

# Pricing Volatility Derivatives with General Risk Functions

Alejandro Balbás  
University Carlos III of Madrid  
[alejandro.balbas@uc3m.es](mailto:alejandro.balbas@uc3m.es)

# Content

- Introduction.
- Describing volatility derivatives.
- Pricing and hedging methods.
- Using infinitely many options.
- Synthetic variance swap.
- Synthetic volatility swap.
- Synthetic variance and volatility swap options.
- Dealing with the available options and risk measures.
- Applications in Nordpool
- Conclusions.

# Introduction

- The **underlying variable** is the **realized volatility** or **variance** of a traded financial asset over the life of the contract, rather than the price of this asset.

# Introduction

- They are traded because:
  - They easily provide diversification. They don't depend on prices evolution, but on how fast prices are changing.
  - As the empirical evidence points out, they provide adequate hedging when facing market turmoil, which usually implies correlations quite close to one.
  - Hedge funds and other risk seeking investors usually sell these products due to their high risk premium.
    - The last two reasons imply very asymmetric returns, with heavy tails.
    - Sellers usually earns money very often, but the profits are really small. They rarely loose money, but loses are very high when they arise.

# Introduction

- The are usually traded in OTC markets, so estimating the total trading volume is problematic.
- But recent estimates for daily trading volume on indices are in the region of \$30 to \$35 millions of notional.

(Broadie, M. and A. Jain, 2008. "Pricing and hedging volatility derivatives". *Journal of Derivatives*. 15, 3, 3-24.)

# Introduction

- Approximately one year and a half ago the *CBOE* (Chicago Board Option Exchange) has been trading volatility derivatives. They have several derivatives:
  - VIX
  - VXN, etc.

[www.CBOE.com](http://www.CBOE.com)
- Besides, there are volatility funds, such as the *Euroption Strategic Fund* ([www.europtionfund.com](http://www.europtionfund.com)), that only deals with volatility derivatives, mainly on European indexes.

# Describing volatility derivatives

- Variance and Volatility Swap:

$$\frac{dS_t}{S_t} = \mu(S_t, t)dt + \sigma(S_t, t)dz$$

is a Brownian Motion reflecting the evolution of the underlying security. The realized variance in  $[0, T]$  is defined by the stochastic integral

$$W^T = \frac{1}{T} \int_0^T \sigma^2(S_t, t)dt$$

and the realized volatility is:

$$V^T = \sqrt{W^T}$$

# Describing volatility derivatives

- Variance and Volatility Swap:

–It may be proved that

$$\text{Lim}_{(\Delta t \rightarrow 0)} \hat{V}^T(n) = \hat{V}^T$$

and

$$\text{Lim}_{(\Delta t \rightarrow 0)} \hat{W}^T(n) = \hat{W}^T$$

with convergence in probability at least.

(Demeterfi, K., E. Derman, M. Karmal, and J. Zou. "A Guide to Volatility and Variance Swaps". *Journal of Derivatives*, 4 (1999), pp. 9 - 32.)

# Describing volatility derivatives

- **Variance and Volatility Swap:**
  - Consider a finite set of trading dates:
  - The estimated variance in  $[0, T]$  is given by:

$$\hat{W}^T(n) = \frac{1}{n(\Delta t)} \sum_{i=1}^n r_i^2$$

where  $r_i = L\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)$

and  $\Delta t = t_i - t_{i-1}, i = 1, 2, \dots, n$ . Similarly, the estimated volatility is given by the square root of the previous expression:

$$\hat{V}^T(n) = \sqrt{\frac{1}{n(\Delta t)} \sum_{i=1}^n r_i^2}$$

# Describing volatility derivatives

- Variance and Volatility Swap:

- A long position in a **Variance Swap** implies the payment of a (numerical) price  $\hat{W}_0^T$  at  $t = 0$ , so as to receive the random amount  $\Phi \hat{W}^T(n)$  at  $t = T$ ,

$\Phi > 0$  being a notional value known by the agents.

- The buyer of a **Volatility Swap** must pay  $\hat{V}_0^T$  at  $t = 0$  so as to receive the random pay-off  $\Phi \hat{V}^T(n)$  at  $t = T$ .

# Describing volatility derivatives

- Variance Swap Future:

- A long position in a Variance Swap Future with maturities  $T_1 < T_2$  implies to accept at  $t = 0$  a commitment so as to purchase a Variance Swap at  $T_1$  with maturity at  $T_2$ . Consider the trading dates

$$0 = t_0 < t_1 < \dots < t_k = T_1 < t_{k+1} < \dots < t_{k+h} = T_2$$

with

$$\Delta t = t_i - t_{i-1}, i = 1, 2, \dots, k + h$$

# Describing volatility derivatives

- Variance Swap Future:

– Then:

$$\hat{W}^{T_2}(k+h) = \frac{1}{(k+h)(\Delta t)} \sum_{i=1}^{k+h} r_i^2$$

$$\hat{W}^{T_1}(k) = \frac{1}{k(\Delta t)} \sum_{i=1}^k r_i^2$$

$$\hat{W}^{(T_1, T_2)}(h) = \frac{1}{(h)(\Delta t)} \sum_{i=k+1}^{k+h} r_i^2$$

manipulating we obtain:

$$\hat{W}^{(T_1, T_2)}(h) = \frac{T_2}{T_2 - T_1} \hat{W}^{T_2}(k+h) - \frac{T_1}{T_2 - T_1} \hat{W}^{T_1}(k)$$

# Describing volatility derivatives

- Variance Swap Future:

–Theorem 1: The Variance Swap Future with maturities at  $T_1$  and  $T_2$  is replicated with:

the purchase of  $\frac{T_2}{T_2 - T_1}$  Variance Swaps with maturity at  $T_2$

the sale of  $\frac{T_1}{T_2 - T_1} e^{-r(T_2 - T_1)}$  Variance Swaps with maturity at  $T_1$   
and borrowing the price of the strategy above

during the time interval  $[0, T_1]$

–Therefore, the price of the Variance Swap Future is given by

$$\hat{W}_0^{(T_1, T_2)} = \left( \frac{T_2}{T_2 - T_1} \hat{W}_0^{T_2} - \frac{T_1}{T_2 - T_1} \hat{W}_0^{T_1} e^{-r(T_2 - T_1)} \right) e^{rT_1}$$

# Describing volatility derivatives

- **Variance Swap Option:**

- We will consider that the buyer of a Variance Swap European Call (Put) with maturities  $T_1 < T_2$  has the right (no obligation) to buy (sell) a Variance Swap at  $T_1$  with maturity at  $T_2$ , and he will pay (receive) the strike  $E$ .
- The price of the option is paid at  $t = 0$ , whereas the strike is paid at  $T_1$

# Describing volatility derivatives

- Covariance Swap:

- Consider a finite set of trading dates:

$$0 = t_0 < t_1 < \dots < t_n = T$$

- Consider also two underlying assets whose price processes will be denoted by  $S_t$  and  $\bar{S}_t$ . Their realized covariance in  $[0, T]$  is given by

$$\hat{C}^T(n) = \frac{1}{n(\Delta t)} \sum_{i=1}^n r_i \bar{r}_i$$

where

$$\bar{r}_i = L \left( \frac{\bar{S}_{t_i}}{\bar{S}_{t_{i-1}}} \right)$$

# Describing volatility derivatives

- Covariance Swap:

- The buyer of a Covariance Swap with maturity at  $T$  will pay at  $t = 0$  the numerical price  $\hat{C}_0^T$ , and he will receive the random pay-off  $\Phi \hat{C}^T(n)$  at  $t = T$ .
- Where  $\Phi$  is the notional price of one covariance point.

# Pricing and hedging methods

- Mainly there are two methods:
  - **Method 1:** using infinitely many options.
  - **Method 2:** using stochastic complete pricing models.

# Pricing and hedging methods

- Method 1:

- It was introduced by:

- Neuberger, A.J. "The Log Contract". *Journal of Portfolio Management*, Vol. 20, No. 2 (1994), pp. 74 – 80.

- And further developments were given by Demeterfi, K. *et al.*

- The first article considered an extension of the Black-Scholes model and proved that the realized volatility equals:

$$g_{vas}(S_T) = \frac{\Phi}{\sqrt{T}} \sqrt{\frac{S_T}{F_0^T} - 1 - L\left(\frac{S_T}{F_0^T}\right)}$$

# Pricing and hedging methods

- **Method 1:**
- Later, the second extended the analysis so as to involve some stochastic volatility models.
- The authors also showed that minor modifications of the formula allow us to accept the existence of jumps in the underlying asset price process.

# Pricing and hedging methods

- Method 1:
- From the formula above and the equality

$$g(S) = (g(h) - hg'(h)) + Sg'(h) + \int_S^h g''(k)(k - S)dk$$

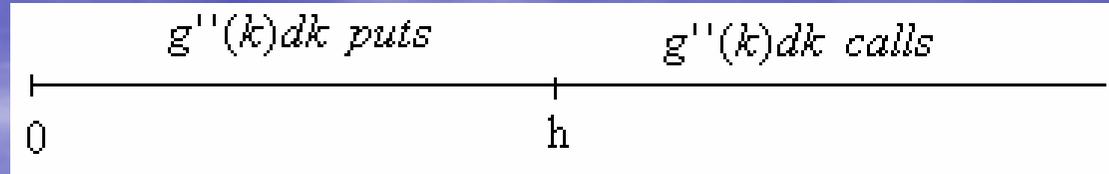
for a tow times differentiable function those papers stated that the variance swap may be replicated:

- Buying  $\frac{\Phi}{T} \frac{1}{k^2} dk$  European Puts with strike  $k$ ,  $0 < k < F_0^T$
- Buying  $\frac{\Phi}{T} \frac{1}{k^2} dk$  European Calls with strike  $k$ ,  $F_0^T < k < \infty$

where the maturity of the options equals the maturity of the variance swap.

# Pricing and hedging methods

- Proof:



- Case 1:



$$\begin{aligned}
 \text{Pay-off} &= \int_0^h \underbrace{\text{put}(k)}_0 g''(k) dk + \int_h^\infty \underbrace{\text{call}(k)}_0 g''(k) dk = \\
 &= \int_0^h \text{put}(k) g''(k) dk = \int_0^S \underbrace{\text{put}(k)}_0 g''(k) dk + \int_S^h (k - S) g''(k) dk \\
 &= \int_S^h (k - S) g''(k) dk
 \end{aligned}$$

- $h = F_0^T$  makes the remaining terms vanish.

# Pricing and hedging methods

- Case 2:



$$\begin{aligned}
 \text{Pay-off} &= \int_0^h \underbrace{\text{put}(k)}_0 g''(k) dk + \int_h^\infty \text{call}(k) g''(k) dk = \\
 &= \int_h^S \text{call}(k) g''(k) dk + \int_S^\infty \underbrace{\text{call}(k)}_0 g''(k) dk \\
 &= \int_h^S (S - k) g''(k) dk = \int_S^h (k - S) g''(k) dk
 \end{aligned}$$

# Pricing and hedging methods

- Method 2:

- Javaheri, Wilmott, and Haug (2002) discussed the valuation of volatility swaps in the GARCH(1, 1) stochastic volatility model.

- (Javaheri, A., P. Wilmott, and E. Haug. "Garch and Volatility Swaps". Working paper, 2002, <http://www.wilmott.com>).

- Little and Pant (2001) developed a finite difference method for the valuation of variance swaps in the case of discrete sampling in an extended Black-Scholes framework.

- (Little, T., and V. Pant. "A Finite Difference Method for the Valuation of variance Swaps". *Journal of Computational Finance*, Vol. 5, No. 1 (2001), pp. 74 - 80).

# Pricing and hedging methods

- Method 2:

–Carr *et al.* (2005) priced options on realized variance by directly modeling the quadratic variation of the underlying process using a Lévy process.

(Carr, P., H. German, D. Madan, and M. Yor. "Pricing Options on Realized Variance" . *Finance and Stochastics*, Vol. 9, No. 4 (2005), pp. 453 - 475).

# Pricing and hedging methods

- Therefore, the variance swap contract has been priced by using infinitely many options, whereas the remaining contracts have been priced by using an stochastic volatility pricing model.
- The first method has advantage since it doesn't depend on the theoretical price we use.
- Option prices are given by the market.
- The drawback is that an infinite number of options can't be traded.

# Pricing and hedging methods

- In a recent paper, Broadie and Jain (2008) use the Heston model and propose to price and hedge volatility swaps and variance swap options by using variance swaps.
- Thus, they hedge a position in volatility by trading variance swaps and consequently by trading infinitely many options.
- They use the pay-off variance as a risk measure to be minimized when approximating the variance swap contract by a finite combination of the available options.

# Pricing and hedging methods

- In this paper we propose to use infinitely many options to replicate every volatility derivative without using the variance swap to connect the options and the asset to be priced and hedged.
- This makes our approach be independent on any pricing model.
- We will use both European and digital options since digital options may be easily priced in practice from the combination provided by the European options.

# Pricing and hedging methods

–Furthermore, the committed error due to the lack of infinitely many available options will be computed by using a general risk function, (an expectation bounded risk measure or a deviation measure).

(Rockafellar, R.T., S. Uryasev and M. Zabarankin, 2006. "Generalized Deviations In Risk Analysis". *Finance & Stochastics*, 10, 51 - 74).

–Since heavy tails and asymmetries are usual when dealing with volatility the use of the variance could be a shortcoming.

(Ogryczak, W. and A. Ruszczyński, 2002. "Dual Stochastic Dominance and Related Mean Risk Models". *SIAM Journal on Optimization*, 13, 60 - 78).

# Using infinitely many options

- From the formulas

$$g(S) = g(h) + \int_a^S g'(k) dk$$

for a differentiable function, and

$$g(S) = (g(h) - hg'(h)) + Sg'(h) + \int_S^h g''(k)(k - S) dk$$

for a 2 times differentiable function, we can prove the following results:

# Using infinitely many options

- **Theorem 2:** Let be  $a, b \in \mathcal{R} \cup \{-\infty, \infty\}$  and  $g : (a, b) \rightarrow \mathcal{R}$  an arbitrary function such that  $g$  and its first derivative exist and are continuous out of a finite set  $D = \{d_1 < d_2 < \dots < d_m\} \subset (a, b)$ . Suppose that  $g$  may be extended and become continuous on  $(a, d_1]$ , and the same property holds if  $(a, d_1]$  is replaced by  $[d_i, d_{i+1}]$ ,  $i = 1, \dots, m - 1$ , or by  $[d_m, b)$ . Denote by  $J_i$  the jump of  $g$  at  $d_i$ ,  $i = 1, \dots, m$ . Then, if  $h \in (a, b)$ ,  $h \leq d_1$  and  $S_T$  is a (random) price at  $T$  such that

$$\mu(S_T \in (a, b)) = 1, \mu(S_T \in D \cup \{h\}) = 0$$

the final pay-off  $g(S_T)$  may be replicated by:

# Using infinitely many options

- Investing  $g(h)^- e^{-r_f T}$  euros in the riskless asset.
- Buying  $J_i$  Digital Calls with strike  $d_i, i = 1, 2, \dots, m$ .
- Buying  $g'(k)dk$  Digital Calls for every strike  $k \in (a, b) \setminus D, k > h$ .
- Selling  $g'(k)dk$  Digital Puts for every strike  $k \in (a, b) \setminus D, k < h$ .

# Using infinitely many options

- Proof:

$$\int_0^{\infty} g'(k) dk \text{ digital calls } h=0$$

$$\begin{aligned} \text{Pay-off} &= \int_0^{\infty} \text{call}(k) g'(k) dk = \\ &= \int_0^S \underbrace{\text{call}(k)}_1 g'(k) dk + \int_S^{\infty} \underbrace{\text{call}(k)}_0 g'(k) dk = \\ &= \int_0^S g'(k) dk \end{aligned}$$

# Using infinitely many options

- **Theorem 3:** Let be  $a, b \in \mathfrak{R} \cup \{-\infty, \infty\}$  and  $g : (a, b) \rightarrow \mathfrak{R}$  an arbitrary function such that  $g$  and its first and second derivatives  $g'$  and  $g''$  exist and are continuous out of a single element  $d \in (a, b)$ . Suppose that  $g$  may be extended and become continuous on  $(a, d]$  and the same property holds if  $(a, d]$  is replaced by  $[d, b)$ . Denote by  $J_d$  and  $J'_d$  the jumps of  $g$  and  $g'$  at  $d$ . Then, if  $S_T$  is a (random) price at  $T$  such that

$$\mu(S_T \in (a, b)) = 1, \mu(S_T = d) = 0$$

the final pay-off  $g(S_T)$  may be replicated by:

# Using infinitely many options

- Investing  $(g(d)^- - hg'(d)^-)e^{-r_f T}$  euros in the riskless asset.
- Buying  $g'(d)^-$  units of the underlying asset.
- Buying  $g''(k)dk$  European Puts with strike  $k$  for every  $k \in (a, d)$ .
- Buying  $g''(k)dk$  European Calls with strike  $k$  for every  $k \in (d, b)$ .
- Buying  $J'_d$  European Calls with strike  $d$ .
- Buying  $J_d$  Digital Calls with strike  $d$ .

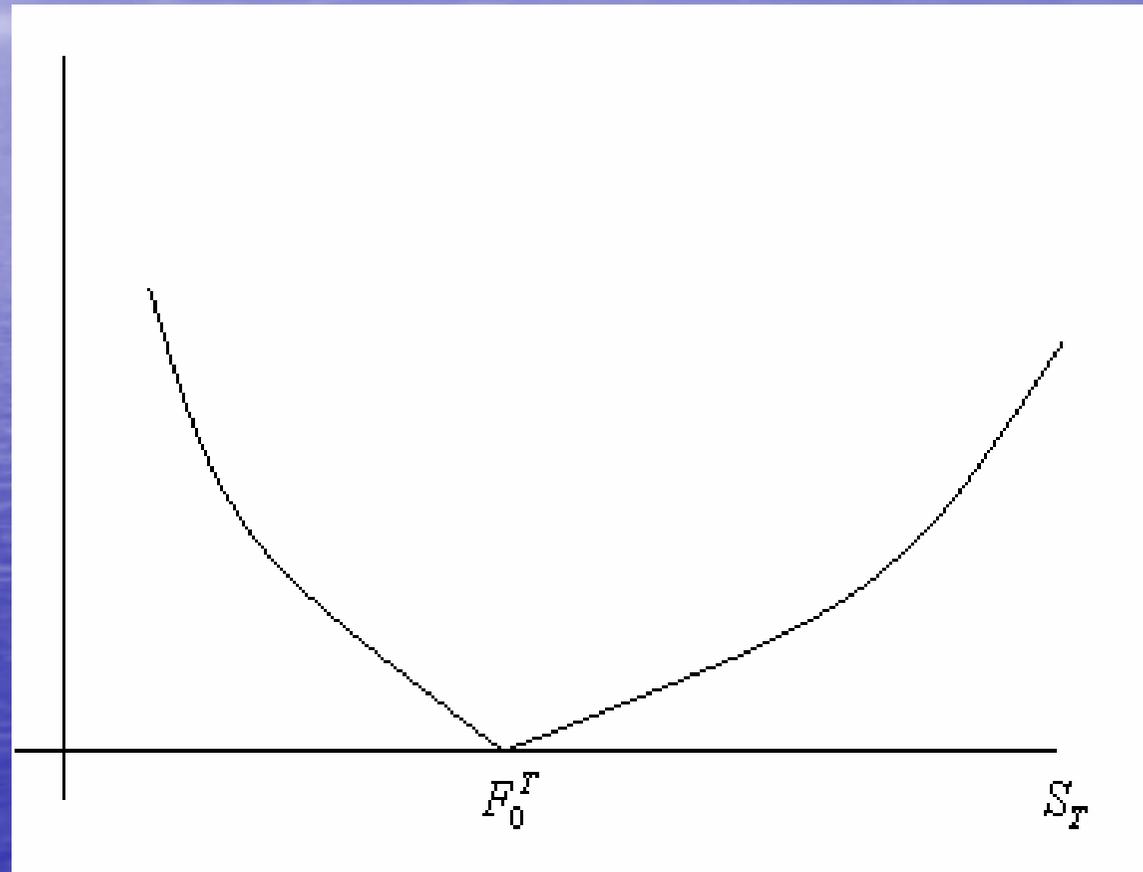
# Using infinitely many options

- These theorems apply to replicate every pay-off  $g$  regular enough. In particular, the variance and volatility swaps.
- For volatility swaps notice that the pay-off function

$$g_{vas}(S_T) = \frac{\Phi}{\sqrt{T}} \sqrt{\frac{S_T}{F_0^T} - 1} - L\left(\frac{S_T}{F_0^T}\right)$$

is continuous, but its first derivative presents a jump at  $S_T = F_0^T$  whose value is  $\frac{2\Phi}{\sqrt{2T}F_0^T}$ .

# Using infinitely many options



# Synthetic Variance Swap

- If  $\mu(S_T > 0) = 1$  holds, the following strategies replicate the variance swap:

- **Strategy 1: (Digital options)**

- Lending  $\frac{\Phi}{T} \left[ \frac{h}{F_0^T} - 1 - L\left(\frac{S_T}{F_0^T}\right) \right] e^{-r_f T}$  euros.

- Buying  $\frac{\Phi}{T} \left[ \frac{1}{F_0^T} - \frac{1}{k} \right] dk$  Digital Calls for every strike  $k$ ,

$$h < k < \infty .$$

- Selling  $\frac{\Phi}{T} \left[ \frac{1}{F_0^T} - \frac{1}{k} \right] dk$  Digital Puts for every strike  $k$ ,

$$0 < k < h .$$

# Synthetic Variance Swap

- Strategy 2: (European options)

- Borrowing  $\frac{\Phi}{T} \left[ L \left( \frac{h}{F_0^T} \right) \right] e^{-r_f T}$  euros.

- Buying  $\frac{\Phi}{T} \left[ \frac{1}{F_0^T} - \frac{1}{h} \right]$  units of the underlying asset.

- Buying  $\frac{\Phi}{T} \frac{1}{k^2} dk$  European Puts with strike  $k$ ,

$$0 < k < h .$$

- Selling  $\frac{\Phi}{T} \frac{1}{k^2} dk$  European Calls with strike  $k$ ,

$$h < k < \infty .$$

# Synthetic Volatility Swap

- If  $\mu(S_T > 0) = 1$  and  $\mu(S_T = F_0^T) = 0$ , then:
- **Strategy 1: (Digital options)**
  - Buying  $\frac{1}{F_0^T} \frac{\Phi}{\sqrt{T}} \frac{k-1}{g_{vas}(k)} dk$  Digital Calls for every strike  
 $k \in (F_0^T, \infty)$ .
  - Selling  $\frac{1}{F_0^T} \frac{\Phi}{\sqrt{T}} \frac{k-1}{g_{vas}(k)} dk$  Digital Puts for every strike  
 $k \in (0, F_0^T)$ .

# Synthetic Volatility Swap

- Strategy 2: (European options)

- Borrowing  $\frac{\Phi}{\sqrt{2T}} \frac{F_0^T - 1}{F_0^T} e^{-r_f T}$  euros.

- Selling  $\frac{\Phi}{\sqrt{2T}} \frac{F_0^T - 1}{(F_0^T)^2}$  units of the underlying asset.

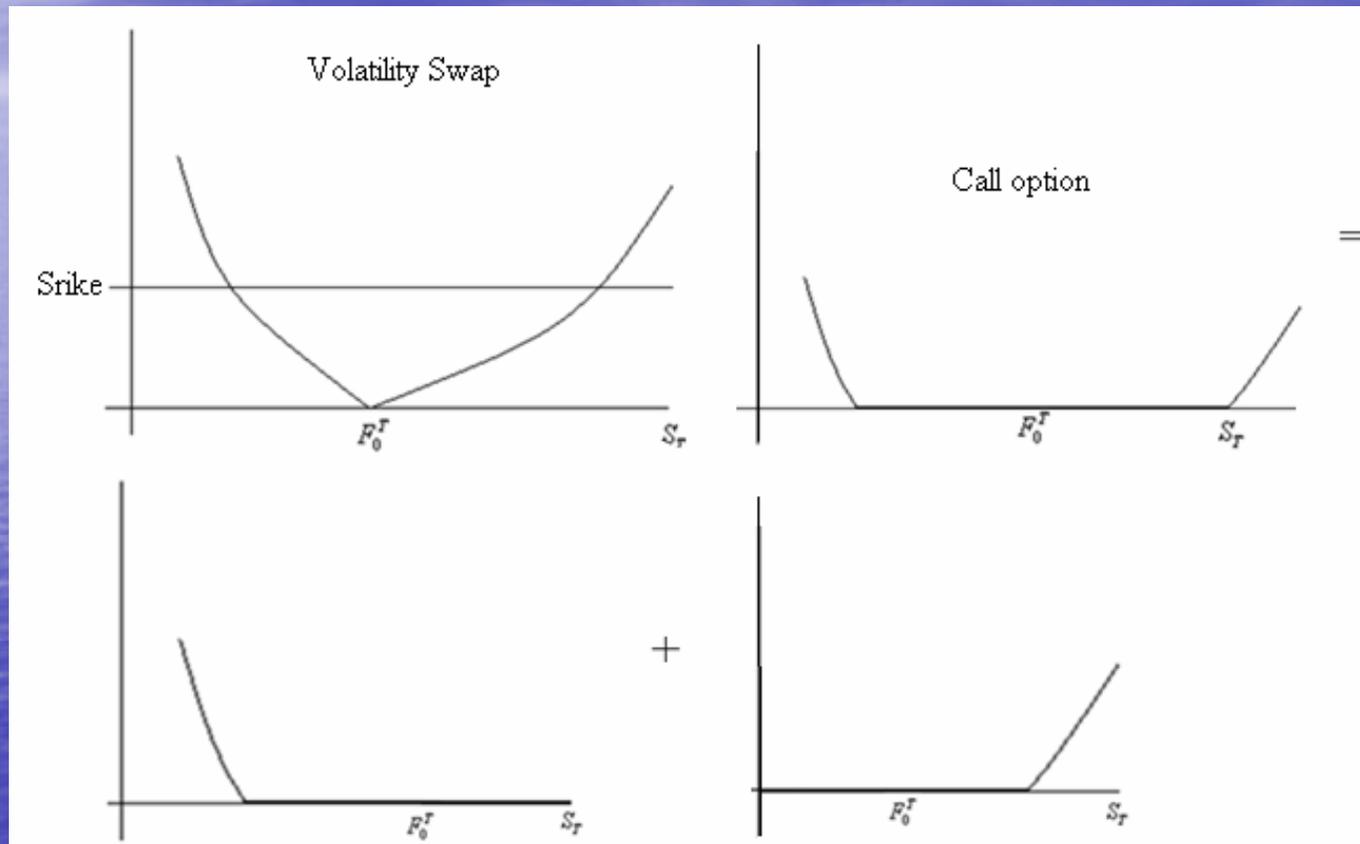
- Buying  $4F_0^T \frac{\Phi}{\sqrt{T}} \frac{g_{vas}(k)^4 - (k - F_0^T)^2}{2k\sqrt{g_{vas}(k)}} dk$  (1)

European Puts with strike  $k$ , for every  $k \in (0, F_0^T)$  .

- Buying (1) European Calls with strike  $k$  for every  $k \in (F_0^T, \infty)$  .

- Buying  $2/\sqrt{2}$  European Calls with strike  $F_0^T$  .

# Synthetic Variance and Volatility Swap Options



- For the sake of simplicity let us assume that the option maturity equals the variance or volatility swap maturity. In the general case, under very weak assumptions we can also prove that the volatility swap final pay-off is a function depending on  $S_T$ , underlying price at the option maturity. 41

# Synthetic Variance and Volatility Swap Options

- The **Variance Swap Option** can be replicated with 2 strategies.
- **Strategy 1: (Digital options):**

– Buying  $\frac{\Phi}{T} \left( \frac{1}{F_0^T} - \frac{1}{k} \right) dk$  Digital Calls for every strike  
 $k > S_2$ .

– Selling  $\frac{\Phi}{T} \left( \frac{1}{F_0^T} - \frac{1}{k} \right) dk$  Digital Puts for every strike  
 $k < S_1$ .

# Synthetic Variance and Volatility Swap Options

- Strategy 2: (European options):

- Investing  $\frac{\Phi}{T} \left( 1 - \frac{S_1}{F_0^T} \right)$  euros in the riskless asset.
- Buying  $\frac{\Phi}{T} \left( \frac{1}{F_0^T} - \frac{1}{S_1} \right)$  units of the underlying asset.
- Buying  $\frac{\Phi}{T} \frac{1}{k^2} dk$  European Puts for every strike  $k < S_1$ .
- Buying  $\frac{\Phi}{T} \frac{1}{k^2} dk$  European Calls for every strike  $k > S_2$ .
- Buying  $\frac{\Phi}{T} \left( \frac{1}{S_1} - \frac{1}{F_0^T} \right)$  European Calls with strike  $S_1$ .
- Buying  $\frac{\Phi}{T} \left( \frac{1}{F_0^T} - \frac{1}{S_2} \right)$  European Calls with strike  $S_2$ .

# Synthetic Variance and Volatility Swap Options

- Volatility Swap Options:
  - Analogously, from theorems 2 and 3, we can see that a **volatility swap option** may be replicated by using infinitely many Digital or European options.

# Dealing with the available options and risk measures

- As already said Demeterfi et al. proposed an heuristic procedure to approximate the variance swap whereas Broadie and Jain solve this problem by minimizing the variance of the committed error.
- This authors don't consider any transaction costs.
- We will provide a general method simultaneously dealing with the lack of infinitely many options, transaction costs and a general risk functions.

# Dealing with the available options and risk measures

- $r_f$  is the risk free rate.
- $S_T$  will be the underlying final pay-off.
- $Y_1, \dots, Y_n$  will be the final pay-offs provided by the available options.
- There are 4 cases:
  - For Digital Calls:  $Y_j = \begin{cases} 0, S_T < E_j \\ 1, S_T > E_j \end{cases}$
  - For Digital Puts:  $Y_j = \begin{cases} 1, S_T < E_j \\ 0, S_T > E_j \end{cases}$
  - For European Calls:  $(S_T - E_j)^+$
  - For European Puts:  $(E_j - S_T)^+$

# Dealing with the available options and risk measures

- $S_0 \leq S^0$  :bid/ask of the underlying asset.
- $y_i \leq y^i$  :bid/ask of  $Y_i$  .
- $\rho$  denotes an expectation bounded risk measure, defined over  $L^p$ , where  $\rho$  satisfies  $S_T \in L^p$ . Thus,  $Y_1, \dots, Y_n \in L^p$  too.
- Usually  $\rho = 1$ .  
 $(x_f, x_0, x^0, x_1, x^1, \dots, x_n, x^n)$  will be the hedging strategy.

# Dealing with the available options and risk measures

- $P(x) = x_f e^{r_f T} + (x^0 - x_0)S_T + (x^1 - x_1)Y_1 + \dots + (x^n - x_n)Y_n$  will be the pay-off.
- $p(x) = x_f + x^0 S^0 - x_0 S_0 + x^1 y^1 - x_1 y_1 + \dots + x^n y^n - x_n y_n$  will be the price.
 
$$\left\{ \begin{array}{l} \min \rho(x_f e^{r_f T} + \sum_{i=1}^n (x^i - x_i)Y_i - g(S_T)) \\ x_f + \sum_{i=1}^n x^i y^i - \sum_{i=1}^n x_i y_i \leq A \\ x \geq 0 \end{array} \right.$$
- We may prove that  $F(A)e^{-r_f T} + A = F(0)e^{-r_f T}$  remains constant and doesn't depend on A. And it will be call the ask price of  $g(S_T)$ .

# Dealing with the available options and risk measures

- The solution  $x$  of the problem doesn't depend on  $A$ , except for the risk-free asset.
- The previous problem may be solved in practice by duality methods. So, suppose that  $q$  is the conjugate of  $p$
- Usually  $p = 1$  and  $q = \infty$ .  $\frac{1}{p} + \frac{1}{q} = 1$
- $\Delta_\rho \subset L^q$  is the sub gradient of  $\rho$  (Rockafellar et al. (2006)). For example if  $\rho = CVaR_{95\%}$  then:

$$\Delta_\rho = \{z(S_T) \in L^\infty; E(z(S_T)) = 1, 0 \leq z(S_T) \leq 20\}$$

# Dealing with the available options and risk measures

- The dual problem is:

$$\left\{ \begin{array}{l} \max E(g(S_T)z(S_T)) \\ E(z(S_T)) = 1 \\ S_0 e^{r_f T} \leq E(z(S_T)S_T) \leq S^0 e^{r_f T} \\ Y_j e^{r_f T} \leq E(z(S_T)Y_j) \leq Y^j e^{r_f T}, j = 1, \dots, n \end{array} \right.$$

- The dual problem is always linear.
- Both problems are analyzed and solved in Balbas, A., R. Balbas and S. Mayoral (2009), Portfolio Choice Problems and Optimal Hedging with General Risk Functions, *European Journal of Operational research*, 192, 2, 603-620.

# Conclusions

- Volatility derivatives are becoming very useful because they make it easy to compose diversified portfolios and also offer protection for market turmoil.
- We have proved that all of them , including volatility swaps and variance or volatility futures and call and put options may be replicated by a combination of infinitely many European options.

# Conclusions

- They can be also replicated with digital options. It seems to be an interesting result since digital options are easily priced and replicated by using the underlying.
- Moreover, to overcome the lack of infinitely many options in the market, we have provided a pricing and hedging method related to Expectation Bounded Risk Measures. The method also incorporates the usual market imperfections.

**Thank you so much  
for your attention**

**Alejandro Balbás  
University Carlos III of Madrid, Spain**

