On the computation of BH entropy in LQG

J. Fernando Barbero G.
Instituto de Estructura de la Materia, CSIC.
Valencia, March 26, 2009

Work in collaboration with I. Agulló, J. Díaz-Polo, E. F.-Borja, and E.J.S. Villaseñor.

Summary

1. Some new results on microscopic black holes.
2. Entropy in loop quantum gravity.
3. Has the last word being said?
4. A different approach.
5. Characterizing the spectrum of the area operator.
6. Black hole degeneracy spectrum.
7. Generating functions.
8. Back to Laplace transforms.
9. Inverse Laplace Transforms and Asymptotics.
10. Conclusions and comments.

Summary

1. Some new results on microscopic black holes.
2. Entropy in loop quantum gravity.
3. Has the last word being said?
4. A different approach.
5. Characterizing the spectrum of the area operator.
6. Black hole degeneracy spectrum.
7. Generating functions.
8. Back to Laplace transforms.
9. Inverse Laplace Transforms and Asymptotics.
10. Conclusions and comments.

Summary

1. Some new results on microscopic black holes.
2. Entropy in loop quantum gravity.
3. Has the last word being said?
4. A different approach.
5. Characterizing the spectrum of the area operator.
6. Black hole degeneracy spectrum.
7. Generating functions.
8. Back to Laplace transforms.
9. Inverse Laplace Transforms and Asymptotics.
10. Conclusions and comments.

Summary

1. Some new results on microscopic black holes.
2. Entropy in loop quantum gravity.
3. Has the last word being said?
4. A different approach.
5. Characterizing the spectrum of the area operator.
6. Black hole degeneracy spectrum.
7. Generating functions.
8. Back to Laplace transforms.
9. Inverse Laplace Transforms and Asymptotics.
10. Conclusions and comments.

Summary

1. Some new results on microscopic black holes.
2. Entropy in loop quantum gravity.
3. Has the last word being said?
4. A different approach.
5. Characterizing the spectrum of the area operator.
6. Black hole degeneracy spectrum.
7. Generating functions.
8. Back to Laplace transforms.
9. Inverse Laplace Transforms and Asymptotics.
10. Conclusions and comments.

Summary

1. Some new results on microscopic black holes.
2. Entropy in loop quantum gravity.
3. Has the last word being said?
4. A different approach.
5. Characterizing the spectrum of the area operator.
6. Black hole degeneracy spectrum.
7. Generating functions.
8. Back to Laplace transforms.
9. Inverse Laplace Transforms and Asymptotics.
10. Conclusions and comments.

Summary

1. Some new results on microscopic black holes.
2. Entropy in loop quantum gravity.
3. Has the last word being said?
4. A different approach.
5. Characterizing the spectrum of the area operator.
6. Black hole degeneracy spectrum.
7. Generating functions.
8. Back to Laplace transforms.
9. Inverse Laplace Transforms and Asymptotics.
10. Conclusions and comments.

Summary

1. Some new results on microscopic black holes.
2. Entropy in loop quantum gravity.
3. Has the last word being said?
4. A different approach.
5. Characterizing the spectrum of the area operator.
6. Black hole degeneracy spectrum.
7. Generating functions.
8. Back to Laplace transforms.
9. Inverse Laplace Transforms and Asymptotics.
10. Conclusions and comments.

Summary

1. Some new results on microscopic black holes.
2. Entropy in loop quantum gravity.
3. Has the last word being said?
4. A different approach.
5. Characterizing the spectrum of the area operator.
6. Black hole degeneracy spectrum.
7. Generating functions.
8. Back to Laplace transforms.
9. Inverse Laplace Transforms and Asymptotics.
10. Conclusions and comments.

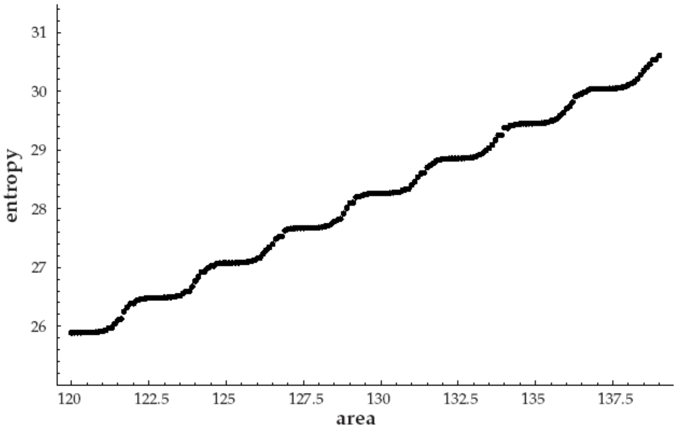
Summary

1. Some new results on microscopic black holes.
2. Entropy in loop quantum gravity.
3. Has the last word being said?
4. A different approach.
5. Characterizing the spectrum of the area operator.
6. Black hole degeneracy spectrum.
7. Generating functions.
8. Back to Laplace transforms.
9. Inverse Laplace Transforms and Asymptotics.
10. Conclusions and comments.

Introduction

- Black hole entropy is a **central topic** in quantum gravity. It may be one of the ways to crack open the problem of quantizing general relativity
- Any self-respecting theory of quantum gravity should **account in detail** for the microscopic degrees of freedom responsible for black hole entropy.
- Within LQG a lot has been understood about this problem in the last years.
- The **Bekenstein-Hawking law is satisfied** by the entropy (after taking a universal value for the Immirzi parameter). This is one of the pillars of LQG as of today.
- It should be reassuring to plot **entropy versus area** to see this beautiful result, right?

[Corichi, Díaz-Polo, Fernández-Borja, CQG **24**:243 (2007)]



Some new results on Microscopic black holes

- Numerical computations of black hole entropy in LQG for small black holes show an **interesting structure** in the black hole degeneracy spectrum or the entropy.
- The results are **quite robust** and extend to the largest values of the area that it has been possible to explore directly.
- Compatibility: the entropy obtained from these microstate countings can be fitted by a linear function of the area (Bekenstein-Hawking area law). The value of γ is numerically **compatible** with the one given by Meissner.

Questions

- Can we understand this phenomenon in fundamental terms?
- Is this newly found structure present for macroscopic black holes?
- Could this have been expected? Could it have been predicted?

DEFINITION OF ENTROPY (Domagala-Lewandowski)

The entropy S of a quantum horizon of the classical area a according to Quantum Geometry and the Ashtekar-Baez-Corichi-Krasnov framework is

$$S = \log \mathfrak{n}(a),$$

where $\mathfrak{n}(a)$ is 1 plus the number of all the finite sequences $(m_1, \ldots, m_n)$ of non-zero elements of $\frac{1}{2}\mathbb{Z}$, such that the following equality and inequality are satisfied:

$$\sum_{i=1}^{n} m_i = 0, \quad \sum_{i=1}^{n} \sqrt{|m_i|(|m_i| + 1)} \leq \frac{a}{8\pi\gamma\ell_P^2}$$

where γ is the Immirzi parameter of Quantum Geometry.

Meissner's approach to **exactly** compute black hole entropy.

- Let us define the sets (in the following $8\pi\gamma\ell_P^2 = 1$, $\mathbb{Z}_* := \mathbb{Z} \setminus \{0\}$)

$$\mathcal{N}_{\leq}(a,p) := \{\vec{m} \in (\mathbb{Z}_*/2)^n : n \in \mathbb{N}, \sum_{i=1}^{n} \sqrt{|m_i|(|m_i|+1)} \leq a, \sum_{i=1}^{n} m_i = p\}$$

$$\mathcal{N}_{\leq}(a) := \{\vec{m} \in (\mathbb{Z}_*/2)^n : n \in \mathbb{N}, \sum_{i=1}^{n} \sqrt{|m_i|(|m_i|+1)} \leq a\}$$

  and let $N_{\leq}(a,p)$ and $N_{\leq}(a)$ be their respective cardinalities.
- The entropy is given by $e^{S(a)} = 1 + N_{\leq}(a,0)$
- At times we will consider the entropy "without projection constraint" S' defined by $e^{S'(a)} = 1 + N_{\leq}(a)$.

The standard approach: functional equations

- A direct argument gives the following functional equation (and a similar one for the $N_{\leq}(a,p)$), here $k_{\mathrm{max}} := \lfloor\sqrt{1+4a^2}-1\rfloor$).

$$N_{\leq}(a) = \sum_{k=1}^{k_{\mathrm{max}}} N_{\leq}^{(k)}(a) = 2\lfloor\sqrt{1+4a^2}-1\rfloor + 2\sum_{k=1}^{k_{\mathrm{max}}} N_{\leq}(a-\sqrt{k(k+2)}/2)\,,$$

$$N_{\leq}(a) = 2\lfloor\sqrt{4a^2+1}-1\rfloor\theta(a-\sqrt{3}/2) + 2\sum_{k=1}^{\infty} N_{\leq}(a-\sqrt{k(k+2)}/2)$$

- Notice that we can extend the sum to infinity. The $\theta(a-\sqrt{3}/2)$ factor is needed in the first term or the r.h.s. to guarantee that it is zero for arbitrary negative values of a.
- This functional equation can be used to obtain a closed form representation for $N_{\leq}(a)$ (a similar one for $N_{\leq}(a,p)$).

$N_{\leq}(a)$ is exponentially bounded and piecewise continuous (Domagala& Lewandowski) hence its Laplace transform exists and is well defined in a half-plane $\{s \in \mathbb{C} : \mathrm{Re}(s) > x_0\}$ for some $x_0 \in \mathbb{R}$. We have then

$$\begin{aligned}
P_{\leq}(s) &:= \int_0^{\infty} N_{\leq}(a) e^{-as}\,\mathrm{d}a \\
&= 2\int_{\frac{\sqrt{3}}{2}}^{\infty} \lfloor \sqrt{4a^2+1} - 1 \rfloor e^{-as}\,\mathrm{d}a + 2\int_{\frac{\sqrt{3}}{2}}^{\infty} \sum_{k=1}^{\infty} N_{\leq}(a - \sqrt{k(k+2)}/2) e^{-as}\,\mathrm{d}a \\
&= \frac{2}{s}\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}/2} + 2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}/2} \int_{-\sqrt{k(k+2)}/2}^{\infty} N_{\leq}(a) e^{-as}\,\mathrm{d}a \\
&= \frac{2}{s}\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}/2} + 2P_{\leq}(s)\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}/2}.
\end{aligned}$$

- We have used the fact that $N_{\leq}(a) = 0$ for $a \leq 0$ to set the lower limits in the integrals equal to zero.
- We finally get

$$P_{\leq}(s) = \frac{1}{s}\left(1 - 2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}/2}\right)^{-1} - \frac{1}{s} = \frac{2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}/2}}{s\left(1 - 2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}/2}\right)}.$$

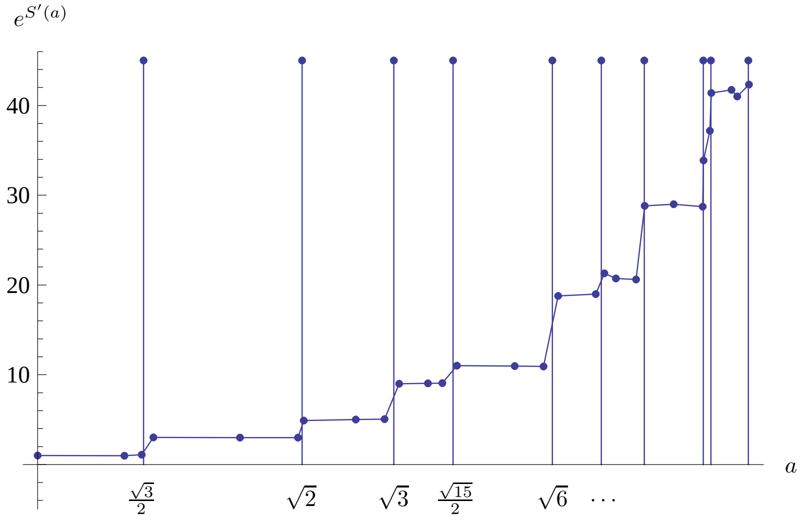
- The fact that $P_{\leq}(s)$ is a proper Laplace transform tells us that we can write

$$N_{\leq}(a) = \frac{1}{2\pi i}\lim_{A\to a^{+}}\int_{x_0-i\infty}^{x_0+i\infty} P_{\leq}(s)e^{As}\,\mathrm{d}s. \tag{1}$$

for some $x_0 \in \mathbb{R}$ chosen in such a way that the singularities in the integrand are to the left of the integration contour.
- A similar approach can be used for $N_{\leq}(a, p)$.

Comments

- The expression that we obtain is **exact**.
- The integrand is a rather complicated function because it encodes a lot of information. Notice that the entropy is a “staircase” function with discontinuities located at the values of the area spectrum.
- It is not very good to compute the entropy numerically.
- It may be useful to obtain the asymptotic behavior of the entropy but this depends on the **analytic structure** of the integrand which is rather non-trivial.
- In any case it suffices to show that **the Bekenstein-Hawking law is satisfied**.
- A similar analysis can be carried out when the projection constraint is also taken into account. [One gets logarithmic corrections].
- A **different strategy** may be useful.



$e^{S'(a)}$
40
30
20
10
a
$\frac{\sqrt{3}}{2}$
$\sqrt{2}$
$\sqrt{3}$
$\frac{\sqrt{15}}{2}$
$\sqrt{6}$
$\cdots$

A NEW STRATEGY

Let us look at the combinatorial problem of computing the entropy from a different point of view.

1. **STEP 1:** For a given value of the area a, consider first the problem of explicitly finding the sequences $(|m_1|, \ldots, |m_n|)$ such that

$$\sum_{i=1}^{n} \sqrt{|m_i|(|m_i|+1)} = a\,.$$

2. **STEP 2:** Allow for signs and determine how many sequences satisfy the *projection constraint*

$$\sum_{i=1}^{n} m_i = 0$$

3. **STEP 3:** Find in this way the black hole degeneracy spectrum from which the entropy can be obtained by integration (in fact, summing for all the area eigenvalues smaller or equal than a).

The spectrum of the area operator (step 1)

- Let us consider only $|m_i|$ (we will reintroduce the signs later).
- Writing $\mathbb{N} \ni k_i := 2|m_i|$ we have $\sum_{i=1}^{n} \sqrt{|m_i|(|m_i|+1)} = a \Rightarrow$

$$\sum_{i=1}^{n} \sqrt{(k_i+1)^2-1} = \sum_{k=1}^{k_{\max}} n_k \sqrt{(k+1)^2-1} = 2a\,.$$

where n_k (possibly 0) tells us the number of times that the label $k \in \mathbb{N}$ appears.

$$(\underbrace{1,\ldots,1}_{n_1},\underbrace{2,\ldots,2}_{n_2},\ldots,\underbrace{k_{max},\ldots,k_{max}}_{n_{k_{max}}})$$

The spectrum of the area operator (step 1)

- Notice that we can always write $\sqrt{(k+1)^2-1}$ as the product of an integer times the square root of a square-free positive integer number p_i by using its prime factor decomposition.
- This means that a must satisfy $2a = \sum_{i=1}^{r} q_i\sqrt{p_i} \quad \text{with } q_i \in \mathbb{N}$
- We have then the following equation

$$\sum_{k=1}^{k_{\max}} n_k\sqrt{(k+1)^2-1} = \sum_{i=1}^{r} q_i\sqrt{p_i}\,.$$

- The previous equation is solved in two steps:

1. First identify the allowed labels k such that $\sqrt{(k+1)^2-1}$ is an integer multiple of the $\sqrt{p_i}$ corresponding to the given a.
2. Determine then the value of n_k that tells us how many times the allowed label k appears.

The spectrum of the area operator (step 1)

1. $\sqrt{(k+1)^2-1} = y\sqrt{p_i} \Leftrightarrow x^2 - p_i y^2 = 1$, with $x := k+1$, $x, y \in \mathbb{N}$.

   - This **quadratic diophantine equation** is the famous **Pell Equation**

$$x^2 - p_i y^2 = 1$$

   - It has a *fundamental solution* (x_1^i, y_1^i) with the smallest value of x. This can be obtained by using continued fractions. An infinite sequence of solutions can be derived from it. They are given by

$$x_\alpha^i = \frac{1}{2}\left[(x_1^i + y_1^i\sqrt{p_i})^\alpha + (x_1^i - y_1^i\sqrt{p_i})^\alpha\right]$$
$$y_\alpha^i = \frac{1}{2\sqrt{p_i}}\left[(x_1^i + y_1^i\sqrt{p_i})^\alpha - (x_1^i - y_1^i\sqrt{p_i})^\alpha\right]$$

   - We label the solutions as $\{(k_\alpha^i, y_\alpha^i) : \alpha \in \mathbb{N}\}$, ($i$ refers to p_i).
   - For instance, for $p_1 = 2$ we have
     $\{(k_\alpha^1, y_\alpha^1) : \alpha \in \mathbb{N}\} = \{(2,2), (16,12), (98,70), (576,408), \ldots\}$

2. Once the values of k are known, the n_k can be found by solving the system of r uncoupled, linear, diophantine equations

$$\sum_{k=1}^{k_{max}} n_k \sqrt{(k+1)^2 - 1} = \sum_{i=1}^{r} \sum_{\alpha=1}^{\infty} n_{k^i_\alpha} y^i_\alpha \sqrt{p_i} = \sum_{i=1}^{r} q_i \sqrt{p_i}$$

so that $\displaystyle\sum_{\alpha=1}^{\infty} y^i_\alpha n_{k^i_\alpha} = q_i, \quad i = 1, \ldots, r.$

- We have used the fact that the $\sqrt{p_i}$ are linearly independent over $\mathbb{Q}$.
- Notice that, once the q_i are fixed, only a finite number of labels k^i_α come into play in these equations.
- It may happen that some of these equations admit no solutions. In this case $\sum_{i=1}^{r} q_i \sqrt{p_i}$ does not belong to the area spectrum.
- **Step 1** is equivalent to giving a **full characterization** of the degeneracy of the area operator.

THE SPECTRUM OF THE AREA OPERATOR (STEP 1)

- Up to this point we have found all the possible choices of labels k (and their multiplicity) compatible with a given value of a,

$$(\underbrace{1,\ldots,1}_{n_1},\underbrace{2,\ldots,2}_{n_2},\ldots,\underbrace{k_{max},\ldots,k_{max}}_{n_{k_{max}}})$$

- The number of different sequences obtained from each of these by reordering is then given by the multinomial coefficient

$$\frac{(\sum_{k=1}^{k_{max}} n_k)!}{\prod_{k=1}^{k_{max}} n_k!}$$

- Now we have to put back the signs and solve the projection constraint.

The spectrum of the area operator (step 2)

- We have to sprinkle signs in the sequences obtained above in such a way that the projection constraint is satisfied in every case

$$\sum_i m_i = 0$$

$$\begin{array}{ccccccccccccccccccc}
(& 1 & , & 2 & , & 5 & , & 2 & , & 2 & , & 3 & , & 1 & , & 4 & , & \dots &) \\
(& + & , & - & , & - & , & + & , & - & , & + & , & - & , & + & , & \dots &) \\
(& z^{+1} & , & z^{-2} & , & z^{-5} & , & z^{+2} & , & z^{-2} & , & z^{+3} & , & z^{-1} & , & z^{+4} & , & \dots &)
\end{array}$$

- A very efficient way to count the number of ways that the signs can be introduced so that the projection constraint is satisfied: look for the coefficient of the constant term in the expansion of

$$\prod (z^{k_i} + z^{-k_i})^{n_{k_i}}$$

- This solves the problem with the help of a **generating function**.

The black hole degeneracy spectrum (step 3)

- The number of states (m-sequences) corresponding to a given value of the area a is obtained by
  1. Finding out the possible values of $|m_i|$ and their multiplicity for a given value of the area a (let us refer to these as **configurations**).
  2. Computing, for each configuration, the number of possible reorderings (multinonial coefficients).
  3. Computing, for each reordering, the number of possibilities to introduce signs in such a way that the projection constraint is satisfied.
- This procedure provides an efficient algorithm to compute the **black hole degeneracy** spectrum [i.e. $N(a,p)$ and, in particular, $N(a,0)$] and check if the entropy "staircase" is really there.
- **Step 3** After doing all this the black hole entropy can be obtained by summing the numbers obtained above for each eigenvalue a' of the area operator such that $a' \leq a$. How can we do this?

- **Generating functions** efficiently encode all the information about a given combinatorial problem.
- They are specially useful to obtain closed form solutions and facilitate the analysis of their asymptotic behavior in relevant regimes.
- They have been widely used, for example, in statistical mechanics.
- **A sample problem**: count the number of non-negative solutions to the diophantine equation $2x_1 + 3x_2 = q$ in terms of $q \in \mathbb{N}$.
- A solution in terms of generating functions: multiply the two following **formal series** associated to the two terms in the equation

$$\begin{aligned} & (x^{2\cdot 0} + x^{2\cdot 1} + x^{2\cdot 2} + x^{2\cdot 3} + x^{2\cdot 4} + x^{2\cdot 5} + x^{2\cdot 6} + \cdots) \\ \times \quad & (x^{3\cdot 0} + x^{3\cdot 1} + x^{3\cdot 2} + x^{3\cdot 3} + x^{3\cdot 4} + x^{3\cdot 5} + x^{3\cdot 6} + \cdots) = \end{aligned}$$

Generating functions

$$= \underbrace{x^{(2\cdot 0+3\cdot 0)}}_{1} + \underbrace{x^{(2\cdot 1+3\cdot 0)}}_{x^2} + \underbrace{x^{(2\cdot 0+3\cdot 1)}}_{x^3} + \underbrace{x^{(2\cdot 2+3\cdot 0)}}_{x^4} + \underbrace{x^{(2\cdot 1+3\cdot 1)}}_{x^5} +$$

$$\underbrace{x^{(2\cdot 0+3\cdot 2)} + x^{(2\cdot 3+3\cdot 0)}}_{2x^6} + \underbrace{x^{(2\cdot 2+3\cdot 1)}}_{x^7} + \underbrace{x^{(2\cdot 4+3\cdot 0)} + x^{(2\cdot 1+3\cdot 2)}}_{2x^8} + \cdots$$

- The coefficient of the term x^q gives the number of solutions to the diophantine equation $2x_1 + 3x_2 = q$ for the chosen value of q.
- The formal series given above actually correspond to meromorphic functions of a complex variable $\mathbb{C}$. In this case

$$(x^{2\cdot 0} + x^{2\cdot 1} + x^{2\cdot 2} + x^{2\cdot 3} + x^{2\cdot 4} + x^{2\cdot 5} + x^{2\cdot 6} + \cdots) = \frac{1}{1-x^2}$$

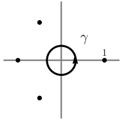
$$(x^{3\cdot 0} + x^{3\cdot 1} + x^{3\cdot 2} + x^{3\cdot 3} + x^{3\cdot 4} + x^{3\cdot 5} + x^{3\cdot 6} + \cdots) = \frac{1}{1-x^3}$$

- The solution to our problem is given by the x^q coefficient of Taylor expansion around $x = 0$ of the function

$$f(x) = \frac{1}{(1-x^2)(1-x^3)}$$

- This can be obtained in closed from from the partial fraction decomposition of $f(x)$. It is also given by the following contour integral

$$\frac{1}{2\pi i}\oint_\gamma \frac{\mathrm{d}z}{z^{q+1}} \frac{1}{(1-z^2)(1-z^3)}$$



- This is specially useful to obtain the asymptotic behavior for $q \to \infty$.

A QUESTION

Can we obtain a generating function for the black hole degeneracy spectrum?

YES!

$$G(z, x_1, x_2, \dots) = \left(1 - \sum_{i=1}^{\infty}\sum_{\alpha=1}^{\infty}(z^{k_\alpha^i} + z^{-k_\alpha^i})x_i^{y_\alpha^i}\right)^{-1}$$

- In the previous formula the variables x_i are associated to squarefree integers p_i and z is an extra variable needed to account for the projection constraint.
- The numbers (k_α^i, y_α^i) are obtained from the solutions to the Pell equation associated to the squarefree p_i.
- The coefficient of the term $z^0 x_1^{q_1} \cdots x_i^{q_i} \cdots$ gives the number of sequences $\vec{m}$ satisfying the projection constraint and such that $2a = \sum_i q_i \sqrt{p_i}$
- Despite the apparent infinite number of terms, for a given value of a **only finite numbers** of variables and terms are needed.

Generating functions

- It is only the fact that the diophantine equations that we need to solve have an effective number of variables that depends on the area, that forces us to introduce a formally infinite number of them.
- The coefficient of the term $z^0 x_1^{q_1} \cdots x_i^{q_i} \cdots$ can be written as a multiple contour integral that may be conceivably used as the starting point to study the black hole degeneracy for macroscopic areas.
- A concrete numerical example: For an area $a = 40\sqrt{2} + 40\sqrt{3}$ the total degeneracy is obtained by taking the generating function

$$G(z,x_1,x_2) = \frac{1}{1-(z^2+z^{-2})x_1^2-(z^{16}+z^{-16})x_1^{12}-(z+z^{-1})x_2-(z^6+z^{-6})x_2^4-(z^{25}+z^{-25})x_2^{15}}\,.$$

- The value of the black hole degeneracy $N(a,0)$ is given by the coefficient of the term $z^0 x_1^{40} x_2^{40}$ in the power series expansion of $G(z,x_1,x_2)$. This is 991809938488860909241077458398212.

Back to Laplace Transforms

One can use these generating functions to obtain $N_{\leq}(a,0)$ from $N(a,0)$.

- The spectrum of the area operator $\mathcal{A}_{\text{IH}} = \{a_n : n \in \mathbb{N}\}$ (a countable, *ordered*, subset of $\mathbb{R}$).
- It is possible, in principle, to build the sequence $\{N(a_n, p) : n \in \mathbb{N}\}$.
- For a fixed value of a_n we can write $N_{\leq}(a_n, p) = \displaystyle\sum_{i=1}^{n} N(a_i, p)$.
- If the $N(a_n, p)$ are encoded in a generating function $g_p(x) = \sum_{n\in\mathbb{N}} N(a_n, p)x^n$ this summation can be carried out by taking.

$$G_p(x) = \frac{g_p(x)}{1-x} = \sum_{n\in\mathbb{N}} N_{\leq}(a_n, p)x^n.$$

- This is difficult now because one would need to have a practical way to find the numbers q_i corresponding to the n^{th} element of $\mathcal{A}_{\text{IH}}$.

Back to Laplace Transforms

- In practice this requires us to solve the following two problems:
  1. Given an eigenvalue of the area $a \in \mathcal{A}_{\mathrm{IH}}$, how many smaller eigenvalues do exist? (we refer to this as the *area ordering problem*).
  2. Given $A \in \mathbb{R}$, what are the values of the q_i corresponding to the largest area eigenvalue $a \in \mathcal{A}_{\mathrm{IH}}$ satisfying $a \leq A$? (alternatively to the closest eigenvalue to A).
- A way out of this is by using Laplace transforms.

$$\mathcal{L}[\sum_{n\in\mathbb{N}} \beta_n \theta(a-a_n); s] = \frac{1}{s}\sum_{n\in\mathbb{N}} \beta_n e^{-a_n s}$$

- If the positions of the jumps (the area eigenvalues a_n) and their magnitudes (the black hole degeneracies β_n) can be encoded in a function that can be expanded as $\sum_{n\in\mathbb{N}} \beta_n e^{-a_n s}$ then we can get an integral representation for the BH entropy as an inverse Laplace transform.

- This can be done by using the BH generating function given above.
- It is enough to substitute the x_i in $G(z; x_1, x_2, \ldots)$ by $x_i = e^{-s\sqrt{p_i}/2}$.
- This is so because $x_1^{q_1} \cdots x_r^{q_r} \mapsto e^{-\frac{s}{2}(q_1\sqrt{p_1}+\cdots+q_r\sqrt{p_r})} = e^{-as}$ when $2a = q_1\sqrt{p_1} + \cdots + q_r\sqrt{p_r}$.
- By doing this we find (without projection constraint)

$$P(s) := \sum_{n\in\mathbb{N}} N(a_n)e^{-a_n s} + 1 = G(1; e^{-s\sqrt{p_1}/2}, e^{-s\sqrt{p_2}/2}, \ldots)$$
$$= \left(1 - 2\sum_{i=1}^{\infty}\sum_{\alpha=1}^{\infty} e^{-sy_\alpha^i\sqrt{p_i}}\right)^{-1}.$$

- The $e^{-sy_\alpha^i\sqrt{p_i}}$ can be simplified by taking into account that the(k_α^i, y_α^i) are solutions to the Pell equation and hence $y_\alpha^i\sqrt{p_i} = \sqrt{k_\alpha^i(k_\alpha^i+2)}$.

- This way ge get

$$P(s)=\left(1-2\sum_{i=1}^{\infty}\sum_{\alpha=1}^{\infty}e^{-s\sqrt{k_\alpha^i(k_\alpha^i+2)/2}}\right)^{-1}=\left(1-2\sum_{k=1}^{\infty}e^{-s\sqrt{k(k+2)/2}}\right)^{-1},$$

- By dividing by s and performing an inverse Laplace transform we arrive *precisely* and the formula found before (without the projection constraint).

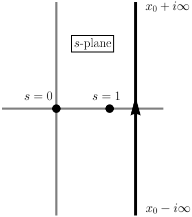
$$e^{S(a)}=\frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty}\mathrm{d}s\,\frac{e^{as}}{s\left(1-2\sum_{k=1}^{\infty}e^{-s\sqrt{k(k+2)/2}}\right)}$$

- This is a **non-trivial check** of the whole approach.
- It allows us to get Meissner's formula in a completely new way.

$$e^{S(a)} = \frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} \mathrm{d}s \frac{e^{as}}{s\left(1-2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}/2}\right)}$$

- The integration contour is a line parallel to the imaginary axis chosen in such a way that **all the singularities are to the left**.
- In some cases this allows us to easily obtain the asymptotic behavior, for example if we consider

$$\frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} \mathrm{d}s \frac{e^{as}}{s(s-1)} = \theta(a)(e^{a}-1)$$



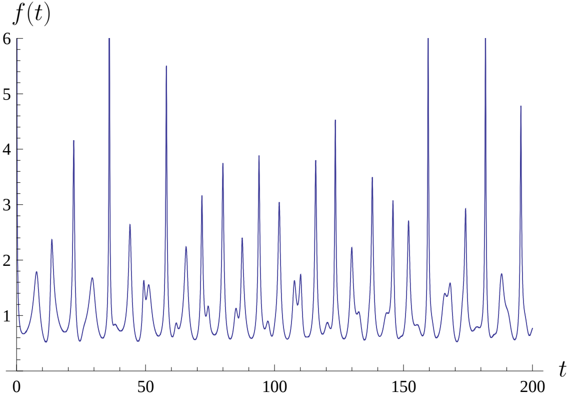
the integrand has poles at $s = 1$ and $s = 0$. Their residues respectively give e^{a} and -1. The leading asymptotic behavior (for large positive a) corresponds to the **pole with the largest real part**.

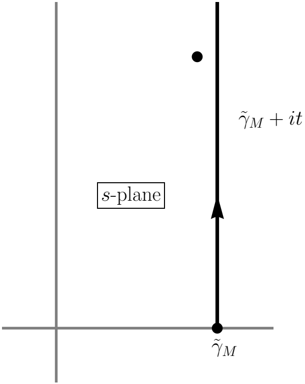
- Where are the singularities of $\dfrac{e^{as}}{s\left(1-2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}/2}\right)}$ located?
  1. There is an **infinite number** of poles.
  2. They are **confined to a band** in the complex plane and their real parts are bounded from above by $s = \tilde{\gamma}_M \approx 1.49246359$ (the value obtained by Meissner), i. e. the only real solution to the equation

$$1 - 2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}/2} = 0\,.$$

  3. There is only **a single pole** of the integrand with real part equal to $\tilde{\gamma}_M$.
  4. The **real parts** of the poles have an **accumulation point** precisely for the value $\tilde{\gamma}_M$ (and maybe others).

A plot of the restriction of the absolute value of the function $1/\left(1-2\sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)/2}}\right)$ to the half-line $\tilde{\gamma}_M + it$, $t \in [0, 200]$





Does the accumulation of the real parts change the asymptotic behavior given by $e^{\tilde{\gamma}_M a}$?

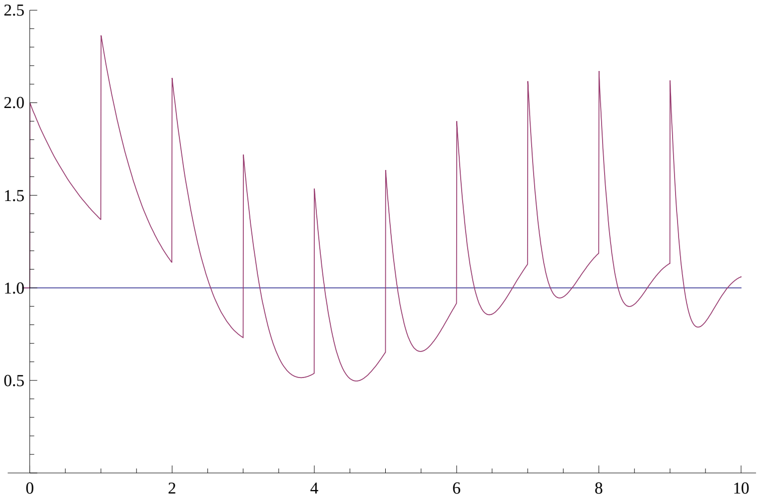
Maybe

- An example

$$\frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty}\mathrm{d}s\, e^{as}\left(\frac{1}{1+s+se^{-s}}+\frac{1}{s}\right)=\theta(a)+\sum_{k=0}^{\infty}\theta(a-k)e^{k-a}L_k(a-k)$$

  where L_k denotes the Laguerre polynomial of degree k. Notice that for a given value of a the previous sum is **finite**.
- The real parts of the (non-real) poles of the integrand are never zero but have an accumulation point at $r=0$, which is the eigenvalue with the largest real part.
- The contribution of the pole at $s=0$ is $\theta(a)$ (asymptotically 1).
- The contribution of the other term for a close enough to the positive integers is larger than 1. Hence **it is not true** that the asymptotic behavior of the integral is controlled by the eigenvalue corresponding to the **largest real part**.

Inverse Laplace Transforms and Asymptotics



Conclusions and perspectives

- We have a very detailed picture for BH entropy in LQG. This allows us to understand the origin of the structure seen in the black hole degeneracy spectrum.
- Our methods are very **flexible**. They can be easily adapted to study other proposals such as the Ghosh-Mitra counting and SO(3) models.
- They can be used to build **efficient algorithms** that confirm and extend previous numerical work.
- We have **generating functions** for all the relevant combinatorial problems. These allow us to reproduce and confirm previous results (Meissner) and may be the starting point to obtain the **asymptotic behavior** of the entropy.
- The **accumulation of the real parts** of the integrand of the inverse Laplace transform giving the entropy leaves room to an oscillatory behavior for macroscopic areas.

[1] M. Domagala and J. Lewandowski, *Black hole entropy from Quantum Geometry*, Class. Quant. Grav. **21** 5233 (2004). arXiv:gr-qc/0407051.

[2] K. Meissner, *Black hole entropy in Loop Quantum Gravity*, Class. Quant. Grav. **21**, 5245 (2004). arXiv:gr-qc/0407052.

[3] A. Corichi, J. Diaz-Polo, E. Fernandez-Borja, *Quantum geometry and microscopic black hole entropy*, Class. Quant. Grav. **24**, 243 (2007). arXiv:gr-qc/0605014.

[4] I. Agulló, J. F. Barbero G., J. Diaz-Polo, E. F.-Borja, E. J. S. Villaseñor, *Black hole state counting in Loop Quantum Gravity: A number theoretical approach*, Phys. Rev. Lett. **100** (2008) 211301. arXiv:0802.4077.

[5] J. F. Barbero G., E. J. S. Villaseñor, *Generating functions for black hole entropy in Loop Quantum Gravity*, Phys. Rev. D **77**, (2008) 121502(R). arXiv:0804.4784.

[6] J. F. Barbero G., E. J. S. Villaseñor *On the computation of black hole entropy in loop quantum gravity*, To appear in CQG. arXiv:0810.1599