Black hole state counting

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Jerzy Lewandowski, Warsaw

The ABCK quantum horizon

- fixed value a of the classical area, prequantized:
  $a \;=\; 4\pi\gamma\ell_{\rm P}k, \qquad k\in\mathbb{N}$
- The horizon Hilbert space $\mathcal{H}^k_{\rm Hor}$ states:
  - $|0\rangle_{\rm Hor}, \quad \ldots \quad, |(b_1,\, \ldots,\, b_n)\rangle_{\rm Hor}, \ldots$
  - $0 \;\neq\; b_i \in \mathbb{Z}_k, \quad i=1,\ldots,n, \quad n\in\mathbb{N}, \quad \sum_{i=1}^n b_i \;=\; 0\in\mathbb{Z}_k.$
- The bulk Hilbert space $\mathcal{H}_{\rm Bul}$ states:
  - $|(0),\, \ldots\rangle_{\rm Bul}, \quad \ldots, \quad |(m_1,j_1,\ldots,m_n,j_n),\, \ldots\rangle_{\rm Bul}$
  - $j_i\in\frac{1}{2}\mathbb{N}$, $i=1,\ldots,n, \quad m_i\in\{-j_i,\, -j_i+1,\, \ldots,\, j_i\}$
  - m_i and j_i represent the bulk geometry at the horizon,
  - "..." stand for the other bulk degrees of freedom
- The Hilbert space $\mathcal{H}_{\rm tot}$ of the horizon coupled with the bulk
  - the vectors $|(b_1,\ldots,b_n)\rangle_{\rm Hor}\otimes|(m_1,j_1,\ldots,m_n,j_n),\ldots\rangle_{\rm Bul}, \quad n\in\mathbb{N}$
  - the constraint $b_i \;=\; -2m_i \bmod k, \;\; \text{for} \;\; i=1,\ldots,n.$

The meaning of js and ms

- js: the LQG bulk quantum area operator
  - defined for any 2-surface contained in the bulk, in particular for the horizon 2-slice
  - $\hat{a}^{\mathrm{LQG}}\,|(m_1, j_1, ..., m_n, j_n), \; ...\rangle_{\mathrm{Bul}} \; = \; 8\pi\gamma\ell_P \sum_{i=1}^{n} \sqrt{j_i(j_i+1)}\,|(m_1, j_1, ..., m_n, j_n), \; ...\rangle_{\mathrm{Bul}}$
  - the spectrum is an interesting application of the number theory (Barbero's talk)
- ms: the flux across the horizon 2-slice of the normal vector field
  - defined only at the horizon by a function r : horizon $\rightarrow$ su(2) given by the ABCK model
  - $a^{\mathrm{flux}}(\tilde{E}, r) \; = \; \frac{1}{2}\int_S |\tilde{E}^a_i r^i \epsilon_{abc} dx^b \wedge dx^c|$
  - $\hat{a}^{\mathrm{flux}}\,|(m_1, j_1, ..., m_n, j_n), \; ...\rangle_{\mathrm{Bul}} \; = \; 8\pi\gamma\ell^2_{\mathrm{Pl}} \sum_{i=1}^{n} |m_i||(m_1, j_1, ..., m_n, j_n), \; ...\rangle_{\mathrm{Bul}}$
  - the spectrum is $8\pi\gamma\ell_P\mathbb{N}$
- Classically $a^{\mathrm{LQG}} = a^{\mathrm{flux}}$.

Entropy: what we want to count

According to the direct definition, the isolated horizon entropy is the logarithm of the number of states $|(b_1, ..., b_n)\rangle_{\rm Hor}$ such that there are states

$$|(b_1, ..., b_n)\rangle_{\rm Hor} \otimes |(m_1, j_1, ..., m_n, j_n), ...\rangle_{\rm Bul} \; \in \; \mathcal{H}_{\rm Tot} \tag{1}$$

which satisfy the condition

$$a(m_1, j_1, ..., m_n, j_n) \; \leq \; a = 4\pi\gamma\ell_P^2 k, \tag{2}$$

where $a(m_1, ..., j_n)$ is the eigenvalue of the quantum area operator.

The map $(m_1, ..., m_n) \;\rightarrow\; (b_1, ..., b_n)$

The counting of the suitable sequences $(b_1, ..., b_n)$ can be translated into a counting of the sequences $(m_1, ..., m_n)$.

$$\sum_{i=1}^{n} b_i \;= 0 \mod k \;\Rightarrow\; \sum_{i=1}^{n} m_i \;=\; 0 \bmod k/2 \tag{3}$$

$$a(m_1, ..., j_n) \le a \;\Rightarrow\; \sum_{i=1}^{n} \sqrt{j_i(j_i+1)} \;\le\; \frac{k}{2} \;\Rightarrow\; \sum_{i=1}^{n} \sqrt{|m_i|(|m_i|+1)} \le \frac{k}{2} \tag{4}$$

$$\Rightarrow \sum_{i=1}^{n} |m_i| \;<\; \frac{k}{2}, \quad \sum_{i=1}^{n} m_i \;=\; 0, \quad |m_i| \;<\; \frac{k}{4}. \tag{5}$$

Now,
$(m_1,\, m_2, ...,\, m_n), (m'_1,\, m'_2, ...,\, m'_n) \;\mapsto\; (b_1, b_2, ..., b_n) \;\Leftrightarrow m_i - m'_i = N_i k/2$

The last enaquality implies, that $(m_1, ...m_n) \;\mapsto\; (b_1, ..., b_n)$ is 1 to 1.

The counting problem in terms of ms only

How to get rid of the js? Given $(m_1, ..., m_n)$ such that

$$\sum_{i=1}^{n} \sqrt{|m_i|(|m_i| + 1)} \leq \frac{k}{2}$$

there is $|(m_1, j_1, ..., m_n, j_n)\rangle_{\mathrm{Bul}}$ such that

$$\sum_{i=1}^{n} \sqrt{j_i(j_i + 1)} \leq \frac{k}{2}$$

Therefore, it is necessary and sufficient to use the former condition, and forget the js.

The set of $(m_1, ..., m_n)$s acounting to the entropy

In summary (Domagala, L 2004):
The entropy $S(a)$ of a quantum ABCK horizon of the classical area $a = 4\pi\gamma\ell_P^2 k$, $k \in \mathbb{N}$, is

$$S(a) = \log\big(1 + N(k)\big)$$

where $N(k)$ denotes the number of all the finite, arbitrarily long, sequences $\vec{m} = (m_1, \ldots, m_n)$ of elements of $\mathbb{Z}_*/2$ (i.e. non-zero elements of $\mathbb{Z}/2$), such that the following equality and inequality are satisfied:

$$\sum_{i=1}^{n} m_i = 0, \quad \sum_{i=1}^{n} \sqrt{|m_i|(|m_1| + 1)} \leq \frac{k}{2}\,.$$