\ln(T_{21}+\frac{1}{\alpha_1^{*FB}}T_{22})-\mu}{\sigma_0}$;
$P = \frac{\ln(T_{31}+\frac{1}{\alpha_1^*}T_{32})-\mu}{\sigma_0} \simeq X_P = \frac{\ln(T_{31}+\frac{1}{\alpha_1^{*FB}}T_{32})-\mu}{\sigma_0}$;

From $\rho(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$ we approximate $\rho(N) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}N^2} \simeq \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}X_N^2} =$
$\frac{1}{\sqrt{2\pi}}(T_{11}+\frac{1}{\alpha_1^{*FB}}T_{12})^{\left(-\frac{1}{2\sigma_0}X_N\right)}e^{\mu\frac{X_N}{2\sigma_0}} = \frac{1}{\sqrt{2\pi}}e^{-\frac{\mu^2}{2\sigma_0^2}}(T_{11}+\frac{1}{\alpha_1^{*FB}}T_{12})^{\frac{\mu}{\sigma_0^2}-\frac{\ln\left(T_{11}+\frac{1}{\alpha_1^{*FB}}T_{12}\right)}{2\sigma_0^2}}$;

Similarly, $\rho(M) \simeq \rho(X_M) = \frac{1}{\sqrt{2\pi}}e^{-\frac{\mu^2}{2\sigma_0^2}}(T_{21}+\frac{1}{\alpha_1^{*FB}}T_{22})^{\frac{\mu}{\sigma_0^2}-\frac{\ln\left(T_{21}+\frac{1}{\alpha_1^{*FB}}T_{22}\right)}{2\sigma_0^2}}$;

$\rho(P) \simeq \rho(X_P) = \frac{1}{\sqrt{2\pi}}e^{-\frac{\mu^2}{2\sigma_0^2}}(T_{31}+\frac{1}{\alpha_1^{*FB}}T_{32})^{\frac{\mu}{\sigma_0^2}-\frac{\ln\left(T_{31}+\frac{1}{\alpha_1^{*FB}}T_{32}\right)}{2\sigma_0^2}}$;

The four integrals can be approximated as:
$Int_1 simplif = \delta\{k_1[\mu\Phi(N)-\sigma_0\rho(N)]+k_2\Phi(N)\} \simeq$
$\simeq \delta\{k_{1sim}\mu\Phi(N)+k_{2sim}\Phi(N)\} = \delta\Phi(N)(\mu k_{1sim}+k_{2sim})$;

$Int_2 simplif = \delta\{k_3 \int_{ThreshNew}^{CornerThr1} \log(\frac{X_1}{X_0} - 1) dX_1 + k_4[\Phi(M) - \Phi(N)]\} \simeq$
$\simeq \delta\{k_{3sim} \int_{ThreshNew}^{CornerThr1} \log(\frac{X_1}{X_0}) dX_1 + k_{4sim}[\Phi(M) - \Phi(N)]\} =$
$\delta\{k_{3sim}\, \{\mu[\Phi\,(M) - \Phi\,(N)] + \sigma_0[\rho\,(M) - \rho\,(N)]\} + k_{4sim}[\Phi(M) - \Phi(N)]\} \simeq$
$\simeq \delta\,(k_{3sim}\, \mu\,(\Phi\,(M) - \Phi\,(N)) + k_{4sim}\,(\Phi(M) - \Phi(N))) =$
$= \delta[\Phi\,(M) - \Phi\,(N)]\,(\mu k_{3sim} + k_{4sim}) =$
$= \delta(X_M - X_N) \left(\frac{\rho(X_M)+\rho(X_N)}{2}\right) (\mu k_{3sim} + k_{4sim})\,;$

$Int_3 simplif = \delta\{k_5\{\mu[\Phi\,(P) - \Phi\,(M)] + \sigma_0[\rho\,(M) - \rho\,(P)]\} + k_6[\Phi(P) - \Phi(M)] \simeq \delta\{k_{5sim}\mu[\Phi\,(P) - \Phi\,(M)] + k_{6sim}[\Phi(P) - \Phi(M)]\} = \delta[\Phi\,(P) - \Phi\,(M)]\,(\mu k_{5sim} + k_{6sim}) = \delta(X_P - X_M)(\frac{\rho(X_P)+\rho(X_M)}{2})\,(\mu k_{5sim} + k_{6sim})\,;$

$Int_4 simplif = \delta\{k_7\{\mu[1 - \Phi\,(P)] + \sigma_0 \rho\,(P)\} + k_8[1 - \Phi(P)]\} \simeq$
$\simeq \delta\{k_{7sim}\mu[1 - \Phi\,(P)] + k_{8sim}[1 - \Phi(P)]\} = \delta[1 - \Phi\,(P)]\,(\mu k_{7sim} + k_{8sim})\,;$

With the trapezoidal rule for N (depending whether it is positive or not)
$\Phi\,(N) = \frac{1}{2} \pm \frac{\rho(N)+\rho(0)}{2}\,(N - 0) = \frac{1}{2} \pm \frac{\rho(X_N)+\frac{1}{\sqrt{2\pi}}}{2} X_N$; for $Int_1 simplif$.
We take for simplicity $\Phi\,(N) \simeq \frac{1}{2} + \frac{\rho(X_N)+\frac{1}{\sqrt{2\pi}}}{2} X_N$.
Also $\Phi\,(P) \simeq \frac{1}{2} + \frac{\rho(X_P)+\frac{1}{\sqrt{2\pi}}}{2} X_P$; Hence $1 - \Phi\,(P) \simeq \frac{1}{2} - \frac{\rho(X_P)+\frac{1}{\sqrt{2\pi}}}{2} X_P$;
We write the simplified integrals:
$Int_1 simplif = \delta\Phi\,(N)\,(\mu k_{1sim} + k_{2sim}) = \delta\left(\frac{1}{2} + \frac{\rho(X_N)+\frac{1}{\sqrt{2\pi}}}{2} X_N\right)(\mu k_{1sim} + k_{2sim})\,;$
$Int_2 simplif = \delta(X_M - X_N)\left(\frac{\rho(X_M)+\rho(X_N)}{2}\right)(\mu k_{3sim} + k_{4sim})\,;$

$Int_3 simplif = \delta(X_P - X_M)\left(\frac{\rho(X_P)+\rho(X_M)}{2}\right)(\mu k_{5sim} + k_{6sim})\,;$
$Int_4 simplif = \delta\,(1 - \Phi\,(P))\,(\mu k_{7sim} + k_{8sim}) = \delta\left(\frac{1}{2} - \frac{\rho(X_P)+\frac{1}{\sqrt{2\pi}}}{2} X_P\right)(\mu k_{7sim} + k_{8sim})\,;$
with the simplified k-s:
$k_{1sim} = \delta\alpha_1^*;$
$k_{2sim} = \delta[\log I_0 + \alpha_1^* R_1^f - r_1^f \alpha_1^* + \frac{1}{2}\frac{\alpha_1^*(1-\alpha_1^*)}{1+R_1^f}\sigma_0^2] + \log I_0 + \log(1-\alpha_1^*) + \log R_1^f + \delta(-\frac{\sigma_1^2}{2}) + \delta(q_2 + \frac{1}{2})\sigma_1^2 + \delta r_2^f;$

$k_{3sim} = k_3 = 1;$
$k_{4sim} = \log\alpha_1^* + (1+\delta)\log I_0 + \delta(R_1^f - \alpha_1^* R_1^f) + \delta r_2^f = \log\alpha_1^* + (1+\delta)\log I_0 + \delta R_1^f(1 - \alpha_1^*) + \delta r_2^f;$

The rest of the k-s are unchanged.

We are looking for the argmax of the sum of the four integrals. We compute the derivative with respect to α_1^* of each of the integrals in the sum.
$\frac{\partial Int_1 simplif}{\partial \alpha_1^*} = \delta(\frac{1}{2} + \frac{\rho(X_N)+\frac{1}{\sqrt{2\pi}}}{2} X_N)[-\frac{1}{1-\alpha_1^*} - \alpha_1^* \frac{\sigma_0^2\delta}{(1+R_1^f)} +$
$+\delta(\mu + R_1^f - r_1^f + \frac{1}{2}\frac{1}{1+R_1^f}\sigma_0^2)];$

$\frac{\partial Int_2 simplif}{\partial \alpha_1^*} = \delta(X_M - X_N)(\frac{\rho(X_M)+\rho(X_N)}{2})(\frac{1}{\alpha_1^*} - \delta R_1^f);$

$\frac{\partial Int_3 simplif}{\partial \alpha_1^*} = \delta(X_P - X_M)(\frac{\rho(X_P)+\rho(X_M)}{2})[\mu(1+\delta) - (1+\delta)r_1^f +$
$+\frac{1+\delta}{2}\sigma_0^2 - \alpha_1^*\sigma_0^2(1+\delta)];$

$\frac{\partial Int_4 simplif}{\partial \alpha_1^*} = \delta(\frac{1}{2} - \frac{\rho(X_P)+\frac{1}{\sqrt{2\pi}}}{2}X_P)[\mu(1+\delta) - (1+\delta)r_1^f +$
$+\frac{1+\delta}{2}\sigma_0^2 - \alpha_1^*\sigma_0^2(1+\delta)];$

We denote by:

$cons_1 = \frac{1}{2} + \frac{\rho(X_N)+\frac{1}{\sqrt{2\pi}}}{2}X_N$; $cons_2 = (X_M - X_N)(\frac{\rho(X_M)+\rho(X_N)}{2})$; $cons_3 = (X_P - X_M)(\frac{\rho(X_P)+\rho(X_M)}{2})$; $cons_4 = \frac{1}{2} - \frac{\rho(X_P)+\frac{1}{\sqrt{2\pi}}}{2}X_P$;

Then the derivative of the sum leads to:

$-\alpha_1^*[cons_1\frac{\sigma_0^2\delta}{(1+R_1^f)} + cons_3\sigma_0^2(1+\delta) + cons_4\sigma_0^2(1+\delta)] + (cons_3 + cons_4)(1+\delta)[\mu - r_1^f + \frac{\sigma_0^2}{2}] - \delta cons_2 R_1^f + \delta cons_1[\mu + R_1^f - r_1^f + \frac{\sigma_0^2}{2(1+R_1^f)}] - \frac{1}{1-\alpha_1^*}cons_1 + \frac{cons_2}{\alpha_1^*} = 0$

To simplify the computations, we can replace $\alpha_1^* = \alpha_1^{*FB}$ in the first term.

This leads to

$-\alpha_1^{*FB}\sigma_0^2[cons_1\frac{\delta}{(1+R_1^f)} + (cons_3+cons_4)(1+\delta)] + (cons_3 + cons_4)(1+\delta)(\mu - r_1^f + \frac{\sigma_0^2}{2}) - \delta cons_2 R_1^f + \delta cons_1[\mu + R_1^f - r_1^f + \frac{\sigma_0^2}{2(1+R_1^f)}] + \frac{cons_2(1-\alpha_1^*) - cons_1\alpha_1^*}{\alpha_1^*(1-\alpha_1^*)} = 0 \Leftrightarrow$

$(cons_3 + cons_4)(1+\delta)(\mu - r_1^f + \frac{\sigma_0^2}{2} - \alpha_1^{*FB}\sigma_0^2) - \alpha_1^{*FB}\sigma_0^2 cons_1\frac{\delta}{(1+R_1^f)} - \delta cons_2 R_1^f + \delta cons_1(\mu + R_1^f - r_1^f + \frac{\sigma_0^2}{2(1+R_1^f)}) + \frac{cons_2 - (cons_1+cons_2)\alpha_1^*}{\alpha_1^*(1-\alpha_1^*)} = 0;$

But $\alpha_1^{*FB} = q_1 + \frac{1}{2} = \frac{\mu - r_1^f}{\sigma_0^2} + \frac{1}{2} = \frac{\mu - r_1^f + \frac{\sigma_0^2}{2}}{\sigma_0^2} \Rightarrow \mu - r_1^f + \frac{\sigma_0^2}{2} - \alpha_1^{*FB}\sigma_0^2 = 0;$

Then the previous expression becomes:

$\alpha_1^*(1-\alpha_1^*)\{-\alpha_1^{*FB}\sigma_0^2 cons_1\frac{\delta}{(1+R_1^f)} + \delta[cons_1(\mu + R_1^f - r_1^f + \frac{\sigma_0^2}{2(1+R_1^f)}) - cons_2 R_1^f]\} + cons_2 - (cons_1 + cons_2)\alpha_1^* = 0;$

This is a second degree equation; we simplify it into a first order one by making: $\alpha_1^*(1-\alpha_1^*) \simeq -\alpha_1^*\alpha_1^{*FB} + \alpha_1^*$;

$\left(-\alpha_1^*\alpha_1^{*FB} + \alpha_1^*\right)\{-\alpha_1^{*FB}\sigma_0^2 cons_1\frac{\delta}{(1+R_1^f)} + \delta\{cons_1[\mu + R_1^f - r_1^f + \frac{\sigma_0^2}{2(1+R_1^f)}] - cons_2 R_1^f\}\} + cons_2 - (cons_1 + cons_2)\alpha_1^* = 0 \Leftrightarrow$

From here it comes:

$\alpha_1^{*HC} = \frac{cons_2}{\left(\alpha_1^{*FB}-1\right)\{-\alpha_1^{*FB}\sigma_0^2 cons_1\frac{\delta}{(1+R_1^f)} + \delta\{cons_1[\mu+R_1^f-r_1^f+\frac{\sigma_0^2}{2(1+R_1^f)}] - cons_2 R_1^f\}\} + (cons_1+cons_2)} = \frac{1}{1+\delta R_1^f\left(1-\alpha_1^{*FB}\right) + \left\{1+\delta\left(\alpha_1^{*FB}-1\right)\left[\mu+R_1^f-r_1^f+\frac{\sigma_0^2(1-2\alpha_1^{*FB})}{2(1+R_1^f)}\right]\right\}\frac{cons_1}{cons_2}}$ is an approximation of the proportion of risky assets for the first period.

11. Proof of Proposition 6

a) For computing the expected consumption we analyze the allocations as in **Proposition 4**.

1) $c_1^{HC} = (I_0 - \alpha_1^{HC} X_0) R_1^f = I_0 R_1^f (1 - \alpha_1^{*HC}); \alpha_2^{*HC} = 1;$

2) $c_1^{HC} = \alpha_1^{HC}(X_1 - X_0) = \alpha_1^{*HC} I_0 (\frac{X_1}{X_0} - 1); \alpha_2^{*HC} = 0;$

3) $c_1^{HC} = \frac{1}{1+\delta}[\alpha_1^{*HC} I_0 \frac{X_1}{X_0} + I_0(1 + R_1^f)(1 - \alpha_1^{*HC})]; \alpha_2^{*HC} = 0;$

4) $c_1^{HC} = \frac{1}{1+\delta}[\alpha_1^{*HC} I_0 \frac{X_1}{X_0} + I_0(1 + R_1^f)(1 - \alpha_1^{*HC})]; \alpha_2^{*HC} = q_2 + \frac{1}{2};$

$E_0(c_1^{HC}) = \int_0^\infty c_1^{HC}(x_1)\rho_{X_1}(x_1)dx_1 = \int_0^{ThreshStr} c_1^{HC}(x_1)\rho_{X_1}(x_1)dx_1 + + \int_{ThreshStr}^{CornerThr1} c_1^{HC}(x_1)\rho_{X_1}(x_1)dx_1 + \int_{CornerThr1}^\infty c_1^{HC}(x_1)\rho_{X_1}(x_1)dx_1 = c_1 + c_2 + c_3;$

$c_1 = \int_0^{ThreshStr} c_1^{HC}(x_1)\rho_{X_1}(x_1)dx_1 = I_0 R_1^f (1 - \alpha_1^{*HC})\Phi(N);$

$c_2 = \int_{ThreshStr}^{CornerThr1} c_1^{HC}(x_1)\rho_{X_1}(x_1)dx_1 = \alpha_1^{*HC} I_0 \{e^{\mu + \frac{\sigma_0^2}{2}}[\Phi(M - \sigma_0) - \Phi(N - \sigma_0)] - \Phi(M) + \Phi(N)\};$

$c_3 = \int_{CornerThr1}^\infty c_1^{HC}(x_1)\rho_{X_1}(x_1)dx_1 = \frac{1}{1+\delta}\{\alpha_1^{*HC} I_0 e^{\mu + \frac{\sigma_0^2}{2}}[1 - \Phi(M - \sigma_0)] + I_0(1 + R_1^f)(1 - \alpha_1^{*HC})[1 - \Phi(M)]\};$

Then

$E_0(c_1^{HC}) = I_0 R_1^f (1 - \alpha_1^{*HC})\Phi(N) + \alpha_1^{*HC} I_0 \{e^{\mu + \frac{\sigma_0^2}{2}}[\Phi(M - \sigma_0) - \Phi(N - \sigma_0)] - \Phi(M) + \Phi(N)\} + \frac{1}{1+\delta}\{\alpha_1^{*HC} I_0 e^{\mu + \frac{\sigma_0^2}{2}}[1 - \Phi(M - \sigma_0)] + I_0(1 + R_1^f)(1 - \alpha_1^{*HC})[1 - \Phi(M)]\};$

b) $\alpha_1^{HC} = \frac{\alpha_1^{*HC} I_0}{X_0}$ and $\alpha_2^{HC} = \frac{\alpha_2^{*HC} Inv_1}{X_1} = \alpha_2^{*HC}[\alpha_1^{*HC}\frac{I_0}{X_0} + \frac{I_0(1+R_1^f)(1-\alpha_1^{*HC})}{X_0}\frac{1}{\frac{X_1}{X_0}} - \frac{c_1^{HC}}{X_1}];$

1) $\frac{c_1^{HC}}{X_1} = \frac{I_0 R_1^f (1 - \alpha_1^{*HC})}{X_0}\frac{1}{\frac{X_1}{X_0}};$

2) $\frac{c_1^{HC}}{X_1} = \alpha_1^{*HC}\frac{I_0}{X_0} - \alpha_1^{*HC}\frac{I_0}{X_0}\frac{1}{\frac{X_1}{X_0}};$

4) $\frac{c_1^{HC}}{X_1} = \frac{1}{1+\delta}(\alpha_1^{*HC}\frac{I_0}{X_0} + I_0(1 + R_1^f)(1 - \alpha_1^{*HC})\frac{1}{\frac{X_1}{X_0}});$

Then:

1) $\alpha_2^{HC} = \alpha_2^{*HC}[\alpha_1^{*HC}\frac{I_0}{X_0} + \frac{I_0(1-\alpha_1^{*HC})}{X_0}\frac{1}{\frac{X_1}{X_0}}];$

2) $\alpha_2^{HC} = \alpha_2^{*HC}\{\frac{I_0}{X_0}[1 + R_1^f(1 - \alpha_1^{*HC})]\frac{1}{\frac{X_1}{X_0}}\};$

4) $\alpha_2^{HC} = \alpha_2^{*HC}\frac{\delta}{1+\delta}[\alpha_1^{*HC}\frac{I_0}{X_0} + \frac{I_0(1+R_1^f)(1-\alpha_1^{*HC})}{X_0}\frac{1}{\frac{X_1}{X_0}}];$

$E_0(\alpha_2^{HC}) = \int_0^{ThreshStr} \alpha_2^{HC}(x_1)\rho_{X_1}(x_1)dx_1 + \int_{X_{FB}^{aprox}}^\infty \alpha_2^{HC}(x_1)\rho_{X_1}(x_1)dx_1 =$

$= \int_0^{ThreshStr}[\alpha_1^{*HC}\frac{I_0}{X_0}+\frac{I_0(1-\alpha_1^{*HC})}{X_0}\frac{1}{\frac{X_1}{X_0}}]\rho_{X_1}(x_1)dx_1+\int_{X_{FB}^{aprox}}^{\infty}(q_2+\frac{1}{2})\frac{\delta}{\delta+1}[\alpha_1^{*HC}\frac{I_0}{X_0}+\frac{I_0(1+R_1^f)(1-\alpha_1^{*HC})}{X_0}\frac{1}{\frac{X_1}{X_0}}]\rho_{X_1}(x_1)dx_1 = \alpha_1^{*HC}\frac{I_0}{X_0}\Phi(N) + \frac{I_0(1-\alpha_1^{*HC})}{X_0}e^{-\mu+\frac{\sigma_0^2}{2}}\Phi(N+\sigma_0)+(q_2+\frac{1}{2})\frac{\delta}{\delta+1}\{\alpha_1^{*HC}\frac{I_0}{X_0}[1-\Phi(P)]+\frac{I_0(1+R_1^f)(1-\alpha_1^{*HC})}{X_0}e^{-\mu+\frac{\sigma_0^2}{2}}[1-\Phi(P+\sigma_0)]\} \Rightarrow$

$E_0(\alpha_2^{HC})-\alpha_1^{HC} = \alpha_1^{*HC}\frac{I_0}{X_0}\{\Phi(N)-1+(q_2+\frac{1}{2})\frac{\delta}{\delta+1}[1-\Phi(P)]\}+\frac{I_0(1-\alpha_1^{*HC})}{X_0}e^{-\mu+\frac{\sigma_0^2}{2}}\{\Phi(N+\sigma_0) + (q_2+\frac{1}{2})\frac{\delta}{\delta+1}(1+R_1^f)[1-\Phi(P+\sigma_0)]\}$;

For $E_0(\beta_2^{HC})-\beta_1^{HC}$ we remark: $\beta_1^{HC} = I_0(1-\alpha_1^{*HC})$, $\beta_2^{HC} = \frac{Inv_1-\alpha_2^{HC}X_1}{1+R_1^f} = \frac{(1-\alpha_2^{*HC})(W_1-c_1^{HC})}{1+R_1^f}$ and $W_1 - c_1^{HC} = \alpha_1^{*HC}I_0\frac{X_1}{X_0} + I_0(1-\alpha_1^{*HC})(1+R_1^f) - c_1^{HC}$.

1) $W_1 - c_1^{HC} = \alpha_1^{*HC}I_0\frac{X_1}{X_0} + I_0(1-\alpha_1^{*HC})$;
2) $W_1 - c_1^{HC} = \alpha_1^{*HC}I_0 + I_0(1-\alpha_1^{*HC})(1+R_1^f)$;
3) $W_1 - c_1^{HC} = \frac{\delta}{1+\delta}[\alpha_1^{*HC}I_0\frac{X_1}{X_0} + I_0(1+R_1^f)(1-\alpha_1^{*HC})]$;

$E_0(\beta_2^{HC}) = \int_0^\infty \beta_2^{HC}(x_1)\rho_{X_1}(x_1)dx_1 = \int_{ThreshStr}^{CornerThr1}\frac{1}{1+R_1^f}[\alpha_1^{*HC}I_0 + I_0(1+R_1^f)(1-\alpha_1^{*HC})]\rho_{X_1}(x_1)dx_1 + \int_{CornerThr1}^{X_{FB}^{aprox}}\frac{1}{1+R_1^f}\frac{\delta}{1+\delta}[\alpha_1^{*HC}I_0\frac{X_1}{X_0} + I_0(1+R_1^f)(1-\alpha_1^{*HC})]\rho_{X_1}(x_1)dx_1+\int_{X_{FB}^{aprox}}^{\infty}\frac{(\frac{1}{2}-q2)}{1+R_1^f}\frac{\delta}{1+\delta}[\alpha_1^{*HC}I_0\frac{X_1}{X_0}+I_0(1+R_1^f)(1-\alpha_1^{*HC})]\rho_{X_1}(x_1)dx_1$;

$E_0(\beta_2^{HC}) - \beta_1^{HC} = \frac{1}{1+R_1^f}\alpha_1^{*HC}I_0\{\Phi(M) - \Phi(N) + \frac{\delta}{1+\delta}e^{\mu+\frac{\sigma_0^2}{2}}[\Phi(P-\sigma_0) - \Phi(M-\sigma_0)+(\frac{1}{2}-q_2)(1-\Phi(P-\sigma_0))]\}+I_0(1-\alpha_1^{*HC})\{\Phi(M)-\Phi(N)+\frac{\delta}{1+\delta}[\Phi(P)-\Phi(M) + (\frac{1}{2}-q2)(1-\Phi(P))]\} - I_0(1-\alpha_1^{*HC})$;

12. Problem FV1

We check first when $W_1 > I_0$.

$W_1 > I_0 \Leftrightarrow \alpha_1 X_1 + \beta_1(1+R_1^f) > I_0 \Leftrightarrow \alpha_1 X_1 > -I_0 R_1^f + \alpha_1 X_0(1+R_1^f) \Leftrightarrow X_1 > X_0(1+R_1^f) - \frac{I_0 R_1^f}{\alpha_1} = Threshold_{FV}$ (this is the threshold that assures $\Pi_1^{FV} > 0$) .

Assuming $W_1 > I_0$, we solve the maximization problem in **Problem FV1, 2)** with Kuhn-Tucker, denoting by f the objective function.

$$\begin{cases} 1) & \frac{\partial f}{\partial c_1} = \delta(\frac{1}{c_1} - \frac{\delta}{W_1 - c_1}) \\ 2) & \frac{\partial f}{\partial \alpha_2^{*FV}} = -\delta\sigma_1^2\alpha_2^* + \delta(q_2+\frac{1}{2})\sigma_1^2 \end{cases}$$

$R_3 : c_1 \leq \alpha_1(X_1 - X_0) + \beta_1 R_1^f$

$$\begin{cases} 1) & \frac{\partial R_3}{\partial c_1} = 1 \\ 2) & \frac{\partial R_3}{\partial \alpha_2^*} = 0 \end{cases}$$

$$\begin{cases} 1) & \delta(\frac{1}{c_1} - \frac{\delta}{W_1 - c_1}) = \lambda_3 \\ 2) & -\delta\sigma_1^2\alpha_2^* + \delta(q_2 + \frac{1}{2})\sigma_1^2 = 0 \\ 3) & \lambda_3(c_1 - \alpha_1(X_1 - X_0) - \beta_1 R_1^f) = 0 \end{cases}$$

with $\lambda_3 \geq 0$.

If $\lambda_3 = 0 \Rightarrow \frac{1}{c_1} - \frac{\delta}{W_1 - c_1} = 0 \Leftrightarrow \frac{W_1 - c_1 - \delta c_1}{c_1(W_1 - c_1)} = 0 \Leftrightarrow W_1 = c_1(1+\delta) \Leftrightarrow c_1^{FV} = \frac{W_1}{1+\delta} = c_1^{FB}$ and $-\alpha_2^{*FV} + q_2 + \frac{1}{2} = 0 \Leftrightarrow \alpha_2^{*FV} = q_2 + \frac{1}{2} = \alpha_2^{*FB}$;

In order to be able to choose the FB, one has to have: (R_3) $c_1^{FB} \leq \Pi_1^{FV} \Leftrightarrow \frac{W_1}{1+\delta} \leq \alpha_1(X_1 - X_0) + \beta_1 R_1^f \Leftrightarrow X_1 \geq X_0(1 + R_1^f) + \frac{I_0(1-\delta R_1^f)}{\delta\alpha_1} = X_0(1 + R_1^f) + \frac{I_0}{\alpha_1}(\frac{1}{\delta} - R_1^f) \stackrel{def}{=} CornerThreshold_{FV}$. Hence, if $X_1 \geq CornerThreshold_{FV}$, the FI can choose FB.

On the other hand, if $X_1 < CornerThreshold_{FV}$, the FI consumes all the profit.

If $\lambda_3 \neq 0 \Rightarrow c_1 = \alpha_1^{FV}(X_1 - X_0) + \beta_1^{FV} R_1^f$ and $\alpha_2^{*FV} = \alpha_2^{*FB}$, we choose this solution if the FB is unavailable.

The discussion reduces to:

If $X_1 \leq Threshold_{FV} \Rightarrow Case1 \Rightarrow c_1^{FV} = 0$ and $\alpha_2^{*FV} = q_2 + \frac{1}{2} = \alpha_2^{*FB}$;

If $X_1 > Threshold_{FV} \Rightarrow Case2 \Rightarrow$ If $X_1 > CornerThreshold_{FV} \Rightarrow Case2.2$ (FI can choose FB) $\Rightarrow c_1^{FV} = \frac{W_1}{1+\delta} = c_1^{FB}$ and $\alpha_2^{*FV} = q_2 + \frac{1}{2} = \alpha_2^{*FB}$;

If $Threshold_{FV} < X_1 \leq CornerThreshold_{FV} \Rightarrow Case2.1$(FI consumes all the profit) $\Rightarrow c_1 = \alpha_1^{FV}(X_1 - X_0) + \beta_1^{FV} R_1^f = \alpha_1^{FV}(X_1 - X_0) + (I_0 - \alpha_1 X_0)R_1^f$ and $\alpha_2^{*FV} = q_2 + \frac{1}{2} = \alpha_2^{*FB}$.

We identify α_1^* satisfying $Threshold_{FV} \leq 0$. This leads to $X_0(1 + R_1^f) - \frac{I_0 R_1^f}{\alpha_1} \leq 0 \Leftrightarrow X_0\left((1 + R_1^f) - \frac{R_1^f}{\alpha_1^*}\right) \leq 0 \Leftrightarrow \frac{R_1^f}{\alpha_1^*} \geq (1 + R_1^f) \Leftrightarrow \alpha_1^* \leq \frac{R_1^f}{1+R_1^f}$. We restricted hence the region where the proportion α_1^* has to belong such that to avoid the bankruptcy possibility: $\alpha_1^* \in \left(0, \frac{R_1^f}{1+R_1^f}\right]$.

We show $Threshold_{FV} < X_0 \Leftrightarrow X_0 + X_0 R_1^f - \frac{I_0 R_1^f}{\alpha_1} < X_0 \Leftrightarrow X_0\alpha_1^{FV} < I_0$ True. This means "FV Good Time" starts under X_0 (because there is a part of the portfolio which surely appreciates -i.e. the risk-free assets).

Also, $CornerThreshold_{FV} > X_0 \Leftrightarrow X_0(1 + R_1^f) + \frac{I_0}{\alpha_1}(\frac{1}{\delta} - R_1^f) > X_0 \Leftrightarrow X_0 R_1^f + \frac{I_o}{\alpha_1\delta}(1 - \delta R_1^f) > 0$ (assuming that $R_1^f < 1$) q.e.d.

13. Proof of **Proposition 7**

The objective function is:

In Case 1, $\delta E_1\{\log(c_1) + \delta\log(W_1 - c_1) + \delta(\frac{-\sigma_1^2}{2})\alpha_2^{*^2} + \delta(q_2 + \frac{1}{2})\sigma_1^2\alpha_2^* + \delta r_2^f\} = -\infty$;

In Case 2.1, $\delta E_1\{\log(c_1)+\delta\log(W_1-c_1)+\delta(\frac{-\sigma_1^2}{2})\alpha_2^{*^2}+\delta(q_2+\frac{1}{2})\sigma_1^2\alpha_2^*+\delta r_2^f\} = \delta E_1\{\log(W_1-I_0)+\delta\log(I_0)+\delta FVct\}$;

In Case 2.2, $\delta E_1\{\log(c_1)+\delta\log(W_1-c_1)+\delta(\frac{-\sigma_1^2}{2})\alpha_2^{*^2}+\delta(q_2+\frac{1}{2})\sigma_1^2\alpha_2^*+\delta r_2^f\} = \delta E_1\{(1+\delta)\log(W_1)+\delta\log(\delta)-(\delta+1)\log(\delta+1)+\delta FVct\}$;

We assume we are starting with an initial pair (α_1,β_1) such that there is no possibility of bankruptcy $\alpha_1^* \in \left(0, \frac{R_1^f}{1+R_1^f}\right]$, hence Case 1 is eliminated.

We define

$$f(X_1) = \begin{cases} f_1^{FV}(X_1) & \text{if } X_1 \in (0, CornerThreshold_{FV}(\alpha_1)) \\ f_2^{FV}(X_1) & \text{if } X_1 \geq CornerThreshold_{FV}(\alpha_1) \end{cases}$$

where

$$f_1^{FV}(X_1) = \delta\{\log(W_1-I_0)+\delta\log(I_0)+\delta FVct\};$$

$$f_2^{FV}(X_1) = \delta\{(1+\delta)\log(W_1)+\delta\log(\delta)-(\delta+1)\log(\delta+1)+\delta FVct\}$$

and $FVct = (\frac{-\sigma_1^2}{2})(q_2+\frac{1}{2})^2+(q_2+\frac{1}{2})\sigma_1^2(q_2+\frac{1}{2})+r_2^f = \frac{\sigma_1^2}{2}(q_2+\frac{1}{2})^2+r_2^f$;
This is $(\frac{-\sigma_1^2}{2})\alpha_2^{*^2}+(q_2+\frac{1}{2})\sigma_1^2\alpha_2^*+r_2^f$ for the case of $\alpha_2^{*FV} = q_2+\frac{1}{2} = \alpha_2^{*FB}$.

Then $E_0 f(X_1)\mid_{(\alpha_1^{FV},\beta_1^{FV})} = \delta\left(I_1^{FV}+I_2^{FV}\right)+\delta^2 FVct$;

where $I_1^{FV} = \int\limits_0^{CornerThreshold_{FV}} [\log(W_1-I_0)+\delta\log(I_0)]\rho(x_1)dx_1$;

$$I_2^{FV} = \int\limits_{CornerThreshold_{FV}}^{\infty} [(1+\delta)\log(W_1)+\delta\log(\delta)-(\delta+1)\log(\delta+1)]\rho(x_1)dx_1;$$

We have to develop formulas for $\int\limits_{a_1}^{a_2}\log(W_1)\rho_{X_1}(x_1)dx_1$ and for $\int\limits_{a_1}^{a_2}\log(W_1-I_0)\rho_{X_1}(x_1)dx_1$.

$$I_1^{FV} = \int\limits_0^{CornerThreshold_{FV}} [\log(W_1-I_0)+\delta\log(I_0)]\rho(x_1)dx_1 =$$

$$= \int\limits_0^{CornerThreshold_{FV}} \log(W_1-I_0)\rho_{X_1}(x_1)dx_1+\delta\log(I_0)\int\limits_0^{CornerThreshold_{FV}} \rho_{X_1}(x_1)dx_1$$

$$I_2^{FV} = (1+\delta)\int\limits_{CornerThreshold_{FV}}^{\infty} \log(W_1)\rho_{X_1}(x_1)dx_1+\int\limits_{CornerThreshold_{FV}}^{\infty} [\delta\log(\delta)-(\delta+1)\log(\delta+1)]\rho_{X_1}(x_1)dx_1$$

For the last two integrals we apply *Lemma 5*.

$$\int_{a_1}^{a_2} \log(W_1)\rho(x_1)dx_1 = \int_{a_1}^{a_2} \log[\alpha_1 X_1 + \beta_1(1+R_1^f)]\rho(x_1)dx_1 =$$

$$= \int_{a_1}^{a_2} \log[\alpha_1 X_1 + (I_0 - \alpha_1 X_0)(1+R_1^f)]\rho(x_1)dx_1 = \int_{a_1}^{a_2} \log(\alpha_1 X_1 + m)\rho(x_1)dx_1;$$

$m = (I_0 - \alpha_1 X_0)(1+R_1^f)$.

$$\int_{a_1}^{a_2} \log(W_1 - I_0)\rho(x_1)dx_1 = \int_{a_1}^{a_2} \log[\alpha_1 X_1 + (I_0 - \alpha_1 X_0)(1+R_1^f) - I_0\rho(x_1)]dx_1 =$$

$$= \int_{a_1}^{a_2} \log(\alpha_1 X_1 + n)\rho(x_1)dx_1;$$

$n = (I_0 - \alpha_1 X_0)(1+R_1^f) - I_0$

In this case the invested quantity is $Inv_0 = \alpha_1 X_0 + \frac{n}{1+R_1^f} = \alpha_1 X_0 + (I_0 - \alpha_1 X_0) - \frac{I_0}{1+R_1^f} = I_0 \frac{R_1^f}{1+R_1^f}$;

The proportion of the risky asset in the "artificial" portfolio is

$\alpha_1^{*p} = \frac{\alpha_1 X_0}{\alpha_1 X_0 + \frac{n}{1+R_1^f}} = \frac{\alpha_1 X_0}{\alpha_1 X_0 + (I_0 - \alpha_1 X_0) - \frac{I_0}{1+R_1^f}} = \alpha_1^* \frac{1+R_1^f}{R_1^f} \in (0,1]$ for any $\alpha_1^* \in (0, \frac{R_1^f}{1+R_1^f}]$.

This leads to $\log(\alpha_1 X_1 + n) = \log(Inv_0) + r_1^f + \alpha_1^{*p}(r_1^x - r_1^f) + \frac{1}{2}\alpha_1^{*p}(1 - \alpha_1^{*p})\sigma_0^2 =$

$= \log(I_0) + \log(R_1^f) + \alpha_1^* \frac{1+R_1^f}{R_1^f} r_1^x - \alpha_1^* \frac{1+R_1^f}{R_1^f} r_1^f + \frac{1}{2}\alpha_1^* \frac{1+R_1^f}{R_1^f}(1 - \alpha_1^* \frac{1+R_1^f}{R_1^f})\sigma_0^2;$

Then $I_1^{FV} = \int_0^{CornerThreshold_{FV}} \log(W_1 - I_0)\rho_{X_1}(x_1)dx_1 +$

$+\delta \log(I_0) \int_0^{CornerThreshold_{FV}} \rho_{X_1}(x_1)dx_1 = k_1^{FV} \int_0^{CornerThreshold_{FV}} r_1^x \rho_{X_1}(x_1)dx_1 +$

$k_2^{FV} \int_0^{CornerThreshold_{FV}} \rho_{X_1}(x_1)dx_1;$

with

$k_1^{FV} = \alpha_1^* \frac{1+R_1^f}{R_1^f};$

$k_2^{FV} = -\frac{1}{2}\alpha_1^{*2}(\frac{1+R_1^f}{R_1^f})^2\sigma_0^2 + \alpha_1^*(\frac{1+R_1^f}{R_1^f}\frac{\sigma_0^2}{2} - \frac{1+R_1^f}{R_1^f} r_1^f) + (\delta+1)\log(I_0) + \log(R_1^f);$

Similarly $I_2^{FV} = (1+\delta) \int\limits_{ThresholdFV_1}^{\infty} \log(W_1)\rho_{X_1}(x_1)dx_1 +$

$+ \int\limits_{ThresholdFV_1}^{\infty} [\delta\log(\delta) - (\delta+1)\log(\delta+1)]\rho_{X_1}(x_1)dx_1 =$

$= k_3^{FV} \int\limits_{CornerThreshold_{FV}}^{\infty} r_1^x \rho_{X_1}(x_1)dx_1 + k_4^{FV} \int\limits_{CornerThreshold_{FV}}^{\infty} \rho_{X_1}(x_1)dx_1;$

with

$k_3^{FV} = (1+\delta)\alpha_1^*$; and $k_4^{FV} = -\frac{1}{2}\alpha_1^{*2}(1+\delta)\sigma_0^2 + \alpha_1^*(1+\delta)(\frac{\sigma_0^2}{2} - r_1^f) + (1+\delta)(\log(I_0) + r_1^f) + \delta\log(\delta) - (\delta+1)\log(\delta+1)$;

$\frac{CornerThreshold_{FV}}{X_0} = T_{41} + \frac{1}{\alpha_1^*}T_{42}$ with $T_{41} = 1 + R_1^f; T_{42} = \frac{1}{\delta} - R_1^f$;

From here we can find the $(\alpha_1^{FV}, \beta_1^{FV})$ for the first period by maximizing with respect to (α_1, β_1). We have to find the *argmax*, in the corresponding interval, of the following (however, we approximated α_1^{*FV} by a constant):

$$E_0 f(X_1)\mid_{(\alpha_1^{FV},\beta_1^{FV})} = \delta\left(I_1^{FV} + I_2^{FV}\right) + \delta^2 FVct =$$
$$= \delta\{k_1^{FV}[\mu\Phi(\frac{\ln(T_{41}+\frac{1}{\alpha_1^*}T_{42})-\mu}{\sigma_0}) - \sigma_0\rho(\frac{\ln(T_{41}+\frac{1}{\alpha_1^*}T_{42})-\mu}{\sigma_0}) +$$
$$+ k_2^{FV}\Phi(\frac{\ln(T_{41}+\frac{1}{\alpha_1^*}T_{42})-\mu}{\sigma_0}) + k_3^{FV}\{\mu[1-\Phi(\frac{\ln(T_{41}+\frac{1}{\alpha_1^*}T_{42})-\mu}{\sigma_0})] +$$
$$+ \sigma_0\rho(\frac{\ln(T_{41}+\frac{1}{\alpha_1^*}T_{42})-\mu}{\sigma_0})\} + k_4^{FV}[1-\Phi(\frac{\ln(T_{41}+\frac{1}{\alpha_1^*}T_{42})-\mu}{\sigma_0})]\} + \delta^2 FVct;$$

14. Proof of **Proposition 8**

a) 1) $X_1 \in (0, CornerThreshold_{FV})$;
$c_1^{FV} = \alpha_1^{FV}(X_1 - X_0) + (I_0 - \alpha_1^{FV}X_0)R_1^f; \alpha_2^{*FV} = q_2 + \frac{1}{2}$;

2) $X_1 \in [CornerThreshold_{FV}, \infty)$;
$c_1^{FV} = c_1^{FB} = W_1\frac{1}{1+\delta}; \alpha_2^{*FV} = q_2 + \frac{1}{2}$;

1) $c_1^{FV} = \alpha_1^{*FV} I_0 \frac{X_1}{X_0} + I_0[(1-\alpha_1^{*FV})R_1^f - \alpha_1^{*FV}]$;
2) $c_1^{FV} = \frac{1}{1+\delta}[I_0(1+R_1^f)(1-\alpha_1^{*FV}) + I_0\alpha_1^{*FV}\frac{X_1}{X_0}]$;

$E_0(c_1^{FV}) = \int_0^\infty c_1^{FV}(X_1)\rho_{X_1}(x_1)dx_1 = \int\limits_0^{CornerThreshold_{FV}} \{\alpha_1^{*FV} I_0\frac{X_1}{X_0} + I_0[(1-\alpha_1^{*FV})R_1^f - \alpha_1^{*FV}]\}\rho_{X_1}(x_1)dx_1 + \int\limits_{CornerThreshold_{FV}}^{\infty} \frac{1}{1+\delta}[I_0(1+R_1^f)(1-\alpha_1^{*FV}) +$

$I_0\alpha_1^{*FV}\frac{X_1}{X_0}]\rho_{X_1}(x_1)dx_1 = \frac{1}{1+\delta}\{I_0\alpha_1^{*FV}e^{\mu+\frac{\sigma_0^2}{2}}[1+\delta\Phi(S-\sigma_0)]+I_0(1-\alpha_1^{*FV})R_1^f[1+\delta\Phi(S)] - \Phi(S)I_0(\alpha_1^{*FV}\delta+1)+I_0(1-\alpha_1^{*FV})\}$;

b)

We compute $E_0(\alpha_2^{FV}) - \alpha_1^{FV}$.

$\alpha_2^{FV} = \alpha_2^{*FV}\frac{W_1-c_1^{FV}}{X_1} = (q_2+\frac{1}{2})[\alpha_1^{*FV}\frac{I_0}{X_0} + \frac{I_0(1+R_1^f)(1-\alpha_1^{*FV})}{X_0}\frac{1}{\frac{X_1}{X_0}} - \frac{c_1^{FV}}{X_1}]$;

1) $\frac{c_1^{FV}}{X_1} = \alpha_1^{*FV}\frac{I_0}{X_0} + \frac{1}{\frac{X_1}{X_0}}\frac{I_0}{X_0}[(1-\alpha_1^{*FV})R_1^f - \alpha_1^{*FV}]$;

2) $\frac{c_1^{FV}}{X_1} = \frac{1}{1+\delta}[\alpha_1^{*FV}\frac{I_0}{X_0} + \frac{I_0(1+R_1^f)(1-\alpha_1^{*FV})}{X_0}\frac{1}{\frac{X_1}{X_0}}]$;

Then:

1) $\alpha_2^{FV} = (q_2+\frac{1}{2})\frac{I_0}{X_0}\frac{1}{\frac{X_1}{X_0}}$;

2) $\alpha_2^{FV} = (q_2+\frac{1}{2})\frac{\delta}{1+\delta}[\frac{I_0\alpha_1^{*FV}}{X_0} + \frac{I_0(1+R_1^f)(1-\alpha_1^{*FV})}{X_0}\frac{1}{\frac{X_1}{X_0}}]$;

$E_0(\alpha_2^{FV}) - \alpha_1^{FV} = \int_0^\infty \alpha_2^{FV}(X_1)\rho_{X_1}(x_1)dx_1 - \alpha_1^{*FV}\frac{I_0}{X_0} =$

$= (q_2+\frac{1}{2})\frac{I_0}{X_0}\int\limits_0^{CornerThreshold_{FV}}\frac{1}{\frac{X_1}{X_0}}\rho_{X_1}(x_1)dx_1 +$

$+(q_2+\frac{1}{2})\frac{\delta}{1+\delta}\frac{I_0\alpha_1^{*FV}}{X_0}\int\limits_{CornerThreshold_{FV}}^{\infty}\rho_{X_1}(x_1)dx_1 +$

$+(q_2+\frac{1}{2})\frac{\delta}{1+\delta}\frac{I_0(1+R_1^f)(1-\alpha_1^{*FV})}{X_0}\int\limits_{CornerThreshold_{FV}}^{\infty}\frac{1}{\frac{X_1}{X_0}}\rho_{X_1}(x_1)dx_1 - \alpha_1^{*FV}\frac{I_0}{X_0} =$

$\frac{I_0}{X_0}\{(q_2+\frac{1}{2})\{e^{-\mu+\frac{\sigma_0^2}{2}}[\Phi(S+\sigma_0) + \frac{\delta}{1+\delta}(1+R_1^f)(1-\alpha_1^{*FV})(1-\Phi(S+\sigma_0))] + \frac{\delta}{1+\delta}\alpha_1^{*FV}(1-\Phi(S))\} - \alpha_1^{*FV}\}$;

We finally compute $E_0(\beta_2^{FV}) - \beta_1^{FV}$.

$\beta_1^{FV} = I_0(1-\alpha_1^{*FV})$;

$\beta_2^{FV} = \frac{1-\alpha_2^{*FV}}{1+R_1^f}[\alpha_1^{*FV}I_0\frac{X_1}{X_0} + I_0(1-\alpha_1^{*FV})(1+R_1^f) - c_1^{FV}]$;

1) $\beta_2^{FV} = \frac{1-\alpha_2^{*FV}}{1+R_1^f}I_0$;

2) $\beta_2^{FV} = \frac{1-\alpha_2^{*FV}}{1+R_1^f}\frac{\delta}{1+\delta}[\alpha_1^{*FV}I_0\frac{X_1}{X_0} + I_0(1-\alpha_1^{*FV})(1+R_1^f)]$;

Then $E_0(\beta_2^{FV}) - \beta_1^{FV} = \int_0^\infty \beta_2^{FV}(X_1)\rho_{X_1}(x_1)dx_1 =$

$= \frac{1-\alpha_2^{*FV}}{1+R_1^f}I_0\Phi(S) + \frac{1-\alpha_2^{*FV}}{1+R_1^f}\frac{\delta}{1+\delta}\alpha_1^{*FV}I_0e^{\mu+\frac{\sigma_0^2}{2}}[1-\Phi(S-\sigma_0)] + \frac{1-\alpha_2^{*FV}}{1+R_1^f}\frac{\delta}{1+\delta}I_0(1-\alpha_1^{*FV})(1+R_1^f)[1-\Phi(S)] - I_0(1-\alpha_1^{*FV})$.

15. Transaction Costs

We first assume for any accounting regime the FI makes the same decisions as in the case without transaction costs, but it subtracts the costs from consumption (at $T = 1$,2) respectively from the available resources to invest in the initial portfolio (at $T = 0$). We assume the transaction costs, known at $T = 0$, 1 and 2 (hence no need here of E_0) are respectively $Cost_0 = s\alpha_1$, $Cost_1 = s\,|\alpha_2 - \alpha_1|$ and $Cost_2 = s\alpha_2$ (proportional with the number of risky assets transacted).

The existence of $Cost_2$ implies at $T = 2$ the FI consumes $c_2^{new} = c_2 - Cost_2 = W_2 - Cost_2 = \alpha_2 X_2 + \beta_2(1 + R_1^f)(1 + R_2^f) - s\alpha_2 = \alpha_2 X_2 + m$, with $m = \beta_2(1 + R_1^f)(1 + R_2^f) - s\alpha_2$.This is equivalent with having invested at $T = 1$ a quantity Inv_1' in a portfolio composed by α_2 risky assets and $\frac{m}{(1+R_1^f)(1+R_2^f)}$ risk-free (this portfolio gives at $T = 2$ $\alpha_2 X_2 + m$). The equivalent invested quantity $Inv_1' = \alpha_2 X_2 + \frac{m}{(1+R_2^f)} = Inv_1 - \frac{s\alpha_2}{(1+R_2^f)}$.The proportion of risky assets in this equivalent portfolio is $\alpha_2^{*p} = \frac{\alpha_2 X_1}{Inv_1'} = \frac{\alpha_2 X_1}{Inv_1}\frac{Inv_1}{Inv_1'} \simeq \frac{\alpha_2 X_1}{Inv_1} = \alpha_2^*$.

At $T = 1$ the FI equivalently invests Inv_1' and consumes $c_1^{new} = c_1 - Cost_1$, due to the transaction costs. Hence the equivalent endowment at $T = 1$ is $W_1' = c_1 - Cost_1 + Inv_1 - \frac{s\alpha_2}{(1+R_2^f)} = W_1 - (Cost_1 + \frac{s\alpha_2}{(1+R_2^f)}) = W_1 - K = \alpha_1 X_1 + \beta_1(1 + R_1^f) - K = \alpha_1 X_1 + n$, with $n = \beta_1(1+R_1^f) - K$. But this is equivalent with having invested at $T = 0$, Inv_0' in a portfolio composed by α_1 risky assets and $\frac{n}{(1+R_1^f)} = \beta_1 - \frac{K}{(1+R_1^f)}$ risk-free. $Inv_0' = \alpha_1 X_0 + \frac{n}{(1+R_1^f)} = I_0 - \frac{K}{(1+R_1^f)}$. The proportion of risky assets in this equivalent portfolio is $\alpha_1^{*p} = \frac{\alpha_1 X_0}{Inv_0'} = \frac{\alpha_1 X_0}{I_0}\frac{I_0}{Inv_0'} \simeq \frac{\alpha_1 X_0}{I_0} = \alpha_1^*$.

Finally, at $T = 0$ we take into account we have to subtract $Cost_0 = s\alpha_1$ from the equivalent invested quantity, hence the final equivalent invested quantity is $Inv_0'' = Inv_0' - s\alpha_1 = I_0 - \frac{K}{(1+R_1^f)} - s\alpha_1 = I_0 - Cost_0 - \frac{Cost_1}{(1+R_1^f)} - \frac{Cost_2}{(1+R_2^f)(1+R_1^f)} = I_0 - s(\alpha_1 + \frac{|\alpha_2 - \alpha_1|}{(1+R_1^f)} + \frac{\alpha_2}{(1+R_1^f)(1+R_2^f)})$.

Hence the additional costs imply (an approximated solution) solving equivalently our initial problem with an initial investment of $I_0 - s(\alpha_1 + \frac{1}{1+R_1^f}\,|E_0\alpha_2 - \alpha_1| + \frac{1}{(1+R_1^f)(1+R_1^f)}E_0\alpha_2)$ instead of I_0. This equivalent invested quantity depends on the accounting regime in force and has to be evaluated ex-ante (at $T = 0$).

q.e.d.