NONLOCAL DENSITY CORRELATIONS AS A SIGNATURE OF HAWKING RADIATION FROM ACOUSTIC BLACK HOLES IN BOSE-EINSTEIN CONDENSATES: THE ANALOGY
Part 2

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Summary

- QFT calculation of density-density correlation function for a BEC analogue of a black hole

- Take into account the effective potential for the quantum field

- Discuss the effects of
  - an infrared cutoff
  - including the potential

Effective metric

- Lab frame

$$ds^2 = \frac{n}{mc}\left[-c^2 dT^2 + (d\vec{x} - \vec{v}dt)\cdot(d\vec{x} - \vec{v}dt)\right]$$

- Let $\vec{v} = -v\hat{x}$

- Transformation to static spherically symmetric coordinates

$$dT = dt + \frac{v}{c^2 - v^2}dx$$

$$ds^2 = \frac{n}{m}\left[-\frac{(c^2 - v^2)}{c}dt^2 + \frac{c}{c^2 - v^2}dx^2 + \frac{dy^2 + dz^2}{c}\right]$$

Solutions to the Gross Pitaevskii Equation

- Background Solution

$$\begin{aligned}\hat{\Psi} &= e^{i\hat{\theta}}\sqrt{\hat{n}} \\ \hat{n} &= n + \hat{n}_1 \\ \hat{\theta} &= \theta + \hat{\theta}_1\end{aligned}$$

with

$$\begin{aligned}n &= \text{constant} \\ \vec{v} &= \frac{\hbar}{m}\vec{\nabla}\theta = -v\hat{x} \\ v &= \text{constant}\end{aligned}$$

- Linearized equation in 3+1 dimensions

$$\begin{aligned}\Box\widehat{\theta}_1 &= 0 \\ \widehat{n}_1 &= -\frac{\hbar}{g}\left(\partial_T\widehat{\theta}_1 + \frac{\hbar}{m}\vec{\nabla}\theta\cdot\vec{\nabla}\widehat{\theta}_1\right)\end{aligned}$$

- Dimensional reduction to 1+1 dimensions

$$\widehat{\theta}_1 = \widehat{\theta}_1^{(2)}\sqrt{\frac{mc}{\hbar n \ell_\perp^2}}$$

$$(\Box^{(2)} + V)\widehat{\theta}_1^{(2)} = 0$$

If $c = c(x)$, then with $dx^* \equiv dx(c^2 - v^2)/c$

$$\begin{aligned}\Box^{(2)} &= \frac{m}{n}\frac{c}{c^2 - v^2}(-\partial_t^2 + \partial_{x^*}^2) \\ V &= \frac{m}{n}\left[\frac{c''}{2}\left(1 - \frac{v^2}{c^2}\right) - \frac{c'^2}{4c} + \frac{5}{4}\frac{v^2}{c^2}c'^2\right]\end{aligned}$$

$$(-\partial_t^2 + \partial_{x^*}^2) + \frac{c^2 - v^2}{c} \frac{n}{m} V \hat{\theta}_1^{(2)} = 0$$

$$V = \frac{m}{n} \left[\frac{c''}{2} \left(1 - \frac{v^2}{c^2} \right) - \frac{c'^2}{4c} + \frac{5}{4} \frac{v^2}{c^2} c'^2 \right]$$

- If the term containing V is zero then solutions are

$$\begin{aligned} \hat{\theta}_1^{(2)} &\sim e^{-i\omega u} \text{ and } e^{-i\omega w} \\ u &= t - x^* \\ w &= t + x^* \end{aligned}$$

- These solutions can be used to define the Unruh state

Unruh State

- Use positive frequency in static spherically symmetric time, t, for left moving modes at $\mathcal{I}^-$

- Define Kruskal coordinates and use positive frequency in Kruskal time for right moving modes on the past horizon

- Then use a Bogolubov transformation to write in terms of $e^{\pm i\omega u}$ on the past horizon

- If left-moving modes are ignored, the two-point function for $\widehat{\theta}_1^{(2)}$ if $x_\ell < 0$ and $x_r > 0$ is

$$\langle\widehat{\theta}_1^{(2)}(t_\ell, x_\ell)\widehat{\theta}_1^{(2)}(t_r, x_r)\rangle =$$

$$\frac{1}{8\pi}\int_0^\infty d\omega \frac{1}{\omega \sinh\left(\frac{\pi\omega}{\kappa}\right)}\left[e^{-i\omega(t_r - t_\ell)}\chi_\omega(x_\ell)\chi_\omega(x_r)\right.$$

$$\left.+e^{i\omega(t_r - t_\ell)}\chi_\omega^*(x_\ell)\chi_\omega^*(x_r)\right]$$

$$\frac{d^2\chi_\omega}{dx^{*2}} + \omega^2\chi_\omega + V_{\text{eff}}\chi_\omega = 0$$

$$V_{\text{eff}} = \frac{c^2 - v^2}{c}\frac{n}{m}V(x)$$

- Potential should be important for modes with $\omega \lesssim |V_{\text{eff}}|_{\text{max}}$

Density-Density Correlation Function

$$G_2(x_\ell, x_r) = \langle \hat{n}(x_\ell)\hat{n}(x_r)\rangle - \langle \hat{n}(x_\ell)\rangle\langle \hat{n}(x_r)\rangle$$

$$= \frac{\hbar n}{m\ell_\perp^2 c^2(x_\ell)c^2(x_r)}$$

$$\times \mathcal{D}\left(\sqrt{c(x_\ell)c(x_r)}\langle \hat{\theta}_1^{(2)}(t_\ell, x_\ell)\hat{\theta}_1^{(2)}(t_r, x_r)\rangle\right)$$

$$\mathcal{D} = \partial_{T_\ell}\partial_{T_r} - v\partial_{x_\ell}\partial_{T_r} - v\partial_{T_\ell}\partial_{x_r} + v^2\partial_{x_\ell}\partial_{x_r}$$

Recall

$$dT = dt + \frac{v}{c^2 - v^2}dx$$

Numerical Computations

- Same parameters as the QM calculation by Carusotto, Fagnocchi, Recati, Balbinot, and Fabbri

- Slight differences:

  Horizon is at $x = 0$

  Black hole is to left of horizon and condensate moves to the left

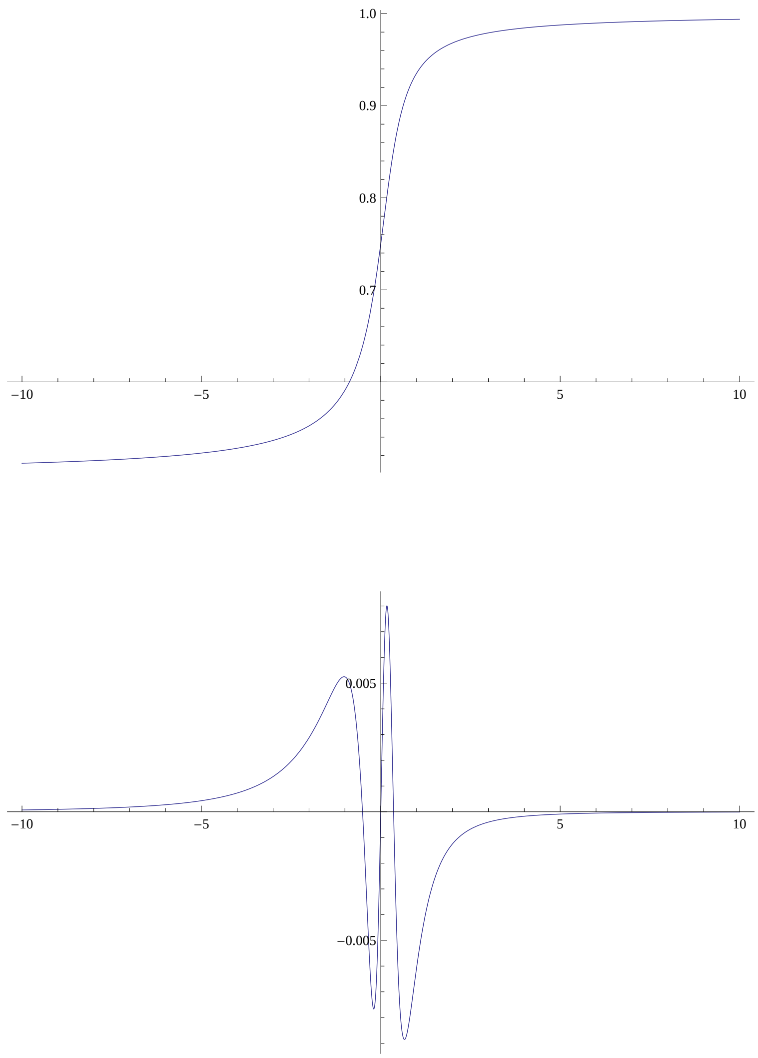
Values

- $\hbar = \xi = c_\ell = 1$
- $v = 3/4$, $c_r = 1/2$
- $\sigma_x = \frac{1}{2}$
- Sound speed is

$$c(x) = \sqrt{c_\ell^2 + \frac{1}{2}\left(c_r^2 - c_\ell^2\right)\left[1 + \frac{2}{\pi}\tan^{-1}\left(\frac{x}{\sigma_x} - 2 + \sqrt{3}\right)\right]}$$

- Surface gravity

$$\begin{aligned}\kappa &= \left[\frac{1}{2}\frac{d}{dx}\left(c(x) - \frac{v^2}{c(x)}\right)\right]_{x=0} \\ &= c'(0) = \frac{1}{8\pi\sigma_x(2-\sqrt{3})} \approx 0.30\end{aligned}$$



The top plot is c (x) and the bottom plot is Veff (x)

Effect of the Potential on the Modes

- Recall the mode equation is

$$\frac{d^2\chi_\omega}{dx^{*2}} + \omega^2\chi_\omega + V_{\mathrm{eff}}\chi_\omega = 0$$

$$V_{\mathrm{eff}} = \frac{c^2 - v^2}{c}\frac{n}{m}V(x)$$

- Expect no significant effect for $\omega^2 \gg |V_{\mathrm{eff}}|_{\mathrm{max}}$

- This discussion on initial conditions for the spatial modes is revised from the original talk. A mistake for the modes which initially go into the region outside the event horizon was pointed out by R. Parentani during the talk and in subsequent discussions. It is corrected here.

  - For solutions on the past horizon which initially go into the region outside the event horizon

    * Fix the behavior so they are right moving in the limit $x \to \infty$

    $$\chi_\omega = e^{i\omega u}$$

    * Fix the amplitude so that the right moving modes originating from the past horizon have unit amplitude on that horizon.

- For the solutions on the past horizon which are always inside the event horizon, fix the behavior so that

$$\chi_\omega = e^{-i\omega u}$$

on the past horizon.

- For $\omega = 0$ nonzero V_{eff} implies that at large $|x|$

$$\chi_0 \rightarrow 1 + ax^*$$

Cutoff

- Recall the 2-point function is

$$\langle \hat{\theta}_1^{(2)}(t_1, x_1) \hat{\theta}_1^{(2)}(t_2, x_2) \rangle =$$

$$\frac{1}{8\pi} \int_0^\infty d\omega \frac{1}{\omega \sinh\left(\frac{\pi\omega}{\kappa}\right)} \left[e^{-i\omega(t_2 - t_1)} \chi_\omega(x_1) \chi_\omega(x_2) \right.$$

$$\left. + e^{i\omega(t_2 - t_1)} \chi_\omega^*(x_1) \chi_\omega^*(x_2) \right]$$

- In $V_{\text{eff}} = 0$ case Balbinot, et. al. subtracted from the two-point function

$$\frac{\kappa}{8\pi^2} \int_0^\infty \frac{d\omega}{\omega^2}$$

- Won't work for $V_{\text{eff}} \neq 0$ so need to impose a cutoff

$$\frac{1}{8\pi} \int_{\omega_c}^\infty d\omega \frac{1}{\omega \sinh\left(\frac{\pi\omega}{\kappa}\right)} \ldots$$

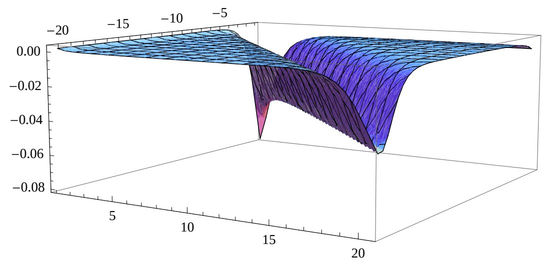
- In principle fix by setting $\lambda_c = 2\pi c/\omega_c \sim 2\pi c_r/\omega_c \sim$ the size of the system

- Effective UV cutoff scale is

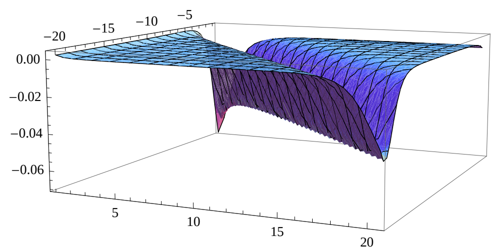
  $\lambda_{uv} \sim 2\pi^2 c/\kappa \sim 60$

  Need $\lambda_c \gg \lambda_{uv}$

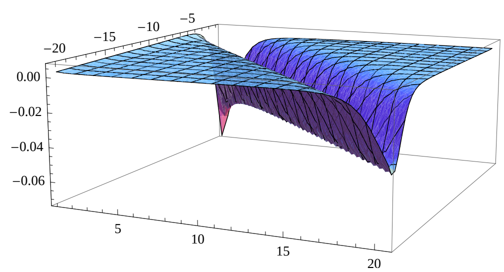
- Specific cases tried: $\lambda_c \approx 3000$, 300



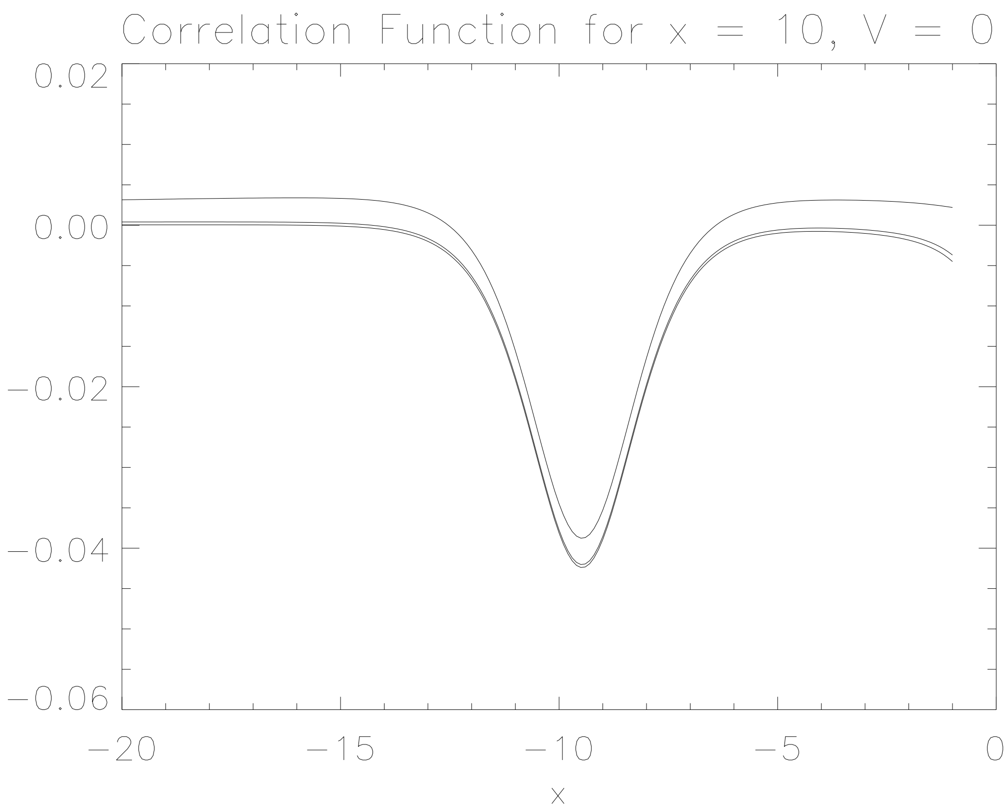
Regularization using the subtraction term



Regularization using a cutoff of ω = 0.002



Regularization using a cutoff of ω = 0.02



- Lowest curve at the minimum: Regularization via the subtraction term

- Next lowest curve: Regularization with a cutoff $\omega_c = 0.002$, $\lambda_c \sim 3000$

- Upper curve: Regularization with a cutoff $\omega_c = 0.02$, $\lambda_c \sim 300$

Explanations

- Recall

$$G_2(x,x') = \frac{\hbar n}{m\ell_\perp^2 c^2(x)c^2(x')}$$
$$\times\mathcal{D}\left(\sqrt{c(x)c(x')}\langle\hat{\theta}_1^{(2)}(t_1,x_1)\hat{\theta}_1^{(2)}(t_2,x_2)\rangle\right)$$

$$\mathcal{D} = \partial_T\partial_{T'} - v\partial_x\partial_{T'} - v\partial_t\partial_{x'} + v^2\partial_x\partial_{x'}$$

- For $V=0$

$$\langle\hat{\theta}_1^{(2)}(t_1,x_1)\hat{\theta}_1^{(2)}(t_2,x_2)\rangle \;=\; \frac{1}{8\pi}\int_{\omega_c}^{\infty} d\omega \frac{\cos(\omega\Delta u)}{\omega\sinh\left(\frac{\pi\omega}{\kappa}\right)}$$

  - For $\Delta u = 0$ high frequency modes contribute more so cutoff is less important

  - Infrared effects most important for term $\sim c'(x_1)c'(x_2)\langle\hat{\theta}_1^{(2)}(t_1,x_1)\hat{\theta}_1^{(2)}(t_2,x_2)\rangle$
    For $c'(x) \ll \omega_c$ this term is small

Conclusions

- Including the potential V requires implementation of a cutoff ω_c

- For physically motivated values find relatively weak dependence of peak of correlation function on the cutoff

  Cutoff dependence is stronger farther from the peak

Future Work

- Consider the case when both points are inside the horizon - QM calculation shows an interesting effect

- Investigate what happens when the event horizon forms in the "sudden" approximation