Kulikov models via the Minimal Model Program.

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Abstract

We give a modern proof of the Kulikov-Pinkham-Persson Theorem for a projective degeneration. We do so by running the Minimal Program and studying the singularities of the remaining log Calabi-Yau pair. The explicit description of these allow us to resolve them using Brieskorn's simultaneous resolution of Du Val singularities and toric resolutions.

Abstract

Wir geben einen modernen Beweis des Kulikov-Pinkham-Persson Satzes einer projektive Degeneration. Mithilfe der «Minimal Model Program» und der Beschreibung der Singularitätetn des verbleibenden log Calabi-Yau-Paares. Bekommt man eine explizite Beschreibung, die lass uns die Singularitäten mit Brieskorns simultaner Auflösung der Du Val Singularitäten und torischen Auflösungen aufzulösen.

Resum

Presentem una prova moderna del Teorema de Kulikov-Pinkham-Persson per a una degeneració projectiva. Mitjançant l'ús del «Minimal Model Program» i la descripció de les seues singularitats del parell log Calabi-Yau resultat. Seguidament, aprofitem la descripció explícita obtinguda per resoldre-les aplicant la resolució simultània de singularitats Du Val de Brieskorn i les resolucions tòriques.

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Contents

0 Introduction

Within the study of degenerations of varieties, a particular breakthrough was obtained in the late 70's by Kulikov. Given a semistable degeneration of K-trivial varieties the objective was to find a nicer birational model that allows for its study. This led to the following theorem.

Theorem 0.1 (Kulikov-Persson-Pinkham)**.** *Let $f' : X' \to B$ be a semistable degeneration of K-trivial smooth surfaces, with all components of the central fiber $X'_0 = f'^{-1}(0)$ algebraic. Then there is a semistable birational model $f : X \to B$ such that $K_X \sim_f \mathcal{O}_X$.*

The property of K-triviality turned out to be very useful thus motivating this definition.

Definition 0.2 (Kulikov model)**.** Let $f : X \to B$ be a semistable degeneration of varieties of dimension n with $mK_{X_t} \sim \mathcal{O}_{X_t}$ over a smooth curve B. We say that $f : X \to B$ is a *Kulikov model* if $mK_X \sim_f \mathcal{O}_X$.

One of the best features of Kulikov models is that they allow for a very explicit description of the degenerate fiber and also control over the monodromy action acting on the $H^2(X_t, \mathbb{Z})$. In particular for K3 surfaces (Def. 4.8) Kulikov provided the following.

Theorem 0.3. *[Kul77, Theorem II] Let $f : X \to B$ be a Kulikov model of $K3$ surfaces. Then the degenerate fiber X_0 must be one of the following 3 types:*

I. $X_0 = V_1$ is a non-singular K3 surface.

II. $X_0 = V_1 + ... + V_n$, where V_1 and V_n are rational surfaces and $V_2, ..., V_{n-1}$ are elliptic ruled surfaces so that $q(V_i) = 1$, $i = 2, ..., n-1$. The dual complex is a chanin and the double curves $C_{1,2}, ..., C_{n-1,n}$ are elliptic curves.

III. $X_0 = V_1 + ... + V_n$, where all V_i are rational surfaces,; the double curves C_i, j are rational and form a cycle on each of the surfaces V_i. The dual complex is a triangulation of the sphere.

These 3 types can be distinguished by means of the monodromy T acting on $H^2(X_t, \mathbb{Z})$. If $N = \log T$, then $N = 0$; $N \neq 0$ but $N^2 = 0$ and $N^2 \neq 0$ but $N^3 = 0$, corresponding respectively to Type I, II and III.

This result was deeply influential not only in the study of degenerations, in which Kulikov models have been studied and applied extensively (see for example [FM83]). But also prominently in birational geometry, as Kulikov described flops for the first time, one simpler version of flips. These are a crucial ingredient that allowed for the grounding and development of the Minimal Model Program.

The original proof of Theorem 0.1 was written by Kulikov in [Kul77], only for the case of K3 surfaces.Nevertheless it was quite obscure and difficult to follow, to the point that even at time of publishing some were left wondering whether it was valid (see the review of his article in Mathematical Reviews 58, # 2208). Moreover the paper featured big mistakes such as a false proof of Theorem 0.1 for Enriques surfaces.

The proof was made more clear and generalized in [PP81]. This cleared doubts about its validity. Nevertheless, this latter proof is still quite technical and challenging, as well as, written in a now slightly outdated language.

In more recent years, running the MMP in [KLSV18] Kollár, Saccà, Laza and Voisin were able to give a weak generalization of this theorem for degenerations of hyper-Kählers of arbitrary dimension.

Theorem 0.4. *[KLSV18, Theorem 1.1] Let $f : X \to C$ be a projective morphism to a smooth projective curve C. Assume that*

i. the generic fiber X_{gen} is irreducible and birational to a K-trivial variety with canonical singularities and

ii. every fiber X_c has at least one irreducible component X_c^* that is not uniruled.

Then there is a finite, possibly ramified, cover $\pi : B \to C$ and a projective morphism $f' : Y \to B$ with the following properties:

1. *Y is birational to $B \times_C X$,*
2. *the generic fiber Y_{gen} is a K-trivial variety with terminal singularities, and*
3. *every fiber Y_b is a K-trivial variety with canonical singularities with (Y, Y_b) being a dlt pair.*

Remark 0.5. The main interest of [KLSV18] is how Theorem 0.4 allows for a description of the degeneration of hyper-Kähler varieties in terms of the dual complex of the degenerate fiber and the finiteness of the monodromy action on the central fiber for degeneration of hyper-Kähler varieties, in a similar fashion as in Theorem 0.3. But this is not relevant to this project, since the Hodge-theoretic part of Theorem 0.3 is quite straight-forward.

Note that there are two main differences between Theorem 0.4 and Theorem 0.1 that make it the first weaker in comparison are that

- every fiber needs to contain at least one irreducible component that is not uniruled. This condition is not satisfied by Kulikov models of Type II and III for K3 surfaces (cf. Theorem 0.3).
- The new fibers are not necessarily snc but simply dlt.

These hypothesis are needed to make the result work in a higher dimensional case, but what happens if we took this methods back to dimension 2? Can we obtain Theorem 0.1 by applying the methods in [KLSV18]? This is the question this Master Thesis aims to give an answer to.

The answer happens to be almost yes. Upon further examination of the resulting minimal model one discovers that singularities are quite mild and one can resolve them leading to a slightly more general version of Theorem 0.1.

Theorem 0.6. *Let $f : X \to C$ a projective flat morphism from a 3-dimensional complex space to the complex disk. Assume that*

1. *X terminal, $\mathbb{Q}$-factorial, (X, X_t) dlt for all $t \in C$.*

2. *X_t is a smooth surface with $K_{X_t} \sim \mathcal{O}_{X_t}$ for $t \neq 0$.*

Then there exists a finite surjective base change $C' \to C$ and a birational map:

$$\begin{array}{ccc} \tilde{X} & \dashrightarrow & X \times_C C' \\ & \searrow \quad \swarrow & \\ & C' & \end{array}$$

Such that $\tilde{X} \to C'$ is a Kulikov model.

The motivation of proving Theorem 0.1 with new methods is double. On the one hand, writing it in more modern terms allows that anyone with familiar with the MMP should be able to follow it much easier than the original proofs and old techniques. On the other hand, the generality of the MMP tools allow the result to be extended to more general settings, for example it extends to general algebraically closed fields of characteristic 0. Moreover, we have reason to believe that similar arguments lead to a generalizations to arbitrary fields of characteristic 0, but this belongs in future work.

The idea we use to prove Theorem 0.6 can be summed up in these steps.

1. Run the Minimal Model program to obtain a dlt minimal model. Similar procedures were already performed in [Fuj11] and [KLSV18]

2. Study the singularities of such models to conclude singularities are very mild, they only occur away from the double loci of the central fiber. This result was proven also in [NXY19], but we give a different proof based on properties of dimension 3.

3. Use simultaneous resolutions of Du Val singularities and toric resolutions to resolve the mild singularities obtaining a Kulikov model.

Unfortunately, there is not much more room for using these arguments to generalize to higher dimensions as it is highly dependent on the explicit description of 3-dimensional varieties.

Moreover we give our own proof of Theorem 0.3 reformulating it in terms of log Calabi-Yau pairs, with no Hodge theory and hopefully in a more comprehensive manner. We also obtain a new counterexample that the MMP cannot be established for compact complex manifolds (Remark 3.14)

Finally we give an outline of what the reader may find in each section.Section 1 provides a brief overview of the Minimal Model Program and explains in more detail the singularities that come up in later sections; Section 2 contains a very brief introduction to the toric geometry needed for Step 3; Section 3 consists of the proof of the Kulikov-Persson-Pinkham theorem and in Section 4 we reprove the classification of the central fiber from Theorem 0.3.

1 Young person's guide to the Minimal Model Program.

> we have heard the opinion expressed that at this particular moment there is no special need to draw up a programme; [...] that it would be better to postpone the elaboration of a programme until such time as when the movement stands on firmer ground; that a programme might, at the moment, turn out to be unfounded.
>
> *A draft of our Party programme*, V. I. Lenin (or maybe Kollár about the MMP)

This section serves as a very brief introduction to the needed knowledge of Minimal Model Program and its singularities for this project. Firstly, we discuss basic definitions of singularities of the MMP and describe more explicitly the singularities and properties which are needed in later sections. After that to give the reader an idea we give a sketch of how the MMP is supposed to run to solve the Minimal Model Conjecture (see Conjecture 1.23).

For this section we use as main references [KM98] and [Kol13], we highly recommend the first one for the unitiated reader.

Remark 1.1. Throughout this section we use the language of schemes but everything also applies to complex analytic spaces.

Basic definitions and notation. Let X be a normal scheme.

- A (Weil) *divisor* is a finite formal $\mathbb{Z}$-linear combination $D = \sum_i m_i D_i$ of integral distinct subschemes of codimension 1. Similarly, define a $\mathbb{Q}$-*divisor* by allowing a $\mathbb{Q}$-linear combination.
- A Weil divisor D is $\mathbb{Q}$-*Cartier* iff mD is Cartier for some integer $m \neq 0$, the minimal $m > 0$ with such property will be called the *index* of D.
- Linear equivalence is denoted by $D_1 \sim D_2$. Two $\mathbb{Q}$-divisors are $\mathbb{Q}$-*linearly equivalent*, $D_1 \sim_{\mathbb{Q}} D_2$, if mD_1 and mD_2 are linearly equivalent $\mathbb{Z}$-divisors for some integer $m \neq 0$.
- Let $f : X \to Y$ be a morphism. Two $\mathbb{Q}$-divisors are *relatively* $\mathbb{Q}$-*linearly equivalent* if there is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor B on Y such that $D_1 \sim_{\mathbb{Q}} D_2 + f^*B$.
- A scheme/complex analytic space is $\mathbb{Q}$-*factorial* if every Weil divisor is $\mathbb{Q}$-*Cartier*.
- We say that a $\mathbb{Q}$-divisor is a *boundary* if $0 \leq \operatorname{coeff}_{D_i} D \leq 1$.
- Let $f : X \to S$ be a proper morphism and C a closed 1-dimensional subscheme of a closed fiber of f. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X and $m > 0$ the index of D, define the *intersection number* of D on C as

$$(D.C) := \frac{1}{m} \deg_C(\mathcal{O}_X(mD)|_C).$$

We say that D is f*-nef* if $(D.C) \geq 0$ for every such curve C.

- A *variety* is an integral separated k-scheme of finite type over k an algebraically closed field which we always assume that $\operatorname{char} k = 0$.

Definition 1.2 (Pairs)**.** We consider pairs (X, Δ) over a smooth variety S satisfying the following conditions.

1. X is normal proper variety normal that has a canonical sheaf $\omega_{X/S}$, i.e. relatively Cohen-Macaulay[1].

2. $\Delta = \sum a_i D_i$ is a $\mathbb{Q}$-linear combination of distinct prime divisors, none of which is contained in $\mathrm{Sing}(X)$.

Definition 1.3 (Simple normal crossings)**.** Let X be a scheme and $p \in X$ a point with ideal sheaf $\mathfrak{m}_p$ and residue field $k(p)$. Then $x_1, ..., x_n \in \mathfrak{m}_p$ if their residue classes $\overline{x}_1, ..., \overline{x}_n$ form a $k(p)$-basis of $\mathfrak{m}_p/\mathfrak{m}_p^2$.

Let $\Delta = \sum a_i D_i$ a Weil divisor we say that (X, Δ) has *simple normal crossing* or is *snc* if at $p \in X$ if X is regular at p and there is a neighborhood $p \in X_p \subset X$ such that $X_p \cap \operatorname{Supp} \Delta = V(x_{i_1} \cdots x_{i_k})$, where $\{x_{i_1}, ..., x_{i_k}\}$ is a possibly empty system of parameters.

We say (X, Δ) is snc if it is snc at every point. Moreover, given (X, Δ) the largest open set $U \subset X$ such that $(U, D|_U)$ is snc is called the *snc locus* and it is denoted by $\mathrm{snc}(X, \Delta)$, its complement $\text{non-snc}(X, \Delta)$.

Let $f : X \to S$ be a morphism, we will say it is *semistable* if for all closed points $t \in S$, $(X, f^{-1}(t))$ is snc.

Definition 1.4 (Log resolution)**.** Let X be a reduced scheme and Δ a Weil divisor on X. A *log resolution* of (X, Δ) is a proper birational morphism $f : X' \to X$ such that X' is regular, $\mathrm{Ex}(f)$ has pure codimension 1 and $(X', \Delta' := \mathrm{Supp}(f^{-1}(\Delta) + Ex(f)))$ is snc.

Theorem 1.5. *[Hir64] Let X be an algebraic variety and D a Weil divisor on X. Then (X, D) has a log resolution.*

1.1 Singularities of the MMP.

Remark 1.6. In the following discussions the role of the base scheme is not very important for this reason we supress it. Consequently, in the notation $\omega_{Y/S}$ and $K_{Y/S}$ we supress the reference to the base but it is implicit.

Let (Y, Δ) be a pair as defined in 1.2 such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier and let $g : Y' \to Y$ be a birational morphism from a normal variety Y'.

[1] Do not worry too much about this condition, it is very technical but it is just ensuring that we can work with the canonical divisor

Let m be the index of $K_Y + \Delta$ and let E_i be the irreducible exceptional divisors. Then one can find rational numbers $a(Y, \Delta, E_i)$ and a natural isomorphism of sheaves:

$$\omega_{Y'}^{[m]}(mg_*^{-1}\Delta) \simeq g^*(\omega_Y^{[m]}(m\Delta)) \left(\sum_i m \cdot a(Y, \Delta, E_i) E_i \right), [2]$$

where $g_*^{-1}\Delta = \sum_j d_j g_*^{-1} D_j$, having $\Delta = \sum_j d_j D_j$. This can be written in terms of divisors as

$$m(K_{Y'} + g_*^{-1}\Delta) \sim g^*(m(K_Y + \Delta)) + \sum_{E_i \text{ exceptional}} m \cdot a(Y, \Delta, E_i) E_i$$

By formally dividing by m one may rewrite the previous expression in terms of $\mathbb{Q}$-equivalence as

$$K_{Y'} + g_*^{-1}\Delta \sim_{\mathbb{Q}} g^*(K_Y + \Delta) + \sum_{E_i \text{ exceptional}} a(Y, \Delta, E_i) E_i. \tag{1}$$

We call the $a(Y, E_i, \Delta)$ the *discrepancy* of E_i. For an exceptional divisor E one writes $\text{center}_Y(E) := \overline{g(E)}$ for the scheme-theoretical closure of the image. Moreover, if we set $a(Y, \Delta, D) = -\operatorname{coeff}_D \Delta$ for non-exceptional divisors $D \subset Y$. Then one can also rewrite the previous formula for arbitrary divisors as

$$K_{Y'} + g_*^{-1}\Delta \sim_{\mathbb{Q}} g^*(K_Y + \Delta) + \sum_{E_j \text{ arbitrary}} a(Y, \Delta, E_j) E_j.$$

Based on the discrepancies one can define the different types of singularities of the MMP. Let (Y, Δ) be a pair as before, for any birational morphism $f : X \to Y$ and any irreducible divisor E on X

$$(Y,\Delta) \text{ is } \begin{cases} \text{terminal} \\ \text{canonical} \\ \text{(purely) log terminal} \\ \text{dlt} \\ \text{log canonical} \end{cases} \text{if } a(Y,\Delta,E) \begin{cases} > 0 \text{ for all } E \text{ exceptional}, \\ \geq 0 \text{ for all } E \text{ exceptional}, \\ > -1 \text{ for all } E \text{ exceptional}, \\ > -1 \text{ if } \text{center}_Y E \subset \mathit{nonsnc}(Y,\Delta), \\ \geq -1 \text{ for all } E. \end{cases}$$

Here dlt stands for *divisorial log terminal.* If $\Delta = 0$, then we will say that Y is terminal, resp. canonical, etc. if $(Y, 0)$ is terminal, resp. canonical, etc.

Remark 1.7. This condition must be checked a priori for all $E \subset X$ possible irreducible exceptional divisors for all possible birational proper morphisms $f : X \to Y$ and for all irreducible divisors on X. Nevertheless, by [KM98, Remark 2.23] the discrepancy $a(Y, \Delta, E)$ does not depend on X or f, but on $v(E, X)$, where $v(-,-)$ denotes the valuation in $k(Y)$ corresponding to the DVR $\mathcal{O}_{E,X}$. If $v(E, Y) = v(E', X')$ for some other $f' : X' \to Y$, then the induced birational map $X \to Y \dashrightarrow X'$ is an isomorphism over the generic point of E and E', hence $a(Y, \Delta, E) = a(Y, \Delta, E')$.

[2] $D^{[m]}$ for a $\mathbb{Q}$-Cartier divisor is defined as $(D^{\otimes m})^{*}*$, where $(-)^*$ denotes the dual. This will not be relevant in the project, hence we do not give it more attention, see [Kol13] for subtleties.

As a consequence in most cases of interest it suffices to check a single log resolution. Now we give some examples for which it is easy to compute.

Example 1.8. There are some cases in which it is enough to check one single resolution (cf. [KM98, Cor. 2.32])

- Let $f : X \to Y$ be any resolution with $\{E_i\}_i$ its irreducible exceptional divisors. Assume $1 \geq \min_i\{a(Y, E_i)\} \geq 0$. Then

  $$\min_i\{a(Y, E_i)\} = \inf_j\{a(Y, E_j) : E_j \text{ exceptional divisor for any proper bir. } g : Z \to Y\}$$

- Let $\Delta = \sum_j a_j D_j$ be a boundary on Y. There exists a log resolution $f : X \to Y$, such that $\sum_j f_*^{-1} D_j$ is smooth. Let f be any such resolution and $\{E_i\}_i$ its irreducible exceptional divisors. Assume $a(Y, \Delta, E_i) \geq -1$, then

  $$\inf_k\{a(Y, E_k, \Delta) : E_k \text{ exceptional divisor }\} = \min\{\min_i\{a(Y, \Delta, E_i)\}, \min_j\{1 - a_j\}, 1\}.$$

  On the left hand side we mean any divisor coming from any resolution.

Example 1.9. Let Y be a smooth variety and consider the blow-up a smooth closed subscheme Z with $\operatorname{codim}(Y, Z) = c > 1$, $\pi : Bl_Z X \to X$. Then one writes

$$K_{Bl_Z Y} \sim \pi^* K_Y + (c - 1)E$$

where E is the exceptional divisor. By the Weak Factorization Theorem any birational morphism factors as a sequence of blow-ups one deduces that a smooth variety is terminal, as $(c - 1) \geq 1$.

Example 1.10. Let Y be a smooth variety and $\Delta = \sum_i c_i C_i$ a boundary such that $\sum_i C_i$ is an snc divisor. Then [KM98, Cor. 2.31] gives an explicit formula to check the type of singularity

$$\inf_E\{a(Y, \Delta, E) : E \text{ excep. div. over } Y\} = \min_i\left\{\min_{i \neq j; D_i \subset D_j \neq \emptyset}\{1 - a_i - a_j\}, \min_i\{1 - a_i\}, 1\right\}$$

In particular observe that if X is a smooth surface

- if $\Delta = 0$, then $(Y, 0)$ is terminal;
- if $\Delta = C$ is an integral smooth curve then (Y, C) is canonical but not terminal and
- if $\Delta = C_1 + C_2$ two smooth integral curves intersecting transversally then (Y, Δ) is log canonical but not log terminal. Moreover it is trivially dlt.

Remark 1.11. The definition of the singularities based on the discrepancies is very subtle. The condition is placed on exceptional divisors, not on log resolutions or birational morphisms! Not always it can be checked on any single log resolution morphism.

We saw in the previous example that the pair (Y, D) with Y smooth and D an integral smooth divisor is canonical. One could think that if one takes the identity $id : (X, D) \to (X, D)$, then as there is no exceptional divisors then the condition to check for terminal is empty. Quite the contrary, the fact that there are no exceptional divisors means that we cannot infer anything from this morphism, as the condition is on exceptional divisors.

Note that by definition every class of singularities contains the later ones. For the first few it is obvious. Now observe that for log terminal, every exceptional divisor has discrepancy > -1, then in particular so does every exceptional divisor with $\text{center}_Y E \subset nonsnc(Y, \Delta)$. Finally for a dlt pair (X, Δ), either one divisor lies over $nonsnc(Y, \Delta)$ or it does not and then as in the previous example [KM98, Cor. 2.31] implies that discrepancy is at least -1.

Remark 1.12. Observe that while $(X, 0)$ is terminal for a smooth variety, we saw that even for very simple cases of a pair with $\Delta \neq 0$, (X, Δ) is not terminal. Intuitively one may think of $(X, 0)$ measuring the singularities of the scheme while a pair (X, Δ) with X smooth would measure the singularities of Δ.

The motivation behind this definition and whether definitions can be checked more or less easily remains quite unclear at the moment. It is difficult to do so without talking about the MMP so the reader will have to wait until Section 1.2.1 to get a better picture.

Remark 1.13 (Local version)**.** Although these definitions are used more naturally globally, one can define them locally the following way. Say $x \in X$ is a point then $(x \in X, \Delta)$ is terminal, resp. canonical, etc. if there is a Zariski open subset $x \in U \subset X$, such that $(U, \Delta|_U)$ is terminal, resp. canonical, etc.

With this small introduction in mind we now give a brief overview on the type of singularities that will appear throughout this project and the important properties that will be needed along the way: Surface singularities, in particular canonical singularities; 3-fold singularities, mostly terminal singularities and dlt singularities and their properties.

1.1.1 Canonical surface singularities.

Let X be a surface and consider the pair $(X, 0)$. Any connected configuration of exceptional curves over it has as intersection matrix which is negative definite. From this one easily deduces that on the canonical bundle formula (1) for non-smooth $x \in X$ either

$$\text{all } a(E_i, X) < 0 \text{ or all } a(E_i, X) = 0$$

In particular by definition one finds that terminal singularities of $(X, 0)$ are smooth.

On the other hand, one considers what happens when $(X, 0)$ is canonical, in this case this will mean that all $a(E_i, X) = 0$. Over $\mathbb{C}$ these are very well known as Du Val singularities. Classified originally in [dV34], under a suitable choice of coordinates they can be locally analytically described by the equations:

$$\begin{array}{ll} A_n & x^2+y^2+z^{n+1}=0 \\ D_n & x^2+y^2z+z^{n-1}=0 \\ E_6 & x^2+y^3+z^4=0 \\ E_7 & x^2+y^3+yz^3=0 \\ E_8 & x^2+y^3+z^5=0 \end{array}$$

This very explicit description has allowed for extensive study of these. In particular we will make use of the possibility of resolving them in a parametrised family. This is known as simultaneous resolution of Du Val singularities.

Theorem 1.14. *[Bri70] Let $f:(x\in X)\to(0\in S)$ be a flat morphism of pointed analytic space germs such that X_0 is a surface with a Du Val singularity at x. Then there exists a finite and surjective ramified covering $g:S'\to S$ such that $f':X':=X\times_S S'\to S'$ has a simultaneous resolution*

$$\begin{array}{ccc} \overline{X}' & \xrightarrow{p} & X' \\ {\scriptstyle \overline{f}'}\downarrow & & \downarrow{\scriptstyle f'} \\ S' & \xrightarrow[=]{} & S' \end{array}$$

One has that $\overline{X}'_s$ is the minimal resolution of $X'_{s'}$ of $X'_{s'}$ for every $s'\in S'$.

Remark 1.15. The simultaneous resolution of rational double points (i.e. Du Val singularities) does not exist in general in the category of schemes, see [Art74] for a counterexample. For a simultaneous resolution of rational double points in the algebraic case one needs to work in the category of algebraic spaces, this was described also in [Art74].

For now we assumed that the divisor is $\Delta=0$. Now one may wonder what happens if we now assume that Δ is a boundary. This is described by the following theorem.

Theorem 1.16. *[Kol13, Theorem 2.29] Let X be a normal variety, $x\in X$ a closed point and $\Delta=\sum c_iC_i$ a boundary. Then $(x\in X,\Delta)$ is canonical if and only if*

1. *either $x\in X$ is regular and $\mathrm{mult}_x\Delta\le 1$,*
2. *or $x\notin \operatorname{Supp}\Delta$, K_X is Cartier and there is a resolution $f:Y\to X$ such that $K_Y\sim f^*K_X$.*

In particular if $\Delta\neq 0$ is reduced then if $(x\in X,\Delta)$ is canonical it is only non-regular away from $\operatorname{Supp}\Delta$, given that $\Delta\neq 0$.

Further one can describe and classify more explicitly log terminal surface and log canonical singularities, this will not be needed for this project so the interested reader may take a look at [Kol13, Section 2.2] and [KM98, Ch. 4].

1.1.2 3-dimensional terminal singularities.

Terminal singularities are the reason the MMP was developed originally to be able to classify 3-dimensional varieties (see Section 1.2). For now it is enough to say that they are the closest class to smooth that is useful within the context of the MMP.

Their theory has also been studied extensively, in particular, terminal singularities over $\mathbb{C}$ were classified by Reid in [Rei85]. Essentially if $x \in X$ is terminal and Gorenstein then it is an isolated *compound Du Val* (cDV) singularity i.e. for a general hyperplane section $x \in H \subset X$, one has that $x \in H$ is a Du Val singularity. If it is not Gorenstein then it is a quotient of a cDV and an explicit list of all possibilities was given in Reid's paper.

Later, [Kol97, 3.4.5] gave a classification of terminal non-hypersurface singularities over non-closed fields, building upon Reid's classification. We will end up not having to actually use either of these descriptions but it helps to have an idea on really how well-behaved and understood this singularities are.

Note that although one thinks of X being terminal as almost smooth, if one takes a smooth scheme and an integral smooth divisor $E \subset X$, then (X, E) is not terminal, but actually canonical as we saw in Example 1.10. This is what we already observed in Remark 1.12, that a pair does not in general measure the singularities of X.

1.1.3 Dlt pairs and adjunction.

Just like terminal is the closest notion to smooth that one can comfortably work with when running the MMP, divisorial log terminal is the closest one can get to snc to be able to comfortably work with MMP tools.

Definition 1.17 (Dlt pair)**.** Let (X, Δ) be a pair with Δ a boundary. Assume that $(K_X + \Delta)$ is $\mathbb{Q}$-Cartier. We say that (X, Δ) is *dlt or divisorial log terminal* iff there is a closed subset $Z \subset X$ such that

1. $X \backslash Z$ is smooth and $\Delta|_{X \backslash Z}$ is an snc divisor.
2. If $f : Y \to X$ is birational and $E \subset Y$ is an irreducible divisor such that $\operatorname{center}_X E \subset Z$ then $a(E, X, \Delta) > -1$.

By taking Z the minimal set with this property, one deduces that $Z = \text{non-snc}(X, \Delta)$. So then it is clear the equivalence with the defintion at the beginning of section 1.1.

In other words (X, Δ) outside of the snc locus the pair is allowed to be at most log terminal. To understand what lies away from the snc locus it is useful to understand the geometry of Δ for this it will come in handy to define the lc centers and how they behave with dlt singularities.

Definition 1.18 (Log canonical center)**.** Let (X, Δ) be a pair with Δ. We say that an irreducible subvariety $Z \subset X$ is a *log canonical center* or *lc center* of (X, Δ) if (X, Δ) is lc at the generic point of Z and there is a divisor E over X such that $a(E, X, \Delta) = -1$ and $\operatorname{center}_X E = Z$.

Theorem 1.19. *[Fuj07, Section 3.9] Let (X, Δ) be a dlt pair and $V_1, ..., V_r$ the irreducible divisors that appear in Δ with coefficient 1.*

1. *The k-codimensional lc-centers of (X, Δ) are exactly the irreducible components of the various $V_{i_1} \cap ... \cap V_{i_k}$.*
2. *Every irreducible component of $V_{i_1} \cap ... \cap V_{i_k}$ is normal of pure codimension k.*
3. *Let $Z \subset X$ be any lc center. Assume that V_i is ($\mathbb{Q}$-)Cartier for some i and $Z \subset V_i$. Then every irreducible component of $V_i|_Z$ is also ($\mathbb{Q}$-)Cartier.*

That is to say, only the reduced components of Δ and their intersections define the stricly log canonical singularities of the dlt pair.

Working with divisors it is always very handy to have some type of adjunction formula. But singularities may pose obstructions to the usual adjunction. Luckily dlt pairs allow for a "fix" of adjunction by adding a correcting term.

Def./Prop. 1.20. *[Fuj07, Rmk. 8.2] Let (X, Δ) be a dlt pair and W an irreducible component appearing with coefficient 1 in Δ. Then there exists a unique $\mathbb{Q}$-divisor on W, defined by the equation*

$$(K_X + \Delta)|_W = K_W + \mathrm{Diff}_W(\Delta).$$

Moreover $(W, \mathrm{Diff}_W(\Delta))$ is a dlt pair.

So this fix of adjunction inherits the property of being dlt, note moreover that as the pair is dlt, W is normal by Theorem 1.19 and if Δ is effective then so is $\mathrm{Diff}_W(\Delta)$ (cf. [KM98, Prop. 4.5]). Computing this correction term is in general not trivial but under good hypotheses it is possible.

Corollary 1.21. *Let (X, Δ) be a dlt pair such that $\Delta = \sum_i V_i$ is a sum of integral distinct divisors with $K_X + \Delta$ Cartier. Then*

$$\mathrm{Diff}_{V_i}(\Delta) = \sum_{i \neq j} (V_i \cap V_j) =: D_i$$

and D_i is Cartier. In other words, the pair (V_i, D_i) is dlt and satisfies the adjunction formula

$$(K_X + \Delta)|_{V_i} = K_{V_i} + D_i$$

This property was previously observed in [KX15, Paragraphs 6 and 15]. We deduce from this corollary from Theorem 1.19 and the following proposition.

Proposition 1.22. *[Fuj07, Prop. 9.2] Let (X, Δ) be a dlt pair, with $\Delta = \sum_i D_i$ a sum of distinct irreducible divisors. Let $W := D_{i_1} \cap ... \cap D_{i_k}$ be a log canonical center, by adjunction, we obtain*

$$K_{D_{i_1}} + \mathrm{Diff}_{D_{i_1}}(D) = (K_X + D)|_{D_{i_1}}$$

and $(D_{i_1}, \mathrm{Diff}_{D_{i_1}}(D))$ is dlt. Note that

- D_{i_1} *is normal,* W *is a lc center for the pair* $(D_{i_1}, \mathrm{Diff}_{D_{i_1}}(D))$,
- $D_{i_j}|_{D_{i_1}}$ *is an irreducible component of* $\mathrm{Diff}_{D_{i_1}}(D))$ *for* $2 \leq j \leq k$, *and*
- W *is an irreducible component of* $(D_{i_2}|_{D_{i_1}}) \cap (D_{i_3}|_{D_{i_1}}) \cap \cdots \cap (D_{i_k}|_{D_{i_1}})$

By applying adjunction k *times repeatedly, we obtain a* $\mathbb{Q}$*-divisor* B *on* W *such that*

$$(K_X + D)|_W = K_W + B$$

and (W, B) *is dlt.*

1.2 Finding minimal models.

Now that we have good knowledge of the singularities occuring in the Minimal Model Program it is time to talk about the program itself.

Classically the classification of algebraic surfaces over $\mathbb{C}$ was completed by Castelnuovo and Enriques around 1900's. In short they proved that every algebraic surface Y was birational to a smooth surface X known as minimal model satisfying one of the following

1. either K_X is not nef and X admits a fibration $X \to C$ structure or
2. K_X is nef, in this case the model is unique.

These properties, in particular nefness of K_X, allow for a very explicit description of the surfaces, the interested reader may see [Bea96].

Question. Is it true that any variety Y of arbitrary dimension is birational to a smooth variety with

1. either K_X is not nef and X admits a fibration $X \to C$ structure or
2. K_X is nef?

Even for dimension 3 the answer to this question remained unsolved for quite some time. The breakthrough was realised by Shigefumi Mori who saw that the obstruction for finding a minimal model was asking for smoothness, which lead him to the define terminal singularities. As we saw in the previous section, these are very mild and close to smooth.

Changing smooth to terminal allowed for a positive answer of the previous question. Proving this and constructing the minimal models in dimension 3 was done by running the Minimal Model Program. Defining it, proving its existence and termination in dimension 3 was only possible thanks to the contributions of many mathematicians: Mori, Kawamata, Kollár, Reid...

This is the motivation behind the singularities defined in the previous section. Intuitively one starts with a pair in a class of singularities and running the MMP will either eliminate all obstructions to nefness of the canonical divisor or find some fibered structure while staying in the initial class of singularities.

The MMP is still very much a work in progress, now the objective is being able to extend this result to higher dimensions and more general settings. In a very general and slightly technical way the conjecture that is waiting to be solved is:

Conjecture 1.23 (Minimal model conjecture). *Let $f : X \to S$ be a proper, dominant morphism between normal, irreducible schemes with generic fiber X_{gen}. Let Δ be an effective $\mathbb{R}$-divisor on X such that (X, Δ) is lc. Then*

- *(X, Δ) has a minimal model (X^{min}, Δ^{min}) if and only if the restriction of $K_X + \Delta$ to the generic fiber X_{gen} is pseudo-effective (that is, it is numerically equivalent to a limit of effective $\mathbb{Q}$-divisors).*
- *If (X, Δ) is dlt (resp. $\mathbb{Q}$-factorial) then one can choose (X^{min}, Δ^{min}) to be dlt (resp. $\mathbb{Q}$-factorial).*

To understand this conjecture we need to be less vague about the definition of a minimal model. We will be satisfied with just saying it is a pair (X^min, Δ^min) such that $K_{X^min} + \Delta^min$ is f-nef, i.e. intersects non-negatively with any curve contained in a closed fiber. For the technical definition see [Kol13].

The research to solve this conjecture is very active and the range of applications is very wide. This project is just one of these applications. But we won't need that much generality for us it will be enough with this part of the already established MMP.

Theorem 1.24 (3-dimensional MMP). *The Minimal Model conjecture holds if* $\dim X \leq 3$ *for schemes and for complex analytic spaces if f is projective and 3-dimensional.*

This result is obtained by running the MMP as in 1.2.2. For a complete proof see [KM98] or [KA92].

Remark 1.25. The hypothesis of projective is necessary for complex analytic spaces. An example by Hironaka reproduced in [Har13, p.443] shows that the behaviour needed for cone contractions does not hold for compact complex manifolds in general. A bit more about this assumption is to be said in Remark 3.5.

1.2.1 Contractions of the Mori cone.

We now give a very short overview of how one actually obtains a minimal model for $f : (X, \Delta) \to S$, for a more detailed account see [KM98]. The idea as we did before is to deal progressively obstructions to f-nef-ness of $K_X + \Delta$ until obtaining that $K_X + \Delta$ "becomes" f-nef or running into a Fano contraction.

Definition 1.26 (Extremal rays and faces). Given V a K-vector space ($K = \mathbb{Q}$ or $\mathbb{R}$). A subset $N \subset M$ is called a *cone* if $0 \in N$ and N is closed under multiplication by positive scalars.

A subcone $M \subset N$ is called extremal or an *extremal face* of N if it satisfies that $u, v \in N$ and $u + v \in M$ imply $u, v \in M$. A 1-dimensional extremal subcone is called an *extremal ray.*

Definition 1.27 (Mori Cone)**.** Let X be a proper variety. A 1-cycle is a formal linear combination of integral and proper curves $C = \sum_i a_i C_i$. 1-cycles with real coefficients modulo numerical equivalence form an $\mathbb{R}$-vector space, denoted $N_1(X)$. Denote the class of a 1-cycle C as $[C]$.

Set

$$\begin{aligned} NE(X) &= \{\sum a_i[C_i] \subset X, 0 \leq a_i \in \mathbb{R}\} \subset N_1(X), \text{ and} \\ \overline{NE}(X) &= \text{ the closure of } NE(X) \text{ in } N_1(X). \end{aligned}$$

$\overline{NE}(X)$ is known as the *Mori cone.*

For any divisor D, set

$$D_{\geq 0} := \{x \in N_1(X) : (x.D) \geq 0\} \text{ and } \overline{NE}(X)_{D \geq 0} := \overline{NE}(X) \cap D_{\geq 0}$$

Analogously one can define relative versions. Let $f : X \to Y$ be a projective morphism then define $N_1(X/Y)$ as the $\mathbb{R}$-vector space generated by irreducible curves $C \subset X$ contained in some closed fiber modulo numerical equivalence on X. From this one defines just the same $\overline{NE}(X/Y)$ and $\overline{NE}(X/Y)_{D \geq 0}$, it is known as the *relative Mori Cone.*

Now we are ready to give a formulation of the Cone and Contraction theorems.

Theorem 1.28 (Cone and contraction Theorem)**.** *[KM98, Theorem 3.35] Let $f : (X, \Delta) \to Y$ be a projective morphism where (X, Δ) is a dlt pair. Then,*

1. *There are (countably many) rational curves $C_j \subset X$ contained in closed fibers of f and*
$$\overline{NE}(X/Y) = \overline{NE}(X/Y)_{(K_X+\Delta)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_j]$$
2. *Let $F \subset \overline{NE}(X/Y)$ be a $(K_X + \Delta)$-negative extremal face. Then there is a unique morphism $cont_F : X \to Z$ over Y, called the contraction of F, such that $(cont_F)_* \mathcal{O}_X = \mathcal{O}_Z$ and an irreducible curve $C \subset X$ is mapped to a point by $cont_F$ iff $[C] \in F$.*

This effectively means that the curves that obstruct the nefness of $(K_X + \Delta)$ are a linear combination of countable rational curves and that any of these $(K_X + \Delta)$-negative face can be effectively be contracted without contracting anything else of the Mori cone. We wish to contract all extremal rays obstructing nef-ness to obtain a minimal model. But we first have to know a bit more about contractions.

Proposition 1.29. *[KM98, Prop. 2.5] Let $f : X \to S$ be a proper morphism. Assume that X is $\mathbb{Q}$-factorial and let $g : X \to Y$ be the contraction of an extremal ray $R \subset \overline{NE}(X/S)$. Then we have one of the following cases:*

1. *(Fano contraction)* $\dim X < \dim Y$.

2. *(Divisorial contraction) g is birational and* $\mathrm{Ex}(g)$ *is an irreducible divisor.*

3. *(Small contraction) g is birational and* $\mathrm{Ex}(g)$ *has codimension* ≥ 2.

When a small contraction occurs a simple argument one shows that K_Y cannot be $\mathbb{Q}$-Cartier, which is a big problem if we want to keep this process. For this reason we do not contract $\mathrm{Ex}(g)$ but rather remove it and compactify it by adding another comdimension ≥ 2 subvariety E^+, this will be called flip.

Definition 1.30 (Flip)**.** Let X be a normal scheme and Δ a $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

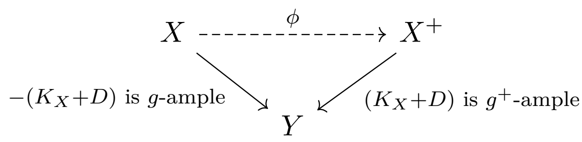
A $(K + \Delta)$-flipping contraction is a proper birational morphism $g : X \to Y$ to a normal scheme Y such that $\mathrm{Ex}(g)$ has codimension at least two in X and $-(K_X + \Delta)$ is g-ample.

A normal scheme X^+ together with a normal proper birational morphism $g^+ : X^+ \to Y$ is called a $(K + \Delta)$-flip of g if

1. $K_{X^+} + \Delta^+$ is $\mathbb{Q}$-Cartier, with $\Delta^+ := \phi_*\Delta$,

2. $K_{X^+} + \Delta^+$ is g^+-ample, and

3. $\mathrm{Ex}(g^+)$ has codimension at least 2 in X^+. By abuse of terminology the induced rational map $\phi : X \dashrightarrow X^+$ is also called a $(K + \Delta)$-flip. A $(K + \Delta)$-flip gives a commutative diagram:



Remark 1.31. It is not clear that flips exist in general, it has only been proven in some special cases. It has been established for the case of interest, i.e. dimension 3.

Example 1.32. The simpler version of a flip is a flop. A flop is defined similarly but assuming that K_X is numerically trivial. It can be shown that flops do not depend on D. Flops were described by Kulikov [Kul77] for the first times and examples of them are given by the so called Type I and II modifications that relate Kulikov models and are an essential part of the original proofs of Theorem 0.1.

A *Type I modification* is constructed as: Given by E be an exceptional curve, but not a double curve, intersecting as in figure 1. Then it can be moved to the adjacent component as in Figure 1.

A *Type II modification* now is given by: E be an exceptional curve which is a double curve as in Figure 2. Then it can be moved to the to the adjacent component as in 2.

In both cases the procedure is done between two Kulikov models, hence K_X is numerically trivial X^+ is the resulting variety and clearly the birational map $\phi : X \dashrightarrow X^+$ induced by this modification has $\mathrm{codim}(\mathrm{Ex}(\phi)) = 2$.

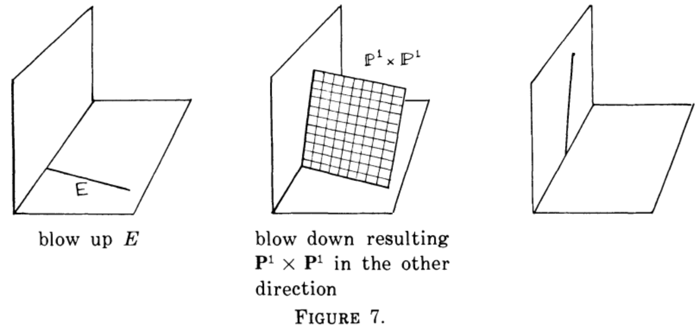


Figure 1: Example of a Type I modification [PP81, Figure 7]

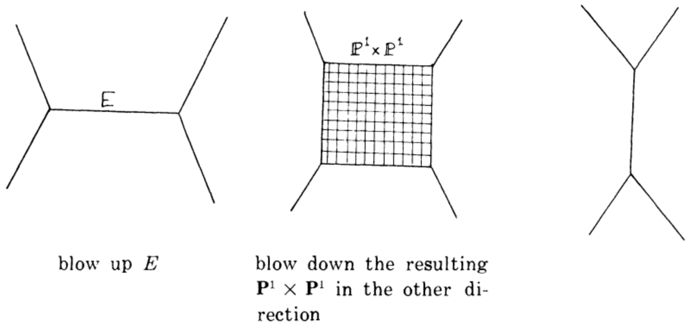


Figure 2: Example of a Type II modification [PP81, Figure 8]

1.2.2 Running the MMP.

We now have all the ingredients to explain how the MMP is expected to run. We describe it for dlt pairs, for a more detailed account see [KM98, 3.31].

Start with a pair $(X, \Delta) = (X_0, \Delta_0)$ a dlt pair where X is a normal $\mathbb{Q}$-factorial scheme over a field (or complex analytic space). Let $f_0 : X \to S$ be a projective morphism.

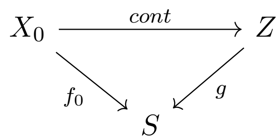
The aim would be to set up a recursive procedure which creates intermediate dlt pairs (X_i, Δ_i) and projective morphisms $f_i : X_i \to S$. This should end at some point a final pair (X^*, Δ^*) and $f^* : X^* \to S$.

Recall that the objective is to find a minimal model or find a some kind of fibered structure for the variety, this will be our criteria to stop.

1. If $K_{X_0} + \Delta_0$ is already f_0-nef then we are done.

2. Else, by Theorem 1.28, there is a $(K_{X_0} + \Delta_0)$-negative extremal ray which can be

contracted. Let *cont* : $X_0 \to Z$ be its contraction.



Then by Proposition 1.29 it is either

(a) A Fano Contraction, in this case we are done as we obtained a fibered structure for (X_0, Δ_0) and so this will be our final model.

(b) A divisorial contraction, in which case we just take $(X_1) := (Z, cont_*\Delta_0)$ and $f_1 = g$.

(c) A flipping contraction, then set $(X_1, \Delta_1) := (X_0^+, \Delta_0^+)$ and $f_1 := g \circ f_0^+$.

3. Repeat this procedure recursively until it terminates. For this one needs to ensure that the theorems cited can be used, for this we note that cone contractions of the Mori cone preserve
   - $\mathbb{Q}$-factoriality,
   - minimal discrepancy[3], being dlt and
   - f_{i+1} as defined above is once again projective.

A priori it is not clear that this algorithm should have finite steps. If X_{i+1} is obtained by a divisorial contraction then the Picard rank drops by one, i.e. $\rho(X_{i+1}) = \rho(X_i) - 1$. But flips are isomorphisms in codimension 2 so $\rho(X_{i+1}) = \rho(X_i)$. Hence the number of divisorial contractions is bounded by $\rho(X_0) - 1$, but flips could theoretically occur infintely. The question of whether this procedure is finite becomes a question of whether infinite sequences of flips exist or not (this is true for dimension 3). In fact, jointly with the existence of flips, termination of flips is the biggest obstruction to establishing the MMP in generality.

Remark 1.33. Here we assumed that the pair is dlt. But one can also assume it is terminal, canonical, log canonical... The MMP runs just the same while staying in the initial class of singularities.

Example 1.34 (MMP for surfaces)**.** Suppose we have a smooth surface X with $\kappa(X) \geq 0$, it is well known that the procedure to obtain a minimal model with nef canonical divisor is contracting the (-1)-curves it contains.

One can do so running the MMP. Let C be a (-1)-curve, then by adjuction $K_X.C = -1$, hence C belongs to a negative extremal ray of the Mori cone. Applying Theorems 1.28 and 1.29 one obtains a morphism *cont* : $X \to X_1$ and it's one of the following:

1. A Fano contraction, if this happened as C is rational it would mean that X admits a ruling but this is not true for $\kappa(X) \geq 0$, hence does not occur.

[3]More specifically, the minimal discrepancy may only stay or get bigger.

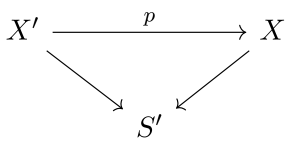
2. A flipping contraction is an isomorphism in codimension 2, hence would be an isomorphism but $cont(C)$ is a point, hence also does not occur.

3. A divisorial contraction is then the only option.

As divisorial contractions are finite one obtains a finite sequence of birational morphisms $X \to X_1 \to ... \to X^*$, with X^* is a terminal (smooth as $\dim X = 2$) surface with K_X nef.

We give one last, easy application of the MMP which is basically a baby version of the proof we give of Theorem 0.1

Corollary 1.35 (KPP for elliptic curves)**.** *Let $Y \to S$ be a flat projective morphism to a smooth projective curve, such that the generic fiber Y_{gen} is a smooth elliptic curve. Then there exists a finite surjective base change $S' \to S$ and $p : (X' := Y \times_S S') \to X$ a birational morphism over S'*



such that $f : X \to S'$ is a Kulikov model (Def. 0.2)

Proof. After applying Mumford's semistable reduction theorem (cf. [KKMSD73]) there exists a finite surjective base change $S' \to S$, so that $Y \times_S S' \to S'$ is a semistable degeneration we may assume that the degeneration is semistable.

Apply the MMP as described above, flips do not occur because of dimension and there is no Fano contraction. The reason for this is that outside the special fibers X'_t is K-trivial and so by adjunction an arbitrary divisor in $|K_{X'}|$ induces a principal divisor on X'_t, so $K_{X'}$ is a linear combination of irreducible components of the special fibers. But by Theorem 1.28 only K-negative curves may be contracted.

This process terminates with a terminal, hence smooth, surface X and $f : X \to S$ a projective morphism. Moreover, a fiber is numerically trivial to curves contained in a closed fiber. So running the MMP for $K_{X'}$ is the same as running it for $K_{X'} + X'_t$. Hence (X, X_t) is a dlt pair.

Let $X_t = \sum_i C_i$ its decomposition into irreducible components, by Theorem 1.19 C_i are normal, hence smooth curves and the double points are lc centers hence part of the snc-locus. So (X, X_t) is snc for all $t \in S'$.

Finally, observe that since K_X is nef, but a linear combination of components of the fibers (so $0 \leq K_X^2$). Then $K_X^2 = 0$ and by Zariski's lemma it must be a linear combination of fibers hence f-trivial. Thus $f : X \to S$ is a K-trivial semistable degeneration, i.e. a Kulikov model. □

2 Brief introduction to toric geometry.

Animal de records, lent i trist animal,
ja no vius, sols recordes.
Llibre de meravelles,
Vicent Andrés Estellés.

In this section we give a very basic and brief introduction to toric geometry intended to cover the minimal theory needed for the techniques later used in Section 3.3 for the proof of Theorem 0.6. The reader will be redirected to [Ful93] for real introduction, which is the main source of this section.

Firstly we will explore the equivalence between lattices and toric varieties and later give a small introduction to toric resolutions via refinement of fans.

2.1 Cones and toric varieties.

Definition 2.1. A semigroup $(S, +)$ is called

- *integral* if there is an embedding $S \hookrightarrow \mathbb{Z}$,
- *affine* if it is an integral finitely generated semigroup.

Given a semigroup $(S, +)$ one constructs an associated semigroup algebra $\mathbb{C}[S]$. It is generated by elements χ^u indexed by elements $u \in S$. The semigroup operation $+$ induces the multiplication of the χ^u in $\mathbb{C}[S]$, thus $\chi^u \cdot \chi^v = \chi^{u+v}$.

Example 2.2. To $\mathbb{N}^n$ one can associate $\mathbb{C}[t_1, ..., t_n]$, where we denote $t_i = \chi^{e_i}$. To Z^n we associate $\mathbb{C}[t_1, t_1^{-1}, ..., t_n, t_n^{-1}]$, the ring of Laurent polynomials.

If S is an affine semigroup, then $\mathbb{C}[S]$ is a finitely generated domain, the embedding $S \hookrightarrow \mathbb{Z}^n$ induces an injective homomorphism $\mathbb{C}[S] \hookrightarrow \mathbb{C}[t_1^{\pm 1}, ..., t_n^{\pm 1}]$ and this is a domain. Hence for an affine semigroup we can take the spectrum $\mathrm{Spec}(\mathbb{C}[S])$ to get an affine variety.

A point of $\mathrm{Spec}(\mathbb{C}[S])$ is given by a $\mathbb{C}$-algebra homomorphism $f : \mathbb{C}[S] \to \mathbb{C}$. This corresponds naturally to a semigroup morphism $\gamma : S \to (\mathbb{C}^*, \cdot)$ satisfying $u \mapsto f(\chi^u)$.

Definition 2.3. A linear algebraic group T is a *torus* if it is isomorphic to some $(\mathbb{C}^*)^n$. The torus inherits the action of $(\mathbb{C}^*)^n$ on itself given by multiplication.

A *toric variety* X is a normal variety that contains a torus T as a Zariski open subset together with an T-action on X

$$T \times X \to X$$

that extends the natural action of T on itself.

Example 2.4. Take $T = \mathbb{C}^* \times .... \times \mathbb{C}^*$ as a torus, then $\mathbb{P}_{\mathbb{C}}$ is a toric variety with subtorus T.

Definition 2.5. Given a lattice N, we define a vector space $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$. A finite set of points S in the lattice defines a *cone* in the corresponding vector space.

$$\sigma = \text{Cone}(S) = \left\{ \sum_{x \in S} \lambda_x x : \ \lambda_x \geq 0 \right\} \subset N_{\mathbb{R}}$$

Let $M := N^{\vee}$ the dual lattice of N, then the *dual cone* of σ is defined as

$$\sigma^{\vee} = \{m \in M_{\mathbb{R}} : \ \langle x, m \rangle \geq 0 \ \forall x \in \sigma\} \subset M_{\mathbb{R}}.$$

Where $\langle -, - \rangle$ denotes the natural pairing $N \times M \to \mathbb{Z}$.

Given a cone $\sigma \subset N_{\mathbb{R}}$ one has that $S_\sigma = \sigma \cap M$ is a finitely generated and hence affine semigroup (cf. [Ful93, Prop. 1]) one can define an affine variety from a cone by setting

$$U_\sigma = \text{Spec}(\mathbb{C}[S_\sigma]).$$

This is in fact a toric variety, if n is the rank of $S_\sigma \mathbb{Z}$ then there is a torus $T \simeq (\mathbb{C}^*)^n$ acting on U_σ. For a lattice $N \simeq \mathbb{Z}^n$ one obtains a torus $T_N \simeq (\mathbb{C}^*)^n$ defined as

$$T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq \text{hom}_{\mathbb{Z}}(N, \mathbb{C}^*)$$

For a cone $\sigma \subset N_{\mathbb{R}}$ the variety U_σ contains the torus T_N as an open subset. The action on U_σ is defined as

$$\begin{array}{ccc} T_N \times U_\sigma & \longrightarrow & U_\sigma \\ (t, \gamma : S_\sigma \to \mathbb{C}) & \longmapsto & (m \mapsto \chi^m(t)\gamma(m)) \end{array}$$

Example 2.6. Consider the cone $\sigma = \text{Cone}(e_2, 2e_1 - e_2)$, one easily sees that S_σ is generated by $(1, 0)$, $(1, 1)$ and $(1, 2)$.

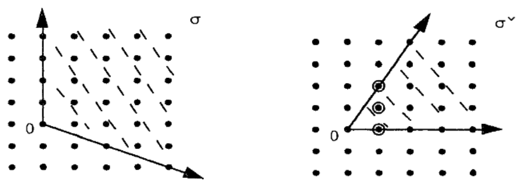


Figure 3: The cone $\sigma = \text{Cone}(e_2, 2e_1 - e_2)$ and its dual cone [Ful93, p.5]

Hence we obtain a toric variety with coordinate ring

$$\mathbb{C}[S_\sigma] = \mathbb{C}[x, xy, xy^2] = \mathbb{C}[u, v, w]/(v^2 - uw).$$

So

$$U_\sigma = \text{Spec}(\mathbb{C}[u, v, w]/(v^2 - uw))$$

2.2 Smoothness and toric resolutions.

Definition 2.7. A cone $\sigma \subset N_{\mathbb{R}}$ is called *smooth* if it is generated by part of a lattice basis.

Theorem 2.8. *[Ful93, p. 29] A cone σ is smooth iff the variety U_σ is smooth.*

Example 2.9. Consider the variety

$$Y = \operatorname{Spec} \mathbb{C}[x, y, z, t]/(xyz - t^n)$$

This variety is clearly non-smooth at the origin. Observe that this variety is toric as it contains the subtorus

$$T = \{(x, y, z, t) \in (\mathbb{C}^*)^4 / \, xyz = t^n\} \subset (\mathbb{C}^*)^4$$

Let σ be the cone spanned by $(1, 0, 0)$, $(0, 1, 0)$ and $(-1, -1, n)$. This defines an inclusion

$$\mathbb{C}[x, y, t, x^{-1}y^{-1}t^n] \hookrightarrow \mathbb{C}[x, x^{-1}, y, y^{-1}t, t^{-1}]$$

clearly $U_\sigma = Y$.

The dual cone is spanned by $(0, 0, 1), (n, 0, 1)$ and $(0, n, 1)$. This is not a $\mathbb{Z}$ basis of $\mathbb{Z}^3$, hence σ is not smooth and also Y is not smooth as expected from the theorem.

One knows by Theorem 1.5 that for a singular variety there is a resolution via blow ups. If we have a toric variety how will this look like? For this we need the concept of fans and subdivision.

Definition 2.10. A *fan* Δ in $N_{\mathbb{R}}$ is a collection of cones in $N_{\mathbb{R}}$ such that

1. a face[4] of cone in Δ is itself a cone in Δ
2. the intersection of two cones in Δ is a face of each.

We already saw that from a cone $\sigma \subset N_{\mathbb{R}}$ one obtains a variety U_σ, then for a fan Δ one has a family of affine varieties which all contain T_N. These can be glued together to obtain a variety U_Δ, the *toric variety of a fan.*

Example 2.11. Consider the fan Δ given by $\sigma_1, \sigma_2 \subset \mathbb{R}^2$. Where σ_1 is the cone spanned by $(1, 0)$ and $(1, 1)$ and σ_2 is spanned by $(0, 1)$ and $(1, 1)$. These meet along the ray ρ generated by $(1, 1)$. In terms of varieties we have

$$U_{\sigma_1} \simeq \operatorname{Spec}(\mathbb{C}[x, y, x^{-1}y]) \text{ and } U_{\sigma_2} \simeq \operatorname{Spec}(\mathbb{C}[x, y, xy^{-1}])$$

These patches are then glued along the Zariski open $U_\rho = \operatorname{Spec}(\mathbb{C}[x, y, xy^{-1}, x^{-1}y])$, corresponding to the ray ρ. The resulting variety from the glueing U_Δ is the variety associated to Δ.

[4] A face is defined in the obvious way.

Def./Prop. 2.12. *A fan Δ is called smooth if all cones in Δ are smooth. Moreover,the associated variety U_Δ is nonsingular if and only if Δ is a smooth fan.*

Proof. Trivial since smooth is a local property. □

From this we deduce that given a toric variety U_Δ defined by a fan Δ which has a singular point $p \in U_\Delta$, then there is a non-smooth cone σ such that $p \in U_\sigma \subset U_\Delta$. Intuitively one then would like to replace the non-smooth cone by a collection of smooth cones. One can do precisely this by subdividing the cone.

Definition 2.13 (Refinement)**.** One says that a fan Δ' *refines* the fan Δ if

1. every cone in Δ is a union of cones in Δ' and
2. the fans have the same support, i.e. $\cup_{\sigma\in\Delta} \operatorname{Supp}(\sigma) = \cup_{\sigma'\in\Delta} \operatorname{Supp}(\sigma')$.

Proposition 2.14. *[Ful93, 2.6] Suppose Δ' is a refinement of Δ, then the morphism $f : U_{\Delta'} \to U_\Delta$ induced by the identity map of $N_\mathbb{R}$ is birational and proper. Moreover it is an isomorphism on the open torus T_N.*

Effectively this means that if we find a way to subdivide the non-smooth cone correctly into a smooth fan then we get exactly a resolution of singularities and it leaves the embedded open torus untouched.

Example 2.15. Let

$$X = \operatorname{Spec} \mathbb{C}[x, y, z, t]/(xy - t^n)$$

As in Example 2.9 this is a singular toric variety containing the subtorus

$$T = \{(x, y, z, t) \in (\mathbb{C}^*)^4 / \, xy = t^n\} \subset (\mathbb{C}^*)^4$$

Similarly to that example the cone σ corresponding to this toric variety is spanned by $(0, 1, 0)$ and $(-1, -n, 0)$ and $(0, 0, 1)$. And so the dual cone $\sigma^\vee$ has as basis by $(0, 1, 0)$ and $(n, 1, 0)$.

To desingularise this variety one wishes to find a smooth refinement of this cone. For this add all rays of the form $\mathbb{R}^+(a, 1)$ with $a \in \mathbb{Z}$, $0 \le a \le n$ and define the cones

$$\sigma_\alpha = \text{ convex hull of } \mathbb{R}^+(a, 1) \text{ and } \mathbb{R}^+(a + 1, 1)$$

Clearly this is a smooth refinement. Let $\tilde{X} := U_\Delta$ where Δ is the fan of $\{\sigma_\alpha\}$. Then by Proposition 2.14 $\tilde{X} \to X$ is a birational proper map which is a resolution of singularities and it an isomorphism on T.

Locally for one of the cones σ_α spanned by $(a, 1)$ and $(a + 1, 1)$, the dual cone $\sigma^\vee$ is spanned by $(-1, a + 1)$ and $(1, -a)$ hence $\tilde{X}$ is obtained by gluing the affine pieces

$$U_{\sigma_\alpha} = \operatorname{Spec}(\mathbb{C}[xt^{-a}, z, x^{-1}t^{a+1}])$$

Set $u_\alpha = xt^{-a}$ and $v_\alpha = x^{-1}t^{a+1}$, then $u_\alpha v_\alpha = t$, so $t = 0$ defines an snc divisor.

We note that after a coordinate change, $xy - t^n = 0$ turns into $(x')^2 + (y')^2 = t^n$. For $t = 0$ this is an A_{n-1} singularity, so essentially we resolved a curve of A_{n-1} singularities.

3 Kulikov-Persson-Pinkham Theorem a new proof.

Bullirà el mar com la cassola en forn,
mudant color e l'estat natural,
e mostrarà voler tota res mal
que sobre si atur un punt al jorn.

Veles e vents, Ausiàs March

The objective of this section is to give a new proof of Theorem 0.1, assuming the morphism is projective. For this we follow a strategy laid out in 3 steps.

Step 1: Running the MMP

1. Start with a dlt pair (Y, Y_t) with Y terminal for a degeneration $f : Y \to C$ of K-trivial surfaces or apply semistable reduction to obtain this set up.

2. Run the semistable MMP program to obtain a new pair (X, X_0), this pair will be dlt, X being terminal and factorial with $K_X \sim_f \mathcal{O}_X$.

Step 2: Analysis of the central fiber

1. Exploit K-triviality and the dlt assumption, to see that X_0 has irreducible components intersecting transversally and it is smooth around the 1 dimensional lc centers. Away from these there are at most a finite amount of isolated canonical singularities.

2. Argue that any singularities from X lie above the Du Val singularities of X_0, concluding it is snc around the 1 lc centers.

Step 3: Resolution of Du Val singularities.

1. After a base change there exists a simultaneous resolution of Du Val singularities, this allows us to deal with the canonical singularities left. At the price of introducing new singularities along the lc centers.

2. These new singularities may be resolved crepantly via toric resolutions. Doing this a finite amount of times gives the desired result.

Remark 3.1. Step 3 is valid only for complex analytic spaces[5]. To perform it for schemes one has to go to the category of algebraic spaces.

[5] Both in [Kul77] and in [PP81] the proof is done for complex analytic spaces.

3.1 Step 1: Semistable reduction and running the MMP.

Everything in this step applies to complex analytic spaces with the same arguments.

Proposition 3.2. *Let $f_0 : Y \to C$ be a flat projective surjective morphism, with*

i. Y be a 3 dimensional terminal $\mathbb{Q}$-factorial variety and C a quasi-projective smooth curve,

ii. (Y, Y_t) dlt for all $t \in C$, where $Y_t := f_0^{-1}(t)$ and

iii. $Y_t := f_0^{-1}(c)$ a smooth K-trivial surface for $t \neq 0$.

Then there exists a birational morphism

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ & {\scriptstyle f_0}\searrow \quad \swarrow{\scriptstyle f} & \\ & C & \end{array}$$

such that

1. *$\pi : Y \to X$ is an isomorphism over $f_0^{-1}(C \backslash \{0\})$,*
2. *X is terminal, Gorenstein and factorial and $\mathrm{Sing}(X) \subset \cup_i \mathrm{Sing}(V_i)$ where V_i are the irreducible components of X_0;*
3. *K_X is f-trivial and every fiber X_t is K-trivial;*
4. *(X, X_b) is a dlt pair for all $t \in C$.*

Proof. Similarly to Corollary 1.35, we run the MMP to $f_0 : Y \to C$, applying Theorem 1.24 the MMP terminates. Observe that a Fano contraction does not occur. Outside the special fibers Y_b is K-trivial and so by the adjunction formula an arbitrary divisor from $|K_Y|$ induces a principal divisor on Y_b. Hence a general element of $|K_Y|$ consists a linear combination of components of the special fiber:

$$K_Y = r_1 V_{1,Y} + ... + r_n V_{n,Y}$$

Note that the contractions of the relative Mori cone (Def. 1.27) only consider curves contained in closed fibers. Hence K_Y intersects trivially with any curve of the relative Mori cone outside of Y_0 and so by Theorem 1.28, no curves outside of the special fiber can be contracted during the MMP, thus a Fano contraction cannot occur.

Thus a composition of flips and divisorial contractions we obtain a birational morphism

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ & {\scriptstyle f_0}\searrow \quad \swarrow{\scriptstyle f} & \\ & C & \end{array}$$

with $f : X \to B$ flat surjective projective morphism such that

- $(X, 0)$ is terminal, normal and $\mathbb{Q}$-factorial.
- K_X is f-nef,
- $\pi : Y \to X$ is an isomorphism over $f_0^{-1}(C\backslash 0)$ as no contractions occur outside of the special fiber.

As we noted before the relative Mori cone only contains curves contained in a closed fiber. Since all fibers are numerically equivalent, the intersection of any fiber with a curve in the relative Mori cone is trivial. Hence every step of the MMP for K_Y is a step of the MMP for $K_Y + Y_t$, hence (X, X_t) is dlt for all t.

Notice that the isomorphism on a general fiber gives that in particular $K_{X_t} \sim \mathcal{O}_{X_t}$. Thus by the same argument as before K_X is f-equivalent to a linear combination of components of the special fiber. By this condition $K_X^2 \leq 0$ but nefness gives that $K_X^2 \geq 0$[6]. Hence by Zariski's lemma it is nef iff it is a linear combination of the fibers and thus $K_X \sim_f \mathcal{O}_X$.

Since X is relatively K-trivial, then it is Gorenstein. A lemma from Kawamata (cf. [Kaw88, Lemma 5.1]) states that for 3-dimensional terminal and Gorenstein variety every $\mathbb{Q}$-Cartier divisor is Cartier. In particular, since X is $\mathbb{Q}$-factorial then it is also factorial. Furthermore, the dlt assumption ensures we can apply adjunction, which shows that the special fiber is also K-trivial

$$\mathcal{O}_{X_0} \sim_f (K_X + X_0)|_{X_0} = K_{X_0}$$

Finally we show that $\operatorname{Sing}(X) \subset \cup_i \operatorname{Sing}(V_i)$. Take a closed point $p \in X\backslash X_0$, then $x \in \operatorname{Supp}(X_t)$ for some $t \neq 0$. But X_t is a smooth Cartier so divisor, so if p was singular in the total space it should also be on the fiber. Hence $p \in X\backslash X_0$ is always smooth. Now take a closed point $p \in X \cap \operatorname{Supp} X_0$, then it lies in some component $V_i \subset X_0$. Since V_i is a Cartier divisor by the same argument it is a necessary condition that p is singular in V_i to be singular on the total space. Hence $\operatorname{Sing}(X) \subset \cup_i \operatorname{Sing}(V_i)$ □

Remark 3.3. By Mumford's semistable reduction theorem (cf. [KKMSD73]) for a projective morphism $Z \to C$ there is a finite possibly ramified cover $\pi : B \to C$ such that $f_B : (B \times_C Z) \to B$ is birational to a projective morphism $g : Y \to B$, with fibers either smooth or snc.

That means that instead of starting with a dlt terminal model one could start with a more general degeneration and perform this operation to start with an snc. But we do not do this because the main objective is just to prove Theorem 0.1 and it makes things simpler.

Remark 3.4. Assuming that only the generic fiber is smooth or with canonical singularities and taking a compact base instead of quasi-projective one can use the same arguments to obtain a similar result. The main difference is mostly presentation-wise

[6] Remark that K_X^2 is well defined as it is compactly supported, since f is proper and a point is compact.

and that the property of MMP being an isomorphism outside of the central fiber is lost. In [KLSV18] it is done in such a way for the interested reader and we also did this in Corollary 1.35.

Remark 3.5 (Projective assumption)**.** In the original statement by Kulikov [Kul77] the morphism is not assumed to be projective just that the fibers are Kähler, this was later improved in [PP81] from Kähler to algebraic. The statement we present is in this sense weaker, nevertheless it is a necessary condition to actually run the MMP.

For complex analytic spaces the assumption that the morphism is projective is necessary as the MMP cannot be stablished for compact complex manifolds in general (cf. [KM98, 2.17]). We later give another example in this direction (see Remark 3.14).

Note that in dimension 3 if X is compact Kähler, normal, $\mathbb{Q}$-factorial with at most terminal singularities the Minimal Model conjecture holds (cf.[HP16]). Hence one could replace the hypothesis of f being projective with f being proper and X compact Kähler in Proposition 3.2.

For schemes by [Kol13, 1.30] it works with only the morphism being proper and surjective. Nevertheless as in the algebraic case the difference between proper and projective is not big we just assume the latter.

3.2 Step 2: Analysis of the central fiber.

As before the same arguments hold for complex analytic spaces.

Once that we have a semistable minimal model, the idea now is to really understand what singularities one finds which will then allow us to reasses the problem of turning this model into a minimal snc model. For this our objective is to prove the following theorem.

Theorem 3.6. *Let X be a 3-dimensional normal variety and (X, Δ) a dlt pair satisfying such that $\Delta = \sum_i V_i$ with V_i irreducible, distinct and Cartier. Suppose $K_X + \Delta$ is Cartier and* $\operatorname{Sing}(X) \subset \operatorname{Supp} \Delta$*.*

Then

$$\operatorname{Sing}(X) \subset \cup_i \operatorname{Sing}(V_i)$$

where $\operatorname{Sing}(V_i)$ *consists only of isolated canonical singularities. Moreover let*

$$D := \sum_{i<j, V_i \cap V_j \neq \emptyset} V_i \cap V_j$$

the double curves on Δ, then around $\operatorname{Supp} D$ *the pair (X, Δ) is snc.*

In particular theorem 3.6 holds for the pair (X, X_0) we obtained in Proposition 3.2. Hence we deduce the following theorem

Theorem 3.7 (Weak Kulikov Models for varieties)**.** *Let $f_0 : Y \to C$ be a flat projective surjective morphism, with* *i.* Y *be a 3 dimensional terminal* $\mathbb{Q}$*-factorial variety and* C *a quasi-projective smooth curve,*

ii. (Y, Y_t) *dlt for all* $t \in C$*, where* $Y_t := f_0^{-1}(t)$ *and*

iii. $Y_t := f_0^{-1}(c)$ *a smooth* K*-trivial surface for* $t \neq 0$*.*

Then there exists a birational morphism

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ & {\scriptstyle f_0}\searrow \quad \swarrow{\scriptstyle f} & \\ & C & \end{array}$$

such that

1. $\pi : Y \to X$ *is an isomorphism over* $f_0^{-1}(C \backslash \{0\})$*,*
2. X *is terminal, Gorenstein and factorial and* $\operatorname{Sing}(X) \subset \cup_i \operatorname{Sing}(V_i)$ *where* V_i *are the irreducible components of* X_0*;*
3. *each* V_i *has at most isolated canonical singularities and is smooth around the double curves,*
4. K_X *is* f*-trivial and every fiber* X_t *is* K*-trivial;*
5. (X, X_0) *is snc outside of* $\cup \operatorname{Sing}(V_i)$

A dlt model satisfying (2-5) will be called a ***weak Kulikov model****.*

Moreover if we assume that X_t *is an Abelian surface (Def. 4.4) then* $\cup_i \operatorname{Sing}(V_i) = \emptyset$ *(see Remark 3.15).*

Remark 3.8. A similar higher dimensional version of this statement was proved in [NXY19, Theorem 4.5]. The proof follows similar logical steps, ours is much quicker and simplified since we take advantage of working in lower dimensions by using Kawamata's result ([Kaw88, Lemma 5.1]), allowing to conclude the irreducible components of the central fiber are Cartier. This lemma does not hold in higher dimensions, the step described is in fact one of the main difficulties in the proof of the aforementioned result.

Let us now explain the proof of Theorem 3.6. First consider the following.

Lemma 3.9. *Let* X *be a normal 3-dimensional variety and* (X, Δ) *be a dlt pair (as in 1.2) such that* $\Delta = \sum_i V_i$ *with* V_i *distinct and irreducible and* $K_X + \Delta$ *Cartier. Then* V_i *have isolated canonical singularities and if*

$$D_i := \sum_{i \neq j, V_i \cap V_j \neq \emptyset} V_i \cap V_j$$

one has that $\operatorname{Supp} D_i$ *is contained in the smooth locus of* V_i*.*

Proof. First of all we remark that by Theorem 1.19, all V_i are normal and so are the irreducible components of D_i. This means in particular that V_i have isolated surface singularities and that the irreducible components of D_i are smooth curves.

Moreover, apply Corollary 1.21 to obtain that (V_i, D_i) are dlt and that $K_{V_i} + D_i$ is Cartier.

By definition of dlt pair there exists a closed subset $Z = \text{non-snc}(V_i, D_i) \subset V_i$ such that any divisor over Z has discrepancy > -1 and that $V_i \backslash Z$ is smooth and $D_i|_{V_i \backslash Z}$ is an snc pair. Note that as $K_{V_i} + D_i$ is Cartier and thus discrepancies are integers.

If Z contains a curve it will be contained in some curve of D_i, but then it is some curve of D_i, this is not possible as these are log canonical centers by 1.19. Suppose there is a canonical closed point $p \in \operatorname{Supp} D_i$ then it is regular by Proposition 1.16. Hence $\operatorname{Supp} D_i$ is in the smooth locus of V_i. □

We are now ready to prove the main theorem of the section.

Proof of theorem 3.6. By Lemma 3.9, the hypothesis that $\operatorname{Sing}(X) \subset \operatorname{Supp} \Delta$ and the fact that V_i are Cartier, we deduce that $\operatorname{Sing}(X) \subset \cup_i \operatorname{Sing}(V_i)$ and $\cup_i Sing(V_i)$ consists of isolated canonical singularities away from the intersection curves.

Let $Z = \text{non-snc}(X, \Delta)$. First observe that V_i, the intersection curves and the triple points are lc centers by Theorem 1.19. As before since V_i are lc centers by Theorem 1.19, so Z does not contain a codimension 1 subset. Moreover, there is no curve in Z as there are no singular curves on V_i and the intersection curves are lc centers. Hence Z contains just points, not the triple points as they are also lc centers, hence $\text{non-snc}(X, \Delta) = \cup_i \operatorname{Sing}(V_i)$ □

3.3 Step 3: A careful simultaneous resolutions of Du Val singularities.

From the work we have already done we obtained a *weak* Kulikov model. The only obstruction to obtain an honest Kulikov model is the remaining canonical singularities lying on the irreducible components of the special fiber. Recall that canonical surface singularities are Du Val singularities (see 1.1.1). These admit a simultaneous resolution after a base change after by Theorem 1.14. We do that exactly.

Around the 1 and 0 dimensional lc centers worse singularities are created by the base change. Nevertheless, this can be fixed via toric resolutions without affecting the canonical divisor so that it is once again snc. Thus we have one Du Val singularity less and retain all other hypothesis, doing this a finite amount of times yields the result.

Remark 3.10. As we noted in Remark 1.15 this resolution might not exist in the category of schemes hence in this section we only work complex analytic spaces.

One could however also perform it for algebraic spaces. In fact, in the category of algebraic spaces steps hold so one also obtains Theorem 3.12 for algebraic spaces.

Definition 3.11. A birational map $f : (X, \Delta_X) \dashrightarrow (Y, \Delta_Y)$ will be called crepant if $K_X + \Delta_X \sim f^*(K_Y + \Delta_Y)$ and $\Delta_Y = f_* \Delta_X$.

We finally have all the ingredients to complete the proof of Theorem 0.6, which we state again now.

Theorem 3.12. *Let $f_0 : Y \to C$ be a projective, flat, surjective morphism of complex analytic spaces. Such that*

1. *Y is a 3-dimensional terminal $\mathbb{Q}$-factorial variety and C a quasi-projective smooth curve,*
2. *(Y, Y_t) dlt for all $t \in C$, where $Y_t := f_0^{-1}(t)$ and*
3. *$Y_t := f_0^{-1}(c)$ a smooth K-trivial surface for $t \neq 0$.*

Then there exists a finite surjective base change $\pi : C' \to C$ and $p : X \to Y \times_C C'$ a birational morphism over C'

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \times_C C' \\ & {}_{f}\searrow \quad \swarrow & \\ & C' & \end{array}$$

such that $f : X \to C'$ is a Kulikov model (Def. 0.2).

Proof. By Theorem 3.2 and Theorem 3.6 it suffices to show we can solve all the isolated Du Val singularities, while retaining the snc locus snc.

Pick a Du Val singularity x there is a neighbourhood $x \in U \subseteq Y$ for which the hypothesis of Theorem 1.14 are satisfied. Hence there exists a finite and surjective base change $\pi : \tilde{C} \to C$ giving a Cartesian square:

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ {\scriptstyle h}\downarrow & \lrcorner & \downarrow{\scriptstyle f_0} \\ \tilde{C} & \xrightarrow[\pi]{} & C \end{array}$$

Notice that Z is h-trivial as the relative canonical divisor does not change for a finite surjective base change. Locally one finds a simultaneous resolution

$$\begin{array}{ccccc} \tilde{Z} & \xrightarrow{q} & V \subset Z & \xrightarrow{g} & x \in W \subset Y \\ {\scriptstyle \tilde{h}}\downarrow & & {\scriptstyle h}\downarrow & \lrcorner & \downarrow{\scriptstyle f_0} \\ \tilde{C} & \xrightarrow[=]{} & \tilde{C} & \xrightarrow[\pi]{} & C \end{array}$$

q gives a minimal resolution fiberwise, therefore q is a birational morphism and an isomorphism in codimension 1, thus $q^* K_Z \sim K_{\tilde{Z}}$. Globally we obtain

$$\begin{array}{ccccc} \tilde{Z} & \xrightarrow{q} & Z & \xrightarrow{g} & Y \\ {\scriptstyle \tilde{h}}\downarrow & & {\scriptstyle h}\downarrow & \lrcorner & \downarrow{\scriptstyle f_0} \\ \tilde{C} & \xrightarrow[=]{} & \tilde{C} & \xrightarrow[\pi]{} & C \end{array}$$

Now it is well known that $\tilde{Z}$ may have singularities arising from the effect of the base change in the lc locus. This is where the toric geometry will help us.

Claim. *There exists a crepant birational morphism*

$$\begin{array}{ccc} Z' & \xrightarrow{\quad j \quad} & \tilde{Z} \\ & {\scriptstyle h'}\searrow \quad \swarrow{\scriptstyle \tilde{h}} & \\ & \tilde{C} & \end{array}$$

such that it is an isomorphism outside of a neighborhood V_0 of the intersection locus of $\tilde{Z}_0$; its a log resolution of singularities over V_0 and $h' : Z' \to \tilde{C}$ is a flat, proper morphism.

Proof of Claim. The proof of this Claim is originally from [Fri83, Proposition 1.2].

Around a triple point call it 0 the base change is analytically isomorphic to

$$U = \operatorname{Spec} \mathbb{C}[x, y, z, t]/(xyz - t^n) \subset V_0$$

Recall that we saw in Example 2.9 exactly this toric variety. With embedded torus

$$T = \{(x, y, z, t) \in (\mathbb{C}^*)^4 / \, xyz = t^n\} \subset (\mathbb{C}^*)^4$$

The dual polyhedron of the cone corresponding to U is spanned by $(0, 0, 1), (n, 0, 1)$ and $(0, n, 1)$. To resolve its singularities we proceed like in Example 2.15. Subdivide σ as follows: add all rays of the form $\mathbb{R}^+(a, b, 1)$ with $a, b \in \mathbb{Z}$, $0 \leq a, b \leq n$, and all planes spanned by such two of the form

$$(a, b, 1) \text{ and } (a, b+1, 1); \text{ or } (a, b, 1) \text{ and } (a+1, b, 1); \text{ or } (a, b+1, 1) \text{ and } (a+1, b, 1)$$

Hence each σ_α is the convex hull of the rays generated by one of the following sets:

$$\sigma_\alpha = \text{ convex hull of } \mathbb{R}^+(a, b, 1) + \mathbb{R}^+(a+1, b, 1) + \mathbb{R}^+(a, b+1, 1)$$

or

$$\sigma_\alpha = \text{ convex hull of } \mathbb{R}^+(a, b+1, 1) + \mathbb{R}^+(a+1, b, 1) + \mathbb{R}^+(a+1, b+1, 1).$$

Let $\tilde{U} = U_\Delta$ where Δ is the fan corresponding to $\{\sigma_\alpha\}$. Then by Proposition 2.14

1. the induced morphism $\tilde{U} \to U$ is a refinement hence it is a proper birational morphism,

2. each σ_α is spanned by a $\mathbb{Z}$-basis of n which implies $\tilde{U}$ is smooth,

3. finally it is an isomorphism over

$$T = \{(x, y, z, t) \in (\mathbb{C}^*)^4 \backslash \, xyz = t^n\}$$

For example let σ_α be as in the first possibility, then the dual polyhedron $\sigma_\alpha^\vee \subset M_\mathbb{R}$ is generated by $(1,0,-a),(0,1,-b)$ and $(-1,1,a+b+1)$. Thus $\tilde{U}$ is obtained by gluing together the affine pieces:

$$U_{\sigma_\alpha} = \operatorname{Spec}\mathbb{C}[xt^{-a}, yt^{-b}, x^{-1}y^{-1}t^{a+b+1}]$$

Set $u_\alpha := xt^{-a}$, $v_\alpha := yt^{-b}$ and $w_\alpha := x^{-1}y^{-1}t^{a+b+1}$. Then $u_\alpha v_\alpha w_\alpha = t$, so that $t=0$ defines a snc divisor.

Away from the triple points, $\tilde{Z}$ has double curves which are locally defined by

$$\operatorname{Spec}\mathbb{C}[x,y,z,t]/(xy-t^n)$$

We already gave a toric resolution of these singularities in Example 2.15. Clearly it is compatible with the resolution we just defined on triple points. Thus all local resolutions along the intersection loci can be patched up together.

Hence we constructed a proper birational morphism $Z' \to \tilde{Z}$ such that it is a log resolution along the intersection loci and it's an isomorphism away from them.

The smooth locus stays smooth and the remaining Du Val singularities stay canonical since the base change does not change the relative canonical divisor.

There is only one thing left to check and that is that the morphism is crepant. For this observe as in [Fri83] that all the newly created components of $\tilde{Z}_0'$ are rational ruled surfaces and all cycles of double curves on them have anticanonical divisors it follows that the morphism is crepant. □

In conclusion, $(Z', (Z_0')$ satisfies being snc away from the Du Val singularities, has one Du Val singularity less than (Y, Y_0) and maintains K-triviality. Repeating this process to (Z', Z_0'), leads inductively to a finite surjective morphism $C' \to C$ and a birational morphism

$$\begin{array}{ccc} X & \xrightarrow{\quad p \quad} & Y \times_C C' \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & C' & \end{array}$$

and a flat proper morphism $f : X \to C'$. Now any terminal singularity of $(x \in X')$ must lie in the central fiber over the interior of the irreducible components V_i but as X_0 is a Cartier divisor. This does not occur since we resolved all singularities hence X is smooth, f-trivial and X_0 is an snc divisor thus we obtained a Kulikov model. □

As a corollary we obtain Theorem 0.1.

Corollary 3.13 (Kulikov-Persson-Pinkham Theorem). *Let $f : X \to C$ be a projective 1-parameter degeneration of smooth K-trivial surfaces over the complex disk. Then there is a finite and surjective base change $C' \to C$ and X' a smooth manifold fitting a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\quad p \quad} & X \times_C C' \\ & {\scriptstyle f'}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & C' & \end{array} .$$

Where p is a birational morphism which is an isomorphism outside the central fiber and $f' : X' \to \Delta'$ is a semistable degeneration with $K_{X'} \sim_{f'} \mathcal{O}_{X'}$.

Remark 3.14 (Limits of the MMP on complex analytic spaces)**.** A degeneration of $K3$ that does not admit a Kulikov model is known ([Nis88]). This degeneration satisfies that the central fiber contains non-algebraic complex analytic surfaces as irreducible component (so called Kodaira class VII surfaces).

One notices that in the proof of the theorem the hypothesis that the morphism is projective has only been used to run the MMP (Theorem 1.24). Hence this provides a new counterexample that the MMP cannot be established in general for compact complex manifolds. As if it did applying this method would give a contradiction.

Remark 3.15 (Canonical singularities for Kulikov models of Abelian surfaces.)**.** For a Kulikov model of Abelian surfaces, each irreducible component of the central fiber is a torus embedding (cf. [FM83, p. 22]). The interior of a toric variety is smooth (cf. [Ful93, Ch. 2]). This shows that Step 3 of the proof was unnecessary and that the weak Kulikov model from Theorem 3.7 was an honest Kulikov model.

4 Classification of the central fiber.

> mais on ne peut les y reconnaître individuellement que par le raisonnement, en laissant le champ libre à toutes les transformations possibles pendant la jeunesse jusquà la limite où ces formes reconstituées empiéteraient sur une autre individualité quil faut identifier aussi
>
> *À la recherche du temps perdu*, Marcel Proust

We now move on to prove the Theorem 0.3 both for degenerations of $K3$ surfaces and Abelian varieties. This is nothing new and was already carried out in [Kul77] and [Per77]. Nevertheless we give our own version of the proof which is more fleshed out, also partly rephrased in terms of log Calabi-Yau pairs. Furthermore for $K3$ surfaces the proof does not use Hodge theory at all.

For this section some basic knowledge about the classification of algebraic surfaces is assumed, the uninitiated reader might find useful either [Bea96] or [Bǎd01].

To begin our proof we first begin with a small introduction to log Calabi-Yau pairs.

Definition 4.1 (log Calabi-Yau)**.** Let (X, Δ) be a pair as in 1.2, we say that (X, Δ) is Calabi-Yau if $K_X + \Delta \sim_{\mathbb{Q}} \mathcal{O}_X$. Moreover, one says that (X, Δ) is log Calabi-Yau or logCY if it is log canonical and Calabi-Yau.

Lemma 4.2. *Let (X, Δ) be a dlt log Calabi-Yau pair, and suppose $\Delta = \sum_i V_i$ with V_i distinct integral divisors. Let $D_i := \sum_{i \neq j} D_i \cap D_j$, then (V_i, D_i) is dlt and logCY.*

Proof. By Corollary 1.19, one has that $D_i = \mathrm{Diff}_{V_i}(\Delta)$. Hence (V_i, D_i) is dlt and furthermore we have equation

$$(K_X + \Delta)|_{V_i} = K_{V_i} + D_i$$

but as (X, Δ) is logCY one deduces that $\mathcal{O}_{V_i} \sim_{\mathbb{Q}} K_{V_i} + D_i$ □

Lemma 4.3. *Let (V, D) be a dlt logCY surface pair, $D = \sum_i C_i$ with C_i irreducible curves. Assume $K_V + D$ is Cartier. Then either*

1. *V is an elliptically ruled surface with $q(V) = 1$*
2. *or V is rational surface.*

Proof. First notice that as in Example 1.35 that dlt surface pair with $K_V + D$ Cartier is snc in a neighborhood of $\operatorname{Supp} D$ and has at most canonical singularities away from these. Assume V is smooth with no (-1)-curves, if not by Theorem 1.16 one can replace V with its minimal resolution and contract the (-1)-curves and all arguments apply, as the canonical divisor does not change.

There are two options either $q(V) = 0$ or $q(V) > 0$. In the first case Castelnuovos rationality criterion (cf. [Bea96, Theorem V.1]) implies that V is rational. In the second case the Albanese morphism defines a genus 0 fibration and so it is birational to a $\mathbb{P}^1 \times C$, for C a smooth curve. Take the projection onto the first coordinate. A generic fiber would be isomorphic to C and so adjunction gives

$$0 \leq -K_V.F - F^2 = 2 - 2g(C).$$

since F and $-K_V$ are effective and $F^2 = 0$, we deduce that $g(C) = 0, 1$. But as $0 < q(V) = g(C)$ the only possibility is $g(C) = 1$ and thus V is an elliptically ruled surface.

□

Definition 4.4 (Abelian variety)**.** An abelian variety X is a proper smooth integral group scheme over a base field k. If $\dim(X) = 2$, it will be called Abelian surface.

One can prove that an Abelian surface satisfies that $\chi(X) = 0$ and $\omega_X \simeq \mathcal{O}_X$.

Lemma 4.5. *[Bǎd01, Prop. 13.5] Let X be a smooth projective surface and let $D = \sum E_i$ be a boundary such that $|D + K_X| = \emptyset$. Suppose that E_i are smooth intersecting transversally.*[7] *Then,*

1. *$q(X) \geq \sum_i p_a(E_i)$, where the p_a is the arithmetic genus.*
2. *The dual graph of D is a disjoint union of trees.*

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

Now take the corresponding LES

$$H^1(X, \mathcal{O}_X) \to H^1(D, \mathcal{O}_D) \to H^2(X, \mathcal{O}_X(-D))$$

By Serre duality, $|D + K_X| = \emptyset$ implies that $h^2(X, \mathcal{O}_X(-D)) = 0$, hence $H^1(X, \mathcal{O}_X) \to H^1(D, \mathcal{O}_D)$ is surjective.

On the other hand, one has the exact sequence

$$0 \to \mathcal{O}_D \xrightarrow{j} \prod_i \mathcal{O}_{E_i} \to \operatorname{coker}(j) \to 0$$

From this one deduces that $h^1(\mathcal{O}_D) \geq \sum h^i(\mathcal{O}_{E_i}) = \sum_i p_a(E_i)$ because $\operatorname{coker}(j)$ is supported in dimension 0.

As for (2). Recall that $H^1(X, \mathcal{O}_X) \cong T_e \operatorname{Pic}_X = T_e \operatorname{Pic}^0_X$, where Pic^0_X denotes the connected component containing the identity in the group scheme Pic_X. Likewise for D. From the LES above one obtains a surjective morphism

$$\operatorname{Pic}^0_X \to \operatorname{Pic}^0_D$$

hence Pic^0_D is an Abelian variety. If D contains a loop, then $Pic(D)$ contains a subgroup which is isomorphic to $\mathbb{G}_m$. This is not possible as Pic^0_D is a smooth Abelian variety. □

[7]This can actually be deduced from the previous hypothesis, but it is not important for us.

Lemma 4.6. *Let (V, D) be a dlt log Calabi-Yau surface pair, $D = \sum_i C_i$ with C_i distinct and irreducible. Assume $K_V + D$ is Cartier. Then either*

1. *V is an elliptically ruled surface with $q(V) = 1$*
   - *D consists of a single elliptic curve, or*
   - *D consists of two disjoint elliptic curves;*
2. *or V is rational surface and either*
   - *D consists of a single elliptic curve, or*
   - *$D = \sum_i C_i$ is a cycle of rational curves.*

Proof. By Lemma 4.3 one only needs to see check the specifics of case 1. and 2.

Suppose V is rational, if $D = C$ then by Lemma 4.2, $K_C \sim \mathcal{O}_C$, hence C is an elliptic curve. If D consists of more than one curve one defines $D_{i_0} = \sum_{i \neq i_0} C_i$ then $D_{i_0} + K_V = -C_{i_0}$. Hence by Lemma 4.5.(1) we deduce that $g(C_i) = 0$ for $i \neq i_0$, since one can choose any other curve conclude that $C_i \cong \mathbb{P}^1$ for all i. Furthermore, Lemma 4.5(2) implies that there is just 1 cycle of rational curves, because otherwise we would find non-trees when choosing C_{i_0} from another cycle.

Suppose V is elliptically ruled. If $D = C$ an integral curve the same arguments conclude that it is elliptic. Else we define again $D_{i_0} = \sum_{i \neq i_0} C_i$. By Lemma 4.5(1) D_{i_0} contains at most one elliptic curve and the rest must be rational curves, so D, contains at most 2 elliptic curves and the rest are rational.

Let C' be an elliptic curve then by the adjunction formula

$$0 = 2g(C') - 2 = K_V.C' + (C')^2 = -(D - C').C' = -\sum_{C_i \neq C'} C_i.C'$$

But the number on the right is minus the number of points intersecting with C'. Hence C' is disjoint to other curves.

We show now that no rational curves can occur in D and consequently D consists of 2 elliptic curves. There cannot be an isolated rational curve since adjunction would prove it's K-trivial. Suppose then that there is a cycle of rational curves. As $\kappa(V) = -\infty$ V admits a ruling $V \to E$. Adjunction formula gives

$$-K_V.F = 2$$

But a fiber along a ruling is rational and the base is an elliptic curve. This would imply that for some rational curve $C_i.F > 0$, so it would define a section which dominates E a curve with higher genus. A contradiction. □

Now we have a good idea of what surface dlt logCY pairs look like. The only ingredient left for the classification of Kulikov models is to be able to relate the geometry of a smooth fiber to the geometry of the special fiber.

Lemma 4.7. *Let $f : X \to C$ be a flat proper morphism, with X a 3 dimensional variety such that $X_0 = f^{-1}(0)$ is an snc divisor with irreducible components V_i, double curves C_{ij} and triple points T_{ijk} then*

$$\chi(X_t) = \chi(X_0) = \sum_i \chi(V_i) - \sum_{i<j} \chi(C_{ij}) + \sum_{i<j<k} \chi(T_{ijk}).$$

Proof. Now as X is flat over the base we have constancy of the Hilbert polynomials and thus

$$\chi(X_t) = \chi(X_0) = \chi(X_0).$$

On the other hand, as X_0 is an snc divisor the Mayer-Vietoris spectral sequence gives that

$$\chi(X_0) = \sum_i \chi(V_i) - \sum_{i<j} \chi(C_{ij}) + \sum_{i<j<k} \chi(T_{ijk}).$$

□

We are now ready to give the classification of the central fiber of a Kulikov model. We note that the only K-trivial smooth projective surfaces are $K3$ surfaces and Abelian surfaces (cf.[Bea96]), hence we only classify those degenerations.

Definition 4.8 ($K3$ surface)**.** A $K3$ surface X over k is a smooth variety of dimension 2 such that

$$\omega_X \simeq \mathcal{O}_X \text{ and } H^1(X, \mathcal{O}_X).$$

One easily deduces from this definition that $\chi(X) = 2$.

Theorem 4.9. *Let $X \to C$ be a weak Kulikov model as in Theorem 3.7. Then if the generic fiber is a $K3$ surface (resp. an Abelian surface). Then the degenerate fiber X_0 must be one of the following 3 types:*

I. $X_0 = V_1$ is a K3 surface (resp. Abelian surface) with canonical singularities .

II. $X_0 = V_1 + ... + V_n$, where V_1 and V_n are rational surfaces and $V_2, ..., V_{n-1}$ are elliptic ruled surfaces so that $q(V_i) = 1$, $i = 2, ..., n-1$ (resp. all V_i are elliptically ruled surfaces, $q(V_i) = 1$). The dual complex is a chain (resp. a cycle) and the double curves $C_{1,2}, ..., C_{n-1,n}$ are elliptic curves (resp. $C_{1,2}, ..., C_{n,1}$ are elliptic curves).

III. $X_0 = V_1 + ... + V_n$, where all V_i are rational surfaces; the double curves $C_{i,j}$ are rational and form a cycle on each of the surfaces V_i. The topological realisation of the dual complex is homeomorphic to the triangulation of S^2 (resp. of $S^1 \times S^1$).

Proof. Note that as the generic fiber is K-trivial it contains no (-1)-curves and as (-1)-curves are deformation invariant the special fibers cannot contain them either.

Assume that all components of X_0 are smooth. In another case they have at most canonical singularities away from the intersection curves. Hence one would take a minimal resolution of the canonical singularities $\mu : \tilde{X}_0 \to X_0$ with $K_{\tilde{X}_0} \sim \mu^* K_{X_0}$ as in Theorem 1.16. Hence,

$$K_{\tilde{X}_0} = \mu^* K_{X_0} = \mu^* \mathcal{O}_{X_0} = \mathcal{O}_{\tilde{X}_0}$$

moreover canonical singularities are rational (cf. [KM98]) , i.e. $R^i\mu_*\mathcal{O}_{\tilde{X}_0} = \mathcal{O}_{X_0}$. Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q\mu_*\mathcal{O}_{\tilde{X}_0}) \Longrightarrow H^{(p+q)}(\tilde{X}_0, \mathcal{O}_{\tilde{X}_0})$$

hence $H^p(\tilde{X}_0, \mathcal{O}_{\tilde{X}_0}) = H^p(X_0, \mathcal{O}_{X_0})$ for all p. In particular $\chi(\tilde{X}_0) = \chi(X_0)$.

Thus we may assume that all components of X_0 are smooth. Let C_{ij} and T_{ijk} the double intersection curves and triple intersection points. By Lemma 4.7 we have

$$\chi(X_t) = \chi(X_0) = \sum_i \chi(V_i) - \sum_{i<j} \chi(C_{ij}) + \sum_{i<j<k} \chi(T_{ijk}).$$

There are now two options either X_0 contains one irreducible component or it contains more than one. In the second case, notice that (X, X_0) is an snc Calabi-Yau pair, hence by Lemma 4.2, $(V_i, D_i := \sum_{j\neq i; V_i\cap V_j\neq\emptyset} V_i \cap V_j)$ is also snc Calabi-Yau. Then Lemma 4.6 gives all the possibilities for each pair (V_i, D_i). We can now give a classification.

Type I. If the central fiber consists of a single irreducible component then by the classification of algebraic surfaces K-triviality and the Euler characteristic uniquely determine the surface. Hence it is a $K3$ surface (resp. Abelian surface) if the general fiber is a $K3$ surface (resp. Abelian surface).

Type II If no rational cycles occur in any components then there are no triple points. The double curves are elliptic and components are elliptically ruled with $q(V) = 1$, both with trivial $\chi(-)$, so they do not contribute to the Euler characteristic.

1. If X_t is a $K3$ surface then $\chi(X_t) = 2$ there are exactly two rational surfaces that admit just one elliptic curve, so we get a chain of surfaces $V_1, ..., V_n$, with V_1, V_n rational and $V_2, ..., V_{n-1}$ elliptically ruled $n \geq 2$.

2. If X_t is an abelian surface then $\chi(X_t) = 0$, we get a chain or a cycle of elliptically ruled surfaces. The first possibility is ruled out using Hodge theory (cf.[Per77]).

Type III If there is a cycle of rational curves in the central fiber then all components are rational with double curves being rational curves forming a cycle. From this we see that $\Pi(X_0)$ is a triangulation of a compact surface without border.

1. If X_t is a $K3$ surface then $\chi(X_t) = 2$, thus we conclude that $\Pi(X_0)$ gives a triangulation of S^2.

2. If X_t is an abelian surface then $\chi(X_t) = 0$, thus we conclude that $\Pi(X_0)$ gives a triangulation of either to $S^1 \times S^1$ or $\mathbb{R}P^2 \# \mathbb{R}P^2$. The second possibility is ruled out using Hodge Theory (cf.[Per77]).

□

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