PROBLEMS OF A LINEAR KIND: FROM VALLEJO TO PEACOCK[1]

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This work is concerned with epistemological conceptions linked to a historical and pedagogical alternatives and approximations in the evolution of teaching of algebra, with respect to the solving of linear problems through algebraic method.

Same of these alternatives are epitomized into two books published at about the same time: Vallejo's Treatise (1812-1841) and Peacock's Arithmetical Algebra (1830-1842). We analyze the proposal for teaching these problems reflected in the first book (Vallejo's), taking into account the historical and cultural background in which it appeared, and then, we contrast this with the proposal for teaching these problems that appears in the second (Peacock's).

Such analyses shed light on pedagogical approaches, which helps us understand the historical roots of the organization of elementary school teaching of algebra, with relation to this topic.

AIM AND METHOD

For a very long time mathematicians have solved a great variety of problems whose common unifying nexus is the theme of linearity. Also, secondary school students advancing from arithmetic to algebra find a wide variety of problems of this kind, with differing teaching focuses, and alternative algebraic and arithmetic methods for solving them.

It is believed that the solution of these problems is elementary and has already been dealt with. However, when studied in depth, it can be seen that in the same way that there is not only one way of solving a linear problem, neither is there only one way of teaching it. The idea of linearity can be treated from different perspectives: through proportionality, through first-degree equations, and through linear function[i]. These three foci correspond to epistemological conceptions linked to a variety of historical and pedagogical alternatives and approximations that reveal the wealth of mathematical thought lying behind such an apparently simple topic.

The historical change from one conception to another caused changes with repercussions in teaching them. One of these occurred on substituting arithmetic methods for algebraic ones to teach and solve linear problems. Another had to do

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with the way of organizing these in textbooks. These changes in focus, in textbooks in the first half of the 19th century, are epitomized in the treatises of Vallejo and Peacock. The former is a key piece of work, due to its influence, for studying the incidence of these changes in mathematical teaching in Spain, and represents a traditional textbook. The latter represents a forefront textbook, an Arithmetic Algebra, this being a bridge between arithmetic and symbolic algebra.

The main goal of this research was to analyze the pedagogical approaches for teaching and solving linear problems, as reflected in Vallejo's Treatise (1821, third edition). For this analysis two main features have been considered: a) the conceptions reflected in Vallejo's text with respect to mathematical concepts linked to algebra, and the way of classifying/organizing/solving the problems of first degree equations; b) its relation to the historical and cultural background in which it appeared, and the proposal for teaching the first degree equations problems that appear in Peacock's *Arithmetic Algebra (1845).*

THE TREATISES OF VALLEJO AND PEACOCK IN THEIR HISTORICAL CONTEXT

At the moment in history when the Treatises of Vallejo and Peacock appeared, an algebra concentrating on finding solutions to equations was giving way to an algebra aimed at studying the conditions necessary for these solutions to exist. Arithmetic began to take on the kinds of approach typical modern or structural of algebra, such as the horizontal language of equalities and parentheses. At the same time the school system began to be reorganized itself into a general, centralized, state system of education.

The *Elementary Treatise of Mathematics* by José Mariano Vallejo (1779 - 1846) was first published in 1813. Written in 5 volumes, it underwent four re-editions, the last being published in 1841 after Vallejo's travels round Europe, for political reasons, which served to give him knowledge with which to broaden and perfect his work. In this period, interest in the study of mathematics in Spain was found not so much in civil society as in military institutions. Although the mathematicians who gave classes in them did not bring anything new to mathematics in itself, one could say that they did add to the teaching of it in that they updated mathematics textbooks, thus renewing the Spanish mathematics bibliography. These authors produced encyclopedic works in which they presented mathematical advances of the 18th century which had not yet been published or widespread. They did this with two aims in mind: to give a general view of mathematics, and to be used as textbooks for official institutions. In these works they synthesized and adapted previous prestigious authors, taking care to produce a strict, intrinsically coherent presentation of ideas that, together with a good job done in reproducing them, made them invaluable and popular.

Vallejo's text on algebra appeared at a transitional moment in the history of 19th century mathematics when steps were being taken towards a general theory of algebra, or an abstract algebra, whose interests lay more in algebraic structures (groups, rings...) than in the theory of equations. These steps were directed at finding and stating the conditions under which solutions to a specific equation or system of equations exist, rather than determining what those solutions are. It seems that this change of direction was influenced by the demonstration that in general no algebraic method exists to demonstrate a fifth degree equation, though Vallejo made a great effort to find one, eventually even believing that he had managed to do so.

Arithmetical Algebra, as Peacock called it, forms part of *A Treatise on Algebra* in its two versions of both 1830 and 1842. This work announces the existence of two different bodies of algebra, "Arithmetical Algebra" and "Symbolical Algebra". In the second version of his *Treatise* Peacock dedicated a volume to each one of these two domains. While Peacock may be considered the driving force behind the birth of current-day Modern Algebra, the most interesting side of him with respect to this study is that of a 19th century author concerned with teaching algebra to students, and in this sense one can take his *Arithmetical Algebra* (Gallardo y Torres, 2005). On reading this (chapter V, "On solving equations") one finds elements of analysis with respect to the way of organizing the teaching of that part of algebra dedicated to solving problems modeled on first-degree equations.

THE CONCEPTIONS REFLECTED IN VALLAJO TEXTBOOKS WITH RESPECT TO MATHEMATICAL CONCEPTS LINKED TO ALGEBRA

Vallejo's textbooks organized the algebra into two parts. In the first, one learns the syntax of algebra; and, in the second, one learns how to use this calculating system to solve problems, by means of equations.

In the first part, we find an explanation of what means a negative solution in an algebraic problem, where the epistemological conception of that period in time are reflected. Vallejo states that although negative it is a correct solution algebraically speaking, it represents a mistake in the original problem statement, because the difference between two numbers must always be less than the greater of the two. "So that if we wish the result to be in positive numbers we must vary the way of working out the problem" (op. cit., p. 251). In stating that the difference between two numbers must be less than the greater of the two, he is not saying anything unusual in the context of the concept of numbers in his time. At that time, a pair formed by a number with its sign was not considered anything more than an adjectival quantity, and when this quantity's adjective meant "opposite" it was called a negative quantity which, considered on its own, meant nothing more than the answer to an opposite question or the objective contrary to that for which the calculation was performed. Specifically, he speaks of negative quantities and not negative numbers, the former being algebraic quantities which arise from algebra and which have not been needed in arithmetic because, there, in solving problems "everything is substituted by words" (op. cit., p. 173). On the other hand, on stating that the solution -6 represents a mistake in the definition of the problem, he is echoing the idea that D'Alembert expressed in the article "Negatif", which appeared in Diderot's *Enciclopedia*.

In the second part of the algebra, Vallejo tries to teach how to solve problems by means of what is called "analysis". Throughout the text Vallejo unveils what "analysis" is for him, following steps in this order:

> Assume that we know that which we are trying to discover in order to find it afterwards. Express the quantities in letters, such that the known values are distinguished from the unknowns. To express through equations the conditions that must be met by the aforementioned quantities. This is called "stating the problem". Determine what known quantities are equal to the unknowns, by means of operations that are performed to leave just one member remaining. This is what is known as "clearing away the unknowns". Translate the formula obtained by clearing away unknowns into ordinary language, so as to obtain the practical rule that can be applied to all similar problems.

These steps expressed the logic that underlies the Cartesian Method. The main features, as Puig y Rojano (2004) have pointed out, are the actions of analysing the statement of the problem and translating it into equations which express, in algebraic language, the relations among quantities.

In Vallejo's opinion, there are no general rules for stating problems, since this depends on the talent of the calculator (op. cit, p. 230), such that his only advice is to ensure a diligent translation from ordinary language to algebraic language. There are two ways to formulate equations from verbal data: either by direct translation of the key words to symbols or by trying to express the meaning of the problem. These ways are referred to in current literature as *syntactic or semantic translation* respectively (MacGregor & Stacey, 1993). Vallejo showed his preference for the former, agreeing with Newton[ii], and the more frequently used method for formulating equations. Here, he coincides with Peacock but this author goes further in his suggestions:

> In some case the conditions may be symbolized in the order in which they present themselves in the problem, by an immediate translation of ordinary into symbolical language; in others they will be involved in such a manner that the discovery of their relation and succession and their consequent symbolical expression will present difficulties, which can only be overcome by close attention and a clear insight into the relations of the numbers and magnitudes which they involve; for such cases general rules are nearly useless, and the student must trust to the diligent and patient study and analysis of examples alone for the acquisition of those habits of mind which will guide his course in their symbolical enunciation (op. cit., p. 250).

In this second part of the algebra, also we find a notion of equation and an algebraic concept of variable.

Equation was related to the comparison of quantities, which in turn was related to relations of equality and inequality. Comparison by equality led to the idea of the equation, and comparison by inequality to the idea of reason. Through this concept, the equation was seen as a way of thinking that need not be restricted to algebra, since an arithmetic equality such as “1+1=2” was an equation as was the algebraic equality “ax+b=c” (Brooks, 1880, p. 194). And need not be a conditioned equality, which is only verified by certain values – the roots of the equation -, since it includes when the two expressions, connected by the sign “=” are identical or are reducible to identity, such x=x, x+5=x+5, … , 3x+4x=7x. Vallejo adopted this broad concept of the equation although he was aware of (and also explicitly noted in a footnote) the more restricted position of other contemporary authors who differentiated between equation and equality (the expression of two quantities separated by an equals sign). The restricted definition of equation to a conditioned equality, is the one that finally prevailed in school textbooks but, as happens all too often in such books, no justification was given as to why one position should be justified and not another.

Although in general the notion of an unknown refers to an unknown number, it can also refer to a variable. This happens when, on defining the problem, indeterminate equations are obtained – those that have fewer equations than unknowns. This is because when we have, for example, “an equation such as: ax + bz = c, there is no other way of finding any of the unknowns x, z than giving values to the other; and since for each value given to z, for example, a different one will appear for x, it is deduced that in an equation of this kind the quantities indicated by the last letters of the alphabet are called variables, because within the same problem there may be as many values as one likes; and those that are indicated by the first letters are called constants because they can have only one value” (op. cit., p. 375).

CLASSIFYING/ORGANIZING THE PROBLEMS TO BE TAUGHT

It is reasonable to suppose that Vallejo aimed to encourage the acquisition of habits that guide the students in stating the problems that lead to equations, and help in familiarizing them with their rules to clear away unknowns. To illustrate these principles and tips, and to familiarize the student with their use, Vallejo sets out and solves a collection of problems. However, in the text there are no comments to help deduce the criteria he followed in their choice and sequence, other than putting them under three different headings, interwoven among others, whose titles are:

> On algebraic analysis and the resolution of first-degree equations. On the rule of three and others that depend on it, such as exchange, fellowship, alligation, false position, etc. On indeterminate first-degree equations.

Note that Vallejo showed his preference for the ancient practices to classify problems according to certain characteristics related to the context, or method. Whereas Peacock classifies the problems choosing a particularly pertinent point of view, according to whether they involve one or more unknowns and the relations between them:

> We shall now proceed to the consideration of the general rules for the symbolical enunciation of problems, in which the unknown number or numbers are more or less involved in the conditions which are required for their determination, and it will tend to

> facilitate this enquiry if we classify, very generally, the problems which present themselves for solution, with reference to the unknown number or numbers which they severally involve (op. cit., p. 249).

Along these lines, he says:

> There are three great classes of such problems to be considered
> First Class. Problems which involve one unknown number only, which is throughout the subject of the conditions proposed.
> Second Class. Problems which involve two or more unknown numbers, which are so related to each other by the conditions of the problem as to be expressed or immediately expressible in terms of one of them only.
> Third Class. Problems which involve two or more unknown numbers which are immediately expressible in terms of one of them, but require to be denoted by distinct symbols" (op. cit., p. 249).

This classification, of problems, implied a change of perspective with other systems for classifying problems, according to their method of solution[iii], given that in many cases there is no single method of solving the same linear problem; or according to certain characteristics of the statement of the problem related to the context[iv], given that problems of a different appearance can have the same structure and methods of solution.

It is clear that some of these were soon displaced by the method of equations, but the way of organizing problems would resist Peacock's proposed change, despite its advantages in methodological clarity and generality, the preference being for the traditional teaching of collections of methods and "curious" problems with no apparent unifying nexus. In any case, the generalization of algebraic method was to bring with it the loss of arithmetic methods based on the study of numerical relations, methods eclipsed by the dominant idea that it was enough to study only one method and this should obviously be the best one. And this was the algebraic one, since it was the most general.

SPECIFICS PROBLEMS OF FIRST DEGREE EQUATIONS

Under the heading dedicated to "solving first-degree equations" Vallejo sets out 18 problems. The order in which they are presented does not follow a flow in keeping with Peacock's three classes; they appear to be shuffled up. It seems clear that Vallejo uses other criteria to organize them. Nor is there a criterion for distributing the problems according to an increasing level of difficulty with respect to the data (whole numbers, fractions), the complexity of the problem's conditions, or the operations or extensions to the rules for clearing away unknowns. Rather, it seems that he wants to attend to some of the elements that constitute the analytical spirit.

In the first one, a Diophantus problem: "To divide a proposed number into two parts whose interval or difference is given", Vallejo introduce and shows how algebra is a language which enables us to move towards generality by means of literal representations and the quantities involved, and how it helps create statements of general rules and formulae for solving families of problems. In the second problem, an abaci problem[v] expressed verbally, with almost nothing else than numbers and abstract relationships of quantities, he carefully explains the syntactic translation process to algebraic symbols. The third problem is an example in which we need to designate all the conditions of a given problem by an equal number of equations. The four following problems show the special interest of Vallejo in enigmatic problem statements[vi], where the difficulty seems to lie more in the language distracters present in the text. The last nine problems are supplementary or for revision. Most of them are problem statements that can be found in texts by contemporary or previous authors, all involving one unknown number only.

Peacock distinguishes two cases of these classes of problems. In one, the conditions can be symbolized in the same order in which they appear in the problem by means of immediate translation from ordinary to symbolic language. In the other, the conditions are so mixed up that it is difficult to discover their relationships and how they succeed one another, and consequently it is also difficult to define them symbolically. Vallejo presents problems of both kinds, without paying much attention to this fact. On the other hand, he stresses the importance of a diligent syntactic translation, and thus shows in the first problems how he substitutes expressions from symbolic language to algebraic language. Among these problems we can highlight one of "God Greet You Problem"[vii], another of "people who buy one thing together"[viii] - both linked traditionally to the rules of false position; and two more with travelers which correspond to the cases: - "they go to meet another", and "they go after another to catch up".

SPECIFICS METHODS TO SOLVE PROBLEMS OF FIRST DEGREE EQUATIONS REFLECTED IN VALLEJO'S TREATISE

The algebraic method enables one to define and resolve rule of three problems in a different way from the traditional one based on the old theory of reason and proportion. To do this, one only has to set about solving the problems of proportionality considering the reasons as fractions and the proportion as an equation. This is what Vallejo does, without abandoning the old theory, to which he dedicates two sections within the algebra, before the rule of three. These sections are: "On reasons and proportions", and "On the transformations that can appear in a proportion while maintaining the existence of the proportion, which is how the analysis was performed in ancient times". Here, as well as the standard problems, Vallejo includes a problem of "cisterns" and one of "clocks". He solves the former through algebraic method and the latter with proportions.

The "faucets filling a cistern" can be seen as an example of "Co-operative work problems" that were very popular in texts of the ancient and medieval world (Kangshen, et al, 1999, p. 338).

> There were two types of such problems: (i) The days required to complete a task by A or B (or more than two) is known and one is asked to find the days required to complete the task by A and B (or more than two) together. (ii)) The number of task of type A or of another task of type b (or more than two task) that one individual can complete in *a* day is known, and it is asked to find the number of sets of task (containing one each of types A and B, etc.) that one individual can complete in a day (Kangshen, et al, 1999, p. 337).

The solution of the first type of problems is obtained through comparison with unity in two ways by using unitary fractions or the minimum common multiple. In the former case, the method taken up by Vallejo, one must consider two parts, and in both a reduction is made to unity: first, the amount of the tank filled in one hour or day is calculated; then knowing this one can calculate how much time it takes to fill a complete tank. In both parts a rule of three is used[ix]. The other case also has two parts. First the minimum common multiple is calculated to find the effect of the taps together. Knowing this, a reduction is made to unity as before[x].

In the "clock" problem[xi], Vallejo does not strictly use the algebraic method, but the one of proportions. This deals with finding at what time the minute hand is above the hour hand, starting from a given time of day. To solve this, he establishes the relationship between the distance covered by the hour hand, x, and that covered by the minute hand, a+x. Then he establishes the proportion at the moment they coincide: 12 : 1 :: a+x : x., reducing through the properties of the proportions and clearing away algebraically, x=a/11. The problem is similar to that of two travelers that reach each other, which has been solved in a previous section, but Vallejo makes no mention of this similarity. Moreover, the method for solving the problem is not strictly algebraic.

Of the three problems that Vallejo solves with the two rules of false position, the first one is of two deficits, while the second and third have one deficit and one excess, and so they are similar. One of these problems is a repetition of another that he already solved by algebraic methods in a previous section[xii].

Thus, one may conclude that once again his criteria for choosing the problems is nearer that of the ancient practices, as have been commented before. Also one might think that he separates the problems of syntactic translation from the problems that are not of this type, when the circumstances of the stated question (or problem) appear unsuitable to be translated directly.

SYNTHESIS AND CONCLUDING REMARKS

Generally speaking, we have seen a conception of the teaching of elementary algebra, reflected in a Spanish textbook that was used as a reference in the first half of the 19th century. The way in which this conception is manifested in questions solved with first-degree equations is contrasted with Peacock's proposal.

Both positions show an interest in pointing out that the general method, the algebraic one, is the best method. But, one way of teaching this topic is to present a collection of apparently arbitrary, but traditional problems, that have more or less interest from the point of view of developing analytical skills, and to organize them in agreement with this. (e.g. distinguishing those that have syntactic translation into algebraic terms from those that do not); without abandoning the ancient practices (Specifics problems and methods). As opposed to this, there is an organization of the questions following a precise, well-defined classification according to the number of unknowns and the relations between them.

In addition, we can look into technical and pedagogical guidelines, sometimes underlying in Newton's tradition, for the teaching process of analysing and solving linear problems. Vallejo, especially offers us points of discussion and ideas quite enlightening related to: The negative solutions when problems are solved by means of equations, the notion of equation and its relation with identity, the notion of variables in algebra and its relation with the unknown quantities, examples of the process of syntactic translation to write equations, the idea of algebra and its relations with equations, letters and unknowns (involves only activity with).

The challenge now is to deduce how these positions and guidelines have been incorporated into subsequent texts, in order to try to understand the roots of the present curriculum. In future, it will be necessary to evaluate how general methods fit in with specific personal ones, and how the powerful tool of algebra sits with the legacy of arithmetic, in relation to new approaches and aims in the teaching of current mathematics.

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[i] From a historical point of view, functions are not formally a part of algebra. So we shall not tackle here.

[ii] “for the solution of questions whose preoccupation is merely with numbers, or the abstract relationships of quantities, almost nothing else is required than that a translation be made from the particular verbal language in which the problem is propounded into one (If I may cal it so) which is algebraic; that is, into characters which are fit to symbolize our concepts regarding the relationships of quantities” (Newton, “Newton’s Lectures on Algebra during 1673-1683”. Whiteside “The Mathematical Papers of Newton”, Ed. D. T. Whiteside. Cambridge. Cambridge University Press. 1972. Vol. V. p. 133).

[iii] reduction to unity, simple and compound rule of three, rules of false position, rules of per cent, of interest, discount, alligation, proportional sharing and fellowship, bartering and exchange

[iv] travelers that split up to meet each other later, taps or cisterns that fill up or empty, clocks that go fast or clock hands that coincide, co-operative jobs, …

[v] “There is asked a such number, which if to the quintuple of the number are added seven times the twelfth part of the same number, and take from of all 17 units, turns out to be 17 added to 203 units” (Vallejo, op. cit., p. 242).

[vi] “On being asked how old Alexander the Great was, Artemidoro the philosopher gave the following reply, according to the bishop Caramuel:

On asking Diodoro
Ambassador of the Prince of Egypt
The Age of the undefeated Macedonian,
Artemidoro
Answered him ingeniously
Two years more has the bellicose
King than his comrade
Efestion, whose father
Four years more than the two he counted,
And the father of Alexander
When ninety six journeys of Apollo
Were all the years these three counted”. (p.244).

[vii] God greet you with you 100 scholars! We are not 100 scholars but our number and the number again and its half and its fourth are 100. How many are we? (Kangshen et al., 1999, p. 161).

[viii] x people buy an item costing y coins. If each one pays a_1, there is an excess of c_1; if each one pays a_2 there is a deficit of c_2.

[ix] One task completed by A and B in p, q days (respectively), is transformed to 1/q, 1/p task respectively in one day. Knowing this one can calculate how much time it takes to complete the whole task with a rule of three (1 day is to 1/q task like x days is to x/q). Therefore we can find the solution solving the equation $x/q + x/p = 1$.

[x] A completes one task in p days, while B takes q days. Suppose they work together for pq days, then A will complete q task, and B, p task. Therefore they will complete p+q task in pq days, i.e. together they will complete one task in pq/(p+q) days.

[xi] Knowing that the handle of a clock is between a given hour, to find what hour will be when the handle of the minutes is on the hours.

[xii] A father in order to stimulate his son whom he studies, says to him: for every day that you know the lesson I give you 10 coins, but for every day that you do not know you have to give me 4; after 15 days the father had to give him 66 coins; he wonders, how many days he studied and how many not?