Spherically symmetric loop quantum gravity

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Reduction in loop quantum gravity

Models necessary to *simplify, understand and develop* dynamics of full theory.

But: Beware of *artefacts.* For reliable results, constructions must be ensured to work equally well in *many different models.*

Key issues for dynamics:

$\longrightarrow$ *Difference equations:* underlying discreteness, step-size affected by *lattice refinement,* directly captured only in inhomogeneous situations. Specific form important for acceptable behavior even of homogeneous models (stable in semiclassical regimes).

$\longrightarrow$ *Effective equations:* intuitive geometrical pictures, but *quantum back-reaction* important in strong quantum regimes (e.g. near singularities).

Kantowski–Sachs

Schwarzschild interior metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

homogeneous for $r < 2M$.
Corresponding invariant connection/triad (SU(2)-gauge fixed):

$$\begin{aligned} A_a^i \tau_i \mathrm{d}x^a &= \tilde{c}\tau_3 \mathrm{d}x + \tilde{b}\tau_2 \mathrm{d}\vartheta - \tilde{b}\tau_1 \sin\vartheta \mathrm{d}\varphi + \tau_3 \cos\vartheta \mathrm{d}\varphi \\ E_i^a \tau^i \frac{\partial}{\partial x^a} &= \tilde{p}_c \tau_3 \sin\vartheta \frac{\partial}{\partial x} + \tilde{p}_b \tau_2 \sin\vartheta \frac{\partial}{\partial \vartheta} - \tilde{p}_b \tau_1 \frac{\partial}{\partial \varphi} \end{aligned}$$

Spatial metric ($x = t$ for interior):

$$\mathrm{d}s^2 = \frac{\tilde{p}_b^2}{|\tilde{p}_c|}\mathrm{d}x^2 + |\tilde{p}_c|\mathrm{d}\Omega^2$$

On shell: $\tilde{p}_c = 0$ at singularity, $\tilde{p}_b = 0$ at horizon.

Phase space

$(\tilde{b}, \tilde{c}, \tilde{p}_b, \tilde{p}_c) \in \mathbb{R}^4$, with $\{\tilde{b}, \tilde{p}_b\} = \gamma G/L_0$, $\{\tilde{c}, \tilde{p}_c\} = 2\gamma G/L_0$ after choosing (x, ϑ, φ)-integration region of size $L_0 \times 4\pi$. Rescale

$$(b, c) := (\tilde{b}, L_o \tilde{c}) \quad , \quad (p_b, p_c) := (L_o \tilde{p}_b, \tilde{p}_c)$$

for coordinate independent variables (but L_0-dependent).

Co-triad: $a_c = |p_b|/\sqrt{|p_c|} = L_0 \tilde{a}_c$, $a_b = \sqrt{|p_c|} = \sqrt{|\tilde{p}_c|}$.

Note: L_0 (or V_0 in isotropic models) key culprit for possible artefacts in minisuperspace constructions. Only coordinate and L_0-independent variables are b, a_b and c/a_c.

Can quantum corrections only depend on the curvature scale c/a_c, or possibly just on the spatial scale a_c (as suggested by inverse triad corrections)?

Here, role of *quantum reduction* becomes important.

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Holonomy-flux algebra

$$h_x^{(\tau)}(A) = \exp\int_0^{\tau L_0} \mathrm{d}x\tilde{c}\tau_3 = \cos\frac{\tau c}{2} + 2\tau_3\sin\frac{\tau c}{2}$$

$$h_\vartheta^{(\mu)}(A) = \exp\int_0^{\mu} \mathrm{d}\vartheta\tilde{b}\tau_2 = \cos\frac{\mu b}{2} + 2\tau_2\sin\frac{\mu b}{2}$$

generate states $|\mu,\tau\rangle = \hat{h}_\vartheta^{(\mu)}\hat{h}_x^{(\tau)}|0,0\rangle$ with $\langle c,b|0,0\rangle = 1$. Triad eigenstates:

$$\hat{p}_b|\mu,\tau\rangle = \frac{1}{2}\gamma\ell_{\mathrm{P}}^2\,\mu|\mu,\tau\rangle, \qquad \hat{p}_c|\mu,\tau\rangle = \gamma\ell_{\mathrm{P}}^2\,\tau|\mu,\tau\rangle$$

Completes kinematical setting. [A. Ashtekar, MB: CQG 23 (2006) 391]

As symmetric states, $|\mu,\tau\rangle$ can be realized as invariant distributions. Key properties of reduction enter at dynamical level: Hamiltonian constraint and lattice refinement.

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Hamiltonian constraint

Standard construction:

$$
\begin{aligned}
\hat{H}^{(\delta)} &= \frac{2i}{\gamma^3\delta^3\ell_{\mathrm{P}}^2}\mathrm{tr}\Big(\sum_{IJK}\epsilon^{IJK}\hat{h}_I^{(\delta)}\hat{h}_J^{(\delta)}\hat{h}_I^{(\delta)-1}\hat{h}_J^{(\delta)-1}\hat{h}_K^{(\delta)}[\hat{h}_K^{(\delta)-1},\hat{V}] \\
&\quad +2\gamma^2\delta^2\tau_3\hat{h}_x^{(\delta)}[\hat{h}_x^{(\delta)-1},\hat{V}]\Big) \\
&= \frac{4i}{\gamma^3\delta^3\ell_{\mathrm{P}}^2}\left(8\sin\frac{\delta b}{2}\cos\frac{\delta b}{2}\sin\frac{\delta c}{2}\cos\frac{\delta c}{2}\right. \\
&\quad\times\left(\sin\frac{\delta b}{2}\hat{V}\cos\frac{\delta b}{2}-\cos\frac{\delta b}{2}\hat{V}\sin\frac{\delta b}{2}\right) \\
&\quad\left.+\left(4\sin^2\frac{\delta b}{2}\cos^2\frac{\delta b}{2}+\gamma^2\delta^2\right)\left(\sin\frac{\delta c}{2}\hat{V}\cos\frac{\delta c}{2}-\cos\frac{\delta c}{2}\hat{V}\sin\frac{\delta c}{2}\right)\right)
\end{aligned}
$$

Real parameter δ

expected to be constant in pure minisuperspace setting.

Difference equation

Derive explicit action of $\hat{H}^{(\delta)}$, transform to triad representation. Resulting difference equation non-singular: evolves across $\tau = 0$ ($p_c = 0$, singularity)

$$
\begin{aligned}
& C_+(\mu,\tau)\left(\psi_{\mu+2\delta,\tau+2\delta} - \psi_{\mu-2\delta,\tau+2\delta}\right) \\
& + C_0(\mu,\tau)\Big((\mu+2\delta)\psi_{\mu+4\delta,\tau} - 2(1+2\gamma^2\delta^2)\mu\psi_{\mu,\tau} \\
& \qquad + (\mu-2\delta)\psi_{\mu-4\delta,\tau}\Big) \\
& + C_-(\mu,\tau)\left(\psi_{\mu-2\delta,\tau-2\delta} - \psi_{\mu+2\delta,\tau-2\delta}\right) = 0
\end{aligned}
$$

with

$$
\begin{aligned}
C_\pm(\mu,\tau) &= 2\delta\left(\sqrt{|\tau \pm 2\delta|} + \sqrt{|\tau|}\right) \\
C_0(\mu,\tau) &= \sqrt{|\tau+\delta|} - \sqrt{|\tau-\delta|}
\end{aligned}
$$

Instability

With constant δ, difference equation *unstable in large region of minisuperspace* ($\mu > 4\tau$): Only exponential rather than oscillating solutions even in supposedly semiclassical regions.
[J. Rosen, J.-H. Jung, G. Khanna: CQG 23 (2006) 7075]

Seen based on analysis of difference equation (von Neumann stability); also indicated by tree-level equation (“holonomized”):

$$H^{(\delta)} = \frac{4}{\gamma^3\delta^3 G}\left(8\sin\frac{\delta b}{2}\cos\frac{\delta b}{2}\sin\frac{\delta c}{2}\cos\frac{\delta c}{2}\sqrt{|p_c|} + \left(4\sin^2\frac{\delta b}{2}\cos^2\frac{\delta b}{2}+\gamma^2\delta^2\right)\frac{|p_b|}{2\sqrt{|p_c|}}\right)$$

easier to analyze numerically. [D.-W. Chiou, L. Modesto, K. Vandersloot]
Incomplete effective equation, but *reliable for inferring problems* in semiclassical regimes (with weak quantum back-reaction).

(Near singularity: state parameters essential.)

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Lattice refinement

[MB, D. Cartin, G. Khanna: PRD 76 (2007) 064018]

Reason for instability: Wave function highly oscillating in semiclassical regimes; discrete lattice must be sufficiently small.

In Hamiltonian constraint, holonomies with constant δ do not take into account necessary *refinement* of discrete structure as geometry grows. *Cannot be seen purely in minisuperspace setting* $\longrightarrow$ extra input required.

Holonomies depend on vertex numbers $\mathcal{N}_I$ determining discreteness:

$$\begin{aligned} h_x^{(\ell_0^x)} &= \exp(\ell_0^x \tilde{c}\tau_3) = \exp(\ell_0^x L_0^{-1} c\tau_3) = \exp(c\tau_3/\mathcal{N}_x)\,, \\ h_\vartheta^{(\ell_0^\vartheta)} &= \exp(\ell_0^\vartheta b\tau_2) = \exp(b\tau_2/\mathcal{N}_\vartheta) \end{aligned}$$

for holonomies along edges of coordinate length ℓ_0^x, ℓ_0^ϑ: $\mathcal{N}_x = L_0/\ell_0^x$ edges of length ℓ_0^x in interval of size L_0.

Improvised uniqueness

Discrete versus classical evolving geometry:

$$\mathcal{N}_x(\lambda)v_x(\lambda) = \frac{|\tilde{p}_b(\lambda)|}{\sqrt{|\tilde{p}_c(\lambda)|}}L_0 \quad , \quad \mathcal{N}_\vartheta(\lambda)v_\vartheta(\lambda) = \sqrt{|\tilde{p}_c(\lambda)|}$$

Allows more freedom since two free functions in discrete geometry for each free function of classical geometry: Number of sites $\mathcal{N}$ as well as sizes v change in (internal) time λ. Precise behavior needed to model full dynamics.

Possible assumption: $v_x(\lambda)$ constant, then $\mathcal{N}_x \propto \tilde{a}_c L_0 = a_c$ and only c/a_c appears in holonomies: *improv[is]ed dynamics.*

Apparently unique, but based on assumption. Gives only one special case, *ruled out* by failure to capture near-horizon dynamics properly (instabilities since $\mathcal{N}_x$ small near $p_b = 0$).

Lattice refining difference equation

General refinement schemes to be considered; gives difference equation whose step-sizes $1/\mathcal{N}_I$ vary:

$$
\begin{aligned}
&C_+(\mu,\tau)\Big(\psi_{\mu+2\delta\mathcal{N}_\vartheta(\mu,\tau)^{-1},\tau+2\delta\mathcal{N}_x(\mu,\tau)^{-1}}\\
&\qquad -\psi_{\mu-2\delta\mathcal{N}_\vartheta(\mu,\tau)^{-1},\tau+2\delta\mathcal{N}_x(\mu,\tau)^{-1}}\Big)\\
&+C_0(\mu,\tau)\Big((\mu+2\delta\mathcal{N}_\vartheta(\mu,\tau)^{-1})\psi_{\mu+4\delta\mathcal{N}_\vartheta(\mu,\tau)^{-1},\tau}\\
&\qquad -2(1+2\gamma^2\delta^2\mathcal{N}_\vartheta(\mu,\tau)^{-2})\mu\psi_{\mu,\tau}\\
&\qquad +(\mu-2\delta\mathcal{N}_\vartheta(\mu,\tau)^{-1})\psi_{\mu-4\delta\mathcal{N}_\vartheta(\mu,\tau)^{-1},\tau}\Big)\\
&+C_-(\mu,\tau)\Big(\psi_{\mu-2\delta\mathcal{N}_\vartheta(\mu,\tau)^{-1},\tau-2\delta\mathcal{N}_x(\mu,\tau)^{-1}}\\
&\qquad -\psi_{\mu+2\delta\mathcal{N}_\vartheta(\mu,\tau)^{-1},\tau-2\delta\mathcal{N}_x(\mu,\tau)^{-1}}\Big)=0
\end{aligned}
$$

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Less symmetry

Lattice refinement of any kind (non-constant δ) goes beyond minisuperspace models. Try to embed homogeneous models in less symmetric ones, such as *spherical symmetry.*

Advantages: Triad representation still available; explicit inhomogeneity; different types of singularities can be studied.

Spherically symmetric gravity

$$A_a^i \mathrm{d}x^a \tau_i = A_x \tau_3 \mathrm{d}x + A_\varphi e^{\frac{1}{2}i\pi\tau} \Lambda_\varphi^A e^{-\frac{1}{2}i\pi\tau} \mathrm{d}\vartheta + A_\varphi \Lambda_\varphi^A \sin\vartheta \mathrm{d}\varphi$$
$$+\tau_3 \cos\vartheta \mathrm{d}\varphi$$

$$E_i^a \frac{\partial}{\partial x^a} \tau_i = E^x \tau_3 \sin\vartheta \frac{\partial}{\partial x} + E^\varphi e^{\frac{1}{2}i\pi\tau} \Lambda_E^\varphi e^{-\frac{1}{2}i\pi\tau} \sin\vartheta \frac{\partial}{\partial\vartheta} + E^\varphi \Lambda_E^\varphi \frac{\partial}{\partial\varphi}$$

with *U(1)-gauge theory* (A_x, E^x) on 1-dimensional Σ,
$A_\varphi : \Sigma \to \mathbb{R}$, $P^\varphi = 2E^\varphi \cos\alpha$ (using $\cos\alpha := \Lambda_\varphi^A \cdot \Lambda_\varphi^E$)
$e^{i\beta} : \Sigma \to \mathrm{U}(1)$, $P^\beta = 2A_\varphi E^\varphi \sin\alpha$ (in $\Lambda_\varphi^A = \cos(\beta)\tau_1 + \sin(\beta)\tau_2$).

Gauss constraint: $\partial_x E^x + 2A_\varphi E^\varphi \Lambda_\varphi^A \times \Lambda_\varphi^E = \partial_x E^x + P^\beta = 0$.
Thus $\sin\alpha = 0$ if homogeneous (Kantowski–Sachs) but new freedom in general spherical symmetry.

Complicated relation between momenta and triad components if $\cos\alpha$ is free. Will make volume in terms of fluxes complicated.

Basic variables and smearing

A simple canonical transformation

$$\begin{array}{lcl} A_\varphi & \longrightarrow A_\varphi \cos\alpha = & -\gamma K_\varphi \\ P^\varphi & \longrightarrow & 2E^\varphi \end{array}$$

gives densitized triad components as momenta.

Use extrinsic curvature component K_φ instead of connection component A_φ, but keep U(1)-connection A_x (for now).
Still allows natural smearing in 1-dimensional model:

$$h_{\mathcal{I}}(A_x) = \exp\left(\tfrac{1}{2} i \int_{\mathcal{I}} A_x(x) \mathrm{d}x\right) \quad , \quad F_v(E^x) = E^x(v)$$

$$h_v(K_\varphi) = \exp(i\gamma K_\varphi(v)) \quad , \quad F_{\mathcal{I}}(E^\varphi) = \int_{\mathcal{I}} \mathrm{d}x E^\phi(x)$$

$$h_v(\beta) = \exp(i\beta(v)) \quad , \quad F_v(P^\beta) = \int_{\mathcal{I}} \mathrm{d}x P^\beta(x)$$

Quantization

Orthonormal basis (1-dimensional graph g, $k_e, k_v \in \mathbb{Z}$, $\mu_v \in \mathbb{R}$)

$$T_{g,k,\mu}(A) = \prod_{e \in E(g)} \exp\left(\tfrac{1}{2} i k_e \int_e A_x \mathrm{d}x\right) \prod_{v \in V(g)} e^{-i\gamma\mu_v K_\varphi(v) + i k_v \beta(v)}$$

Flux operators:

$$\hat{E}^x(x) f(A) = -i\gamma \sum_e \frac{\delta h_e}{\delta A_x(x)} \frac{\partial f}{\partial h_e} = \frac{1}{2}\gamma \sum_{x \in e} h_e \frac{\partial f}{\partial h_e}$$

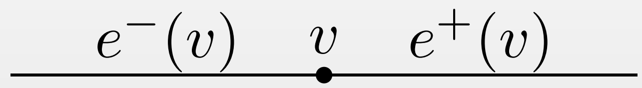
such that $\hat{E}^x(x) T_{g,k,\mu} = \frac{1}{2}\gamma k_{e(x)} T_{g,k,\mu}$.
Similarly:

$$\int_{\mathcal{I}} \hat{E}^\varphi T_{g,k,\mu} = \gamma \sum_{v \in \mathcal{I}} \mu_v T_{g,k,\mu} \quad , \quad \int_{\mathcal{I}} \hat{P}^\beta T_{g,k,\mu} = \gamma \sum_{v \in \mathcal{I}} k_v T_{g,k,\mu}$$

Constraints

Gauss constraint: $(E^x)' + P^\beta = 0$ implies

$k_v = -\frac{1}{2}(k_{e^+(v)} - k_{e^-(v)})$ such that



$$T_{g,k,\mu}(A) = \prod_{e\in E(g)} \exp\left(\tfrac{1}{2} i k_e \int_e (A_x + \beta')\mathrm{d}x\right) \prod_{v\in V(g)} \exp(-i\gamma\mu_v K_\varphi(v))$$

Now only extrinsic curvature: $\gamma K_x = A_x + \beta'$ (U(1)-invariant).

Diffeomorphism constraint: moves vertices.

Hamiltonian constraint:

$$H[N] = \int_\Sigma \mathrm{d}x N(x)|E^x|^{-1/2}\left((1-\Gamma_\varphi^2 + K_\varphi^2)E^\varphi - 2K_\varphi K_x E^x + 2E^x\Gamma'_\varphi\right)$$

with $\Gamma_\varphi = -\frac{(E^x)'}{2E^\varphi}$.

Hamiltonian constraint

Operator constructed from holonomy and flux operators following general scheme [T. Thiemann]

$$\mathrm{tr}(h_x h_\vartheta(v_+) h_x^{-1} h_\vartheta(v)^{-1} h_\varphi \{h_\varphi^{-1}, V\}) \sim \delta\gamma^2 K_\varphi(v) K_x(v) \sqrt{|E^x|}$$

and

$$\mathrm{tr}(h_\vartheta(v) h_\varphi(v) h_\vartheta(v)^{-1} h_\varphi(v)^{-1} h_x \{h_x^{-1}, V\}) \sim \delta\gamma^2 K_\varphi(v)^2 \frac{E^\varphi}{\sqrt{|E^x|}}$$

Radial edge length δ gives discretized integration measure; h_x, h_ϑ and h_φ suitable SU(2)-holonomies.

Operator changes labels through holonomies, may add new vertices to graphs as source for lattice refinement.

Physical states

Without explicit refinement: Decomposition in basis states ($k_e \in \mathbb{Z}$, $0 \le \mu_v \in \mathbb{R}$)

$\psi(A,\phi) = \sum_{\vec{k},\vec{\mu}} \tilde{\psi}(\vec{k},\vec{\mu})$  (A,ϕ)

gives rise to $\hat{H}\psi(A,\phi) = 0$ as set of *coupled difference equations* (one for each edge)

$$\hat{C}_0(\vec{k})\tilde{\psi}(\ldots,k_-,k_+,\ldots) + \hat{C}_{R+}(\vec{k})\tilde{\psi}(\ldots,k_-,k_+-2,\ldots)$$
$$+\hat{C}_{R-}(\vec{k})\tilde{\psi}(\ldots,k_-,k_++2,\ldots) + \hat{C}_{L+}(\vec{k})\tilde{\psi}(\ldots,k_--2,k_+,\ldots)$$
$$+\hat{C}_{L-}(\vec{k})\tilde{\psi}(\ldots,k_-+2,k_+,\ldots) + \cdots = 0$$

with difference operators $\hat{C}_0$ and $\hat{C}_\pm$ acting on the μ-dependence.

Non-singular: evolution across $k_i = 0$. [PRL 95 (2005) 061301]

Bounded inverse triad. [V. Husain, O. Winkler: CQG 22 (2005) L127]

Simpler dilaton models?

[MB, J. Reyes: CQG 26 (2009) 035018]

Use canonical transformation to introduce arbitrary dilaton potential (e.g. higher-dimensional black holes, BF [JT]): Hamiltonian constraint

$$K_\varphi^2 E^\varphi + 2K_\varphi K_x |E^x| + E^\varphi \left(-|E^x|^{\frac{1}{2}} V(\tfrac{1}{4}|E^x|) - \Gamma_\varphi^2 \right) + 2|E^x|\Gamma'_\varphi = 0$$

where again $\Gamma_\varphi = -(E^x)'/2E^\varphi$.

Only one term affected, which turns out to be the simplest one directly quantized in terms of flux operators.
Terms containing extrinsic curvature or spatial derivatives (spin connection) do not depend on the potential.

Advantage: Most details of loop quantization (holonomies, spatial discretization of derivatives) model independent.

Disadvantage: No simplifications from alternative potentials.

Consistent deformations

Inverse triad corrections can be incorporated in anomaly-free way:

$$H^{\rm inv}_{\rm grav}[N] = \int_\Sigma \mathrm{d}x N(x) f[E^x] |E^x|^{-1/2} \Big((1-\Gamma_\varphi^2 + K_\varphi^2) E^\varphi - 2K_\varphi K_x E^x + 2E^x \Gamma_\varphi' \Big)$$

together with matter Hamiltonian

$$H^{\rm inv}_{\rm matter}[N] = \int_\Sigma \mathrm{d}x N(x) (\nu[E^x] \mathcal{H}_{\rm kin} + \sigma[E^x] \mathcal{H}_{\rm grad} + \mathcal{H}_{\rm pot})$$

provided that $f^2 = \nu\sigma$.

Constraint algebra deformed, but remains first class.

[To be compared with partially gauge-fixed version: V. Husain et al.]

LTB models

[MB, T. Harada, R. Tibrewala: PRD 78 (2008) 064057]

Alternatively, consistent formulation with (partial) holonomy corrections under LTB-like condition $[(E^x)' = 2g[K_\varphi]E^\varphi]$.

Constraint equation (for $R = \sqrt{|E^x|}$, in terms of mass function $F(x)$):

$$2R\ddot{R} + \dot{R}^2\sqrt{1-\gamma^2\delta^2\dot{R}^2} = 0$$

together with the evolution equation

$$\begin{aligned} & 4\dot{R}^2R'\sqrt{1-\gamma^2\delta^2\dot{R}^2} + 8R\dot{R}\dot{R}' \\ = \ & F'\left(1+\sqrt{1-\gamma^2\delta^2\dot{R}^2}\right)^2\sqrt{1-\gamma^2\delta^2\dot{R}^2}\,. \end{aligned}$$

Spherical lattice refinement: $\delta[R]$, more complicated analysis.

Potentially naked singularities. No indication of resolution yet.

Conclusions

General features of *black hole dynamics* in models of loop quantum gravity available, but not unique.

Consistency conditions (e.g. stability) can be checked; provide conditions also for acceptable behavior in more general cases or full theory. *Several models already ruled out!*

Spacelike singularities in spherical symmetry resolved (even inhomogeneous ones), but no clear effective picture yet: *quantum back-reaction* still to be captured reliably.
For constraints, not all corrections implemented consistently *(anomaly-free)* yet.