Quantum measuring processes for trapped ultracold atoms

Roberto Floreanini

INFN, Sezione di Trieste

Work done in collaboration with:
S. Anderloni, F. Benatti,
G.G. Guerreschi, A. Trombettoni

Outline

- The behaviour of ultracold atoms in optical lattices is analyzed by means of absorption images of the atomic sample after trap release
- The density profile obtained by averaging over multiple shots is usually intended as representing the mean value of the density operator in the given many-body state
- An alternative interpretation is instead based on coherente states and makes use of a generalized quantum measure (POVM))
- This observation might have experimental relevance

Double-well potential

In a suitable approximation, the dynamics of cold atoms in an double-well potential can be described by a two-mode Bose-Hubbard hamiltonian:

$$H = E\left(a_1^\dagger a_1 + a_2^\dagger a_2\right) + U\left((a_1^\dagger a_1)^2 + (a_2^\dagger a_2)^2\right) - T\left(a_1^\dagger a_2 + a_2^\dagger a_1\right)$$

- Trapping potential term $\propto E$;
- On-site boson-boson repulsive interaction term $\propto U$
- Hopping term $\propto T$;

The total number N of particles is conserved: the Hilbert space is thus $(N+1)$-dimensional.

Number states

The $N+1$-dimensional Hilbert space can be spanned by Fock states

$$|k,\, N-k\rangle = \frac{(a_1^\dagger)^k (a_2^\dagger)^{N-k}}{\sqrt{k!(N-k)!}}|0\rangle$$

with k particles in the first well and $N-k$ in the second.

Number states are:

- Orthonormal: $\langle k,\, N-k|k',\, N-k'\rangle = \delta_{kk'}$
- Complete: $\sum_{k=0}^{N} |k,\, N-k\rangle\langle k,\, N-k| = \mathbb{1}_{N+1}$

Phase states

Alternatively one can introduce coherent-like states

$$|N;\,\varphi,\,\xi\rangle = \frac{1}{\sqrt{N!}}\left(\sqrt{\xi}\,e^{i\varphi/2}a_1^\dagger + \sqrt{1-\xi}\;\;e^{-i\varphi/2}a_2^\dagger\right)^N|0\rangle$$

in which all N particles are in a coherent superposition, with definite

- relative phase $\varphi \in [0, 2\pi]$,
- mean occupation number $\xi \in [0, 1]$,

$$\langle N;\varphi,\xi|a_1^\dagger a_1|N;\varphi,\xi\rangle = N\xi \qquad \langle N;\varphi,\xi|a_2^\dagger a_2|N;\varphi,\xi\rangle = N(1-\xi)$$

Phase states

Phase states are:

- Normalized,

$$\langle N; \varphi, \xi | N; \varphi, \xi \rangle = 1\,;$$

- Near-orthogonal (for large N),

$$\langle N; \varphi, \xi | N; \varphi', \xi' \rangle \approx 0 \text{ unless } \varphi = \varphi',\ \xi = \xi';$$

- Overcomplete,

$$\int_0^1 d\xi \int_0^{2\pi} \frac{d\varphi}{2\pi} |N; \varphi, \xi\rangle\langle N; \varphi, \xi| = \frac{1}{N+1} \mathbb{1}_{N+1}\,.$$

Quantum phase transition in optical lattice

The Bose-Hubbard Hamiltonian describes a cross-over between a superfluid and insulator pahse, driven by the ratio T/U:

$T/U \ll 1$

Insulator phase,
the ground state is a Fock state

$$|MI\rangle \sim |N/2,\, N/2\rangle$$

$T/U \gg 1$

Superfluid phase,
the ground state is a coherent state

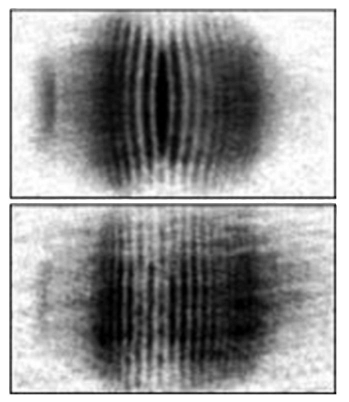
$$|SF\rangle \sim |N;\varphi,1/2\rangle$$

The measuring process

In general, it is very hard to perform density measurements inside the trapping potential. An indirect procedure is then adopted:

- The trapping potential is switched off
- The atoms expand freely (no interaction)
- The two fractions of condensate once contained in the two wells can overlap and interfere with each other
- After a certain time t, the expanding cloud is illuminated and the corresponding absorption image collected; it shows interference fringes, irrespectively from the system initial state
- The average density profile is obtained by superimposing many absorption images

Absorption images



from M.R. Andrews *et al.*, Science **275** (1997) 637

The many-body model

Introduce a complete set of single-particle atom states

$$\{|w_i\rangle\}_{i=1}^{\infty}\,, \qquad |w_i\rangle = a_i^\dagger|0\rangle$$

The bosonic creation operator can then be decomposed as

$$\psi^\dagger(x) = \sum_i w_i^*(x)\, a_i^\dagger$$

$$[a_i^\dagger,\, a_j] = \langle w_i|w_j\rangle = \delta_{ij}$$
$$[\psi^\dagger(x),\, \psi(y)] = \delta(x-y)$$

where $w_i(x) = \langle x|w_i\rangle$ are the corresponding wavefunctions

The free evolution after trap release is described by a unitary operator U_t:

$$|w_i\rangle \to |w_i(t)\rangle = U_t|w_i\rangle$$
$$|w_i(t)\rangle := a_i^\dagger(t)|0\rangle\,, \qquad a_i^\dagger(t) \equiv U_t\, a_i^\dagger\, U_t^\dagger$$

Density profiles after free expansion

At the time of trap release, prepare the system in the condensed state $|N;\varphi,\xi\rangle$; then, using

$$\psi(x)|N;\varphi,\xi,t\rangle = \sqrt{N}\Big(\sqrt{\xi}\,e^{i\frac{\varphi}{2}}\,w_1(x,t)+\sqrt{1-\xi}\,e^{-i\frac{\varphi}{2}}\,w_2(x,t)\Big)|N-1;\varphi,\xi,t\rangle$$

the average of the density operator $n(x)=\psi^\dagger(x)\psi(x)$ at time t will be given by

$$\begin{aligned}\langle n(x,t)\rangle_{\varphi,\xi} \;=\; & N\Big[\xi\,|w_1(x,t)|^2+(1-\xi)\,|w_2(x,t)|^2\\ & +2\sqrt{\xi(1-\xi)}\,\Re e\big(w_1(x,t)w_2^*(x,t)\,e^{i\varphi}\big)\Big]\end{aligned}$$

showing the expected interference fringes, modulated as

$$\Re e\big[w_1(x,t)w_2^*(x,t)\,e^{i\varphi}\big] \propto \cos\Big(\frac{md}{t}x+\varphi\Big)$$

where d is the distance between the wells, while m is the atom mass

On the other hand, preparing the system in a number state $|k,\, N-k, t\rangle$, one gets

$$\begin{aligned} \langle n(x,t)\rangle_k &= \langle k,\, N-k, t|\, \psi^\dagger(x)\psi(x)\, |k,\, N-k, t\rangle \\ &= k\,|w_1(x,t)|^2 + (N-k)\,|w_2(x,t)|^2 \end{aligned}$$

and no interference fringes should be observed

Nevertheless, in actual data one notices:

Experimental results

- every one-shot image shows a density profile compatible with that of a phase state, *i.e.* $\langle n(x,t)\rangle_{\varphi,\xi}$, the better, the larger N is
- the space between fringes is the same in each shot, but the offset (given by the value of relative phase φ) changes randomly from image to image, unless one already starts with $|N;\varphi,\xi\rangle$

The theoretical interpretation

- In quantum mechanics, mean values refer to statistical averages over many experimental runs; and indeed, superimposing multiple shots, the interference fringes disappear
- However, for large N, one can assimilate ensamble abverages with mean values with respect to macroscopically occupied many-body states
- The observation that the experimentally obtained one-shot density profiles reproduce the mean $\langle n(x,t)\rangle_{\varphi,\xi}$ even starting with a number state $|k, N-k\rangle$ suggests an interpretation in terms of a quantum generalized measure

Experimental average

After collecting $\mathcal{N}$ single shots, all obtained starting from the same initial number state $|k, N-k\rangle$, the experimental average density profile is obtained through

$$\overline{n(x,t)} = \sum_{(\varphi_i,\xi_i)} \frac{\mathcal{N}(\varphi_i,\xi_i)}{\mathcal{N}} \langle n(x,t)\rangle_{\varphi_i,\xi_i}$$

where $\mathcal{N}(\varphi_i,\xi_i)$ enumerates the number of times a pair (φ_i,ξ_i) with the corresponding pattern $\langle n(x,t)\rangle_{\varphi_i,\xi_i}$ is obtained

A natural theoretical prediction for the weights $\mathcal{N}(\varphi_i,\xi_i)/\mathcal{N}$ is, for sufficiently large $\mathcal{N}$, given by the overlap probabilities

$$\frac{\mathcal{N}_k(\varphi,\xi)}{\mathcal{N}} = |\langle\varphi,\xi;N|k,N-k\rangle|^2 = \binom{N}{k}\xi^k(1-\xi)^{N-k}$$

Quantum generalized measure

For sufficiently large number $\mathcal{N}$ of single-shot picuters, the average density is thus given by

$$\begin{aligned} \overline{n_k(x,t)} &= \int_0^1 \mathrm{d}\xi \int_0^{2\pi} \mathrm{d}\varphi \; |\langle \varphi, \xi; N | k, N-k \rangle|^2 \; \langle n(x,t) \rangle_{\varphi,\xi} \\ &= \mathrm{Tr}\big[\rho'_k(t)\, n(x)\big] \end{aligned}$$

where the transformed density matrix ρ'_k is obtained from the initial one $\rho_k = |k, N-k\rangle\langle k, N-k|$ through the action of the map

$$\rho_k \to \rho'_k = \int_0^1 \mathrm{d}\xi \int_0^{2\pi} \mathrm{d}\varphi \; P_{\varphi,\xi} \; \rho_k \; P_{\varphi,\xi} \qquad P_{\varphi,\xi} = |\varphi, \xi; N\rangle\langle \varphi, \xi; N|$$

The set $\Big\{P_{\varphi,\xi}\Big\}_{(\varphi,\xi)}$ form a *Positive Operator Valued Measure* (POVM), that generalizes the von Neumann projective measure

Remarks

- The average $\overline{n_k(x,t)}$ in general differs from the expression $\langle n(x,t)\rangle_k$ of the mean density usually adopted to fit experimental data:

$$\langle n(x,t)\rangle_k = k|w_1(x,t)|^2 + (N-k)|w_2(x,t)|^2$$

$$\overline{n_k(x,t)} = \frac{N}{N+2}\Big[(k+1)|w_1(x,t)|^2 + (N-k+1)|w_2(x,t)|^2\Big]$$

- Although the difference becomes of order one for large N

$$\overline{n_k(x,t)} - \langle n(x,t)\rangle_k \approx |w_1(x,t)|^2 - |w_2(x,t)|^2 + O\left(\frac{1}{N}\right)$$

  it is suppressed by a factor $1/N$ with respect to the dominant contribution $N|w_1|$

- For the state $|N/2,\, N/2\rangle$ the two expression coincide

Density correlations in optical lattices

When the N atoms are confined in a one-dimensional lattice with M sites, it is more convenient to look at density-density correlations $n(x,x')$ as averages of the two-point operator

$$\psi^\dagger(x)\,\psi^\dagger(x')\,\psi(x)\,\psi(x')$$

After integration with respect to the barycenter coordinate, $R=(x+x')/2$, and a suitable normalization, one is lead to study the behaviour of the following observable:

$$\mathcal{G}(r,t) \equiv \frac{\int \mathrm{d}R\ n(R-\frac{r}{2}, R+\frac{r}{2}, t)}{\int \mathrm{d}R\ n(R-\frac{r}{2}, t)\ n(R+\frac{r}{2}, t)}$$

It measures the conditional probability of finding two atoms at points separated by a distance r, averaged over all positions; in absence of correlations, it takes a constant value equal to one

Density correlations averages

When the system is initiallly prepared in a number state $|k_1, k_2, \ldots, k_M; N\rangle$, with k_i representing the occupation number of the i-th site, the generalized quantum measure based on the POVM gives

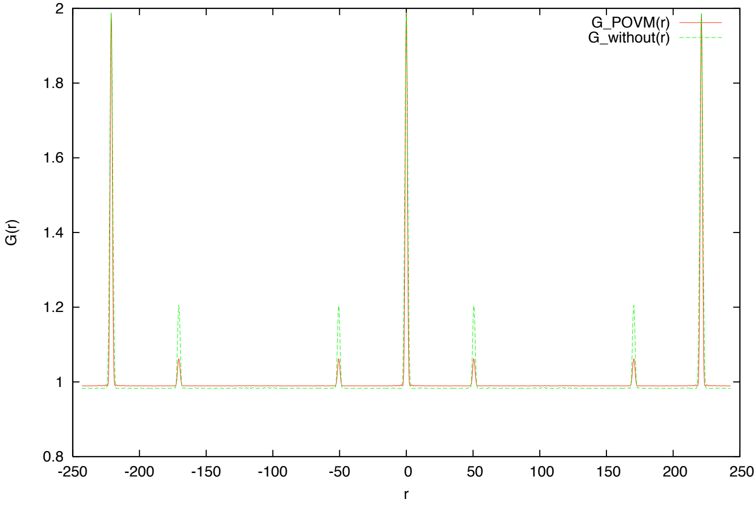
$$\overline{\mathcal{G}_{\vec{k}}(r,t)} = \frac{N(N-1)}{N^2}\left\{1 + \frac{1}{(N+M)(N+M-1)}\sum_{i\neq j=1}^{M}(k_i+1)(k_j+1)e^{iQ(i-j)r}\right\}$$

with $Q = md/t$, while the standard trace formula would yield

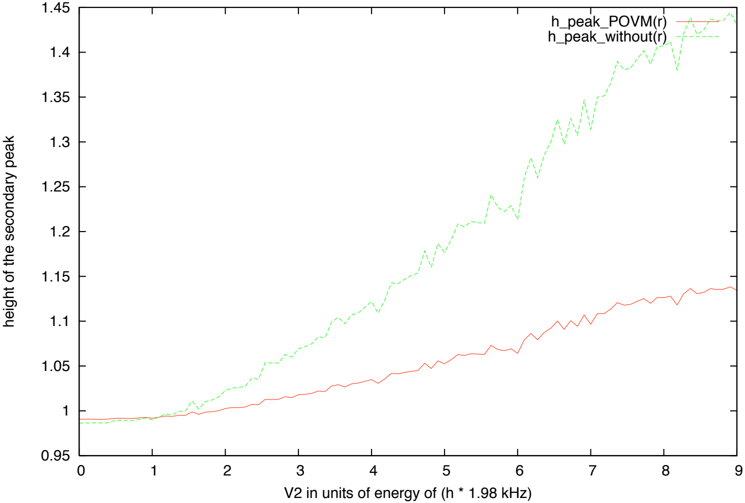
$$\langle\mathcal{G}(r,t)\rangle_{\vec{k}} = \frac{N(N-1)}{N^2}\left\{1 \;+\; \frac{1}{N(N-1)}\sum_{i\neq j=1}^{M}k_i k_j e^{iQ(i-j)r}\right\}$$

Using a bicromatic lattice to fill the M sites with unequal number of atoms, the two averages are seen to give different predictions

Simulations: $\overline{\mathcal{G}_{\vec{k}}}$ vs $\langle \mathcal{G} \rangle_{\vec{k}}$



Height of secondary peak



Outlook

- Density profiles obtained superimposing absorption images obtained after the release of the confining optical lattice can be theoretically described in terms of a generalized quantum measure based on coherent-like states
- This result can be naively understood by interpreting the formation of the absorption image as the result of the interaction of the system with a classical, macroscopic measuring apparatus: many atoms concur to the formation of a single pixel in the image and this is possible only if all atoms are in a same coherent superposition
- Coherent states are much more stable against the decohering effects due to the presence of an external environment:

$$\frac{\Gamma_{\text{Fock}}}{\Gamma_{\text{coherent}}} \simeq N$$

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