Large $SU(2)$ gauge transformations in LQG: effects on black hole entropy

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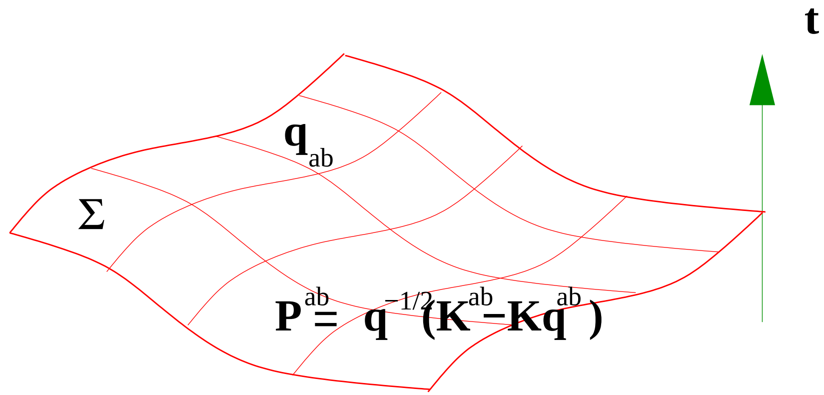
A remarkable feature of general relativity (GR) is that it admits a connection formulation with a (unconstrained) phase space isomorphic to that of $SU(2)$ Yang Mills theory [Ashtekar, Barbero].

From ADM variables to Ashtekar-Barbero variables

$$(q_{ab}, P^{ab}) \rightarrow (A^i_a, E^a_i)$$



From the (densitized) triad $qq^{ab} = E^a_j E^b_i \delta^{ij}$ and $K^i_a = q^{-\frac{1}{2}} K_{ab} E^{bi}$ define

$$^{\gamma}P^a_i = (\kappa\gamma)^{-1} E^a_i \qquad A^i_a = \delta W_1[E]/\delta E^a_i + \gamma K^i_a.$$

$$W_1[E] = \int_\Sigma \epsilon_{bcd} E^a_{[i} E^b_{j]} \partial_a \frac{E^{ci} E^{dj}}{\det(E)} \quad \text{which gives} \quad \Gamma^i_a = \delta W_1[E]/\delta E^a_i.$$

$$G_i = \epsilon_{ijk} E^{aj} K^k_a \approx 0 \rightarrow G_i = D_a {}^{\gamma}P^a_i \approx 0.$$

Are there more general connection variables than the ones obtained above? Yes, take

$$W_1'[E] = W_1[E] + \\ + \int_\Sigma \lambda_1 \mathscr{L}_{CS}(\Gamma) + \lambda_2 \sqrt{E} + \lambda_3 R[E]\sqrt{E} + \lambda_4 R_{abcd}R^{abcd}[E]\sqrt{E} + \cdots$$

Another way: given a background independent functional $W_2[A]$

$${}^\gamma P_i^a \rightarrow {}^\gamma P_i^a + W_2[A]/\delta A_a^i.$$

Only possibility

$$W_2[A] = \theta S_{CS}[A] = \frac{\theta}{16\pi^2}\int_\Sigma \mathrm{Tr}[A \wedge dA + \frac{2}{3} A \wedge A \wedge A].$$

where θ is a real parameter. Taking $\lambda_n = 0$ and defining $B_i^a = \epsilon^{abc} F_{bc}^i$ we get

$$\boxed{{}^{\gamma\theta}P_i^a = (\kappa\gamma)^{-1} E_i^a + \tfrac{\theta}{8\pi^2} B_i^a \quad A_a^i = \Gamma_a^i + \gamma K_a^i}$$

There is a more geometric way to get the previous variables

Large $SU(2)$ gauge transformations[Ashtekar-Balachandran]

$$\text{Dirac procedure} \quad D_a E^a_i \triangleright \Psi[A] = 0$$

i.e., gauge invariance under $\mathscr{G}_0 \subset \mathscr{G}$ ($\mathscr{G}_0$ gauge transformations connected to the identity). As $\mathscr{G}/\mathscr{G}_0 \approx \mathbb{Z}$. Elements $[g(x)] \in \mathscr{G}/\mathscr{G}_0$ are characterized by

$$w[g] = \frac{1}{24\pi^2} \int_\Sigma \mathrm{tr}[g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg].$$

Therefore, physical ($\mathscr{G}_o$-invariant) are in $\mathscr{H} = \oplus_\theta \mathscr{H}_\theta$ with $\theta \in [0, 2\pi]$ such that

$$\Psi[A] \in \mathscr{H}_\theta, \quad \text{and} \quad \alpha \in \mathscr{G} \quad \Rightarrow \quad \alpha \triangleright \Psi[A] = e^{i\theta w[\alpha]}\Psi[A].$$

Since local physical observables are $\mathscr{G}$ invariant $\Rightarrow \mathscr{H}_\theta =$ super selected sectors. The non-trivial transformation rule for states in $\mathscr{H}_\theta$ can be shifted to operators

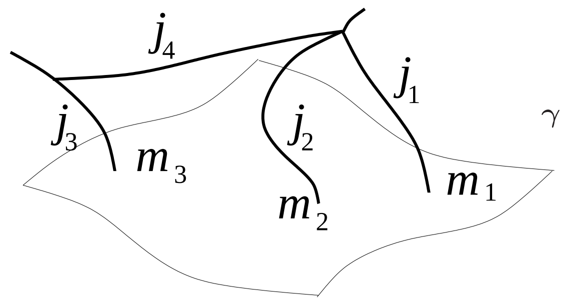
$$\Psi_0[A] = \exp\left(-i\theta S_{CS}[A]\right)\Psi[A] \in \mathscr{H}_0 \qquad \Rightarrow$$

$$^{\gamma\theta}P^a_i \equiv \exp\left(-iW_2[A]\right) {}^{\gamma}P^a_i \exp\left(iW_2[A]\right)$$

$$\boxed{^{\gamma\theta}P^a_i = {}^{\gamma}P^a_i + \frac{\theta}{8\pi^2}\, B^a_i}$$

Effects on quantum geometry

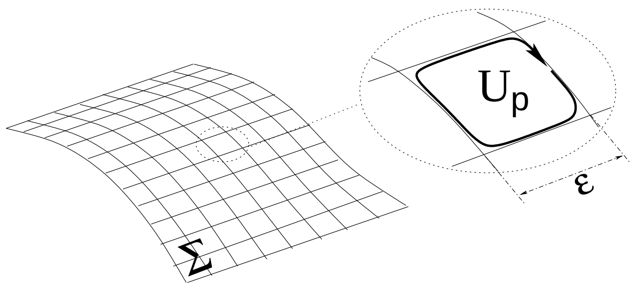
The flux operators ${}^{\gamma\theta}P(r,S) = \int_S r \cdot (\epsilon^{\gamma\theta}P)$ for $r \in su(2)$ have discrete spectrum



$$
\begin{aligned}
{}^{\gamma\theta}P(r,S) \triangleright |n; \{j_i, m_i\}_{i=1}^n\rangle &= \\
&= \sum_{i=1}^n m_i \ |n; \{j_i, m_i\}_{i=1}^n\rangle
\end{aligned}
$$

Area and volume are ill-defined (IR divergent) for $\theta \neq 0$

$$
A(S) = \int_S \sqrt{E_i^a E^{bi} n_a n_b} = \kappa\gamma \int_S \left[{}^{\gamma\theta}P \cdot {}^{\gamma\theta}P - \frac{\theta}{4\pi^2} B \cdot {}^{\gamma\theta}P + \frac{\theta^2}{(8\pi^2)^2} B \cdot B \right]^{1/2}
$$



$$
\begin{aligned}
A(S) \triangleright 1 &= \lim_{\epsilon \to 0} \sum_{n,m} \sqrt{E(S^{nm}, \tau^i) E(S^{nm}, \tau_i)} \triangleright 1 \\
&= \tfrac{\kappa\gamma\theta}{8\pi^2} \lim_{\epsilon \to 0} \sum_{n,m} \sqrt{B(S^{nm}, \tau^i) B(S^{nm}, \tau_i)} \triangleright 1 \\
&= \tfrac{\kappa\gamma\theta}{4\pi^2} \lim_{\epsilon \to 0} \sum_{n,m} \sqrt{\mathrm{Tr}[U^{nm}\tau_i]\mathrm{Tr}[U^{nm}\tau^i]} \triangleright 1,
\end{aligned}
$$

$$
||A_\epsilon(S) \triangleright 1|| > K\epsilon^{-1}
$$

Isolated horizons boundary condition

There are non-trivial degrees of freedom at the horizon encoded in the pull back of the bulk connection on the horizon $H = \Delta \cap \Sigma$; a $U(1)$-connection $A = A^i r_i$ and

$$F_{ab}(A) = -\frac{2\pi}{a_H}\epsilon_{abc}E^c{}_i r^i \quad \text{where } a_H \text{ is the macroscopic area of the horizon}$$

The simplectic structure [Ashtekar-Corichi-Krasnov]

$$\Omega(\delta_1,\delta_2) = \frac{1}{8\pi G\gamma}\int_\sigma \mathrm{Tr}[\delta_1 A\wedge\delta_2(\epsilon\cdot E)-\delta_2 A\wedge\delta_1(\epsilon\cdot E)] - \frac{a_H}{16\pi^2 G\gamma}\int_H \delta_1 A\wedge\delta_2 A,$$

where $(\epsilon\cdot E)^i_{ab} \equiv \epsilon_{abc}(E^c)^i$, and the horizon contribution is a $U(1)$ Chern-Simons simplectic form of level $k = a_H/(4\pi\gamma G)$. The previous simplectic structure can be obtained as the curl of the simplectic potential

$$\Theta(\delta) = -\frac{1}{8\pi G\gamma}\int_\Sigma \mathrm{Tr}[\delta A\wedge(\epsilon\cdot E)] + \frac{a_H}{32G\pi^2\gamma}\int_H \delta A\wedge A.$$

Effect of θ on the simplectic structure: introducing a new potential

$$\tilde{\Theta} = \Theta - \delta W[A] =$$
$$= -\int_\Sigma \mathrm{Tr}\delta A \wedge (\frac{1}{8\pi G\gamma}\epsilon \cdot E + \frac{\theta}{8\pi^2}F(A)) + \left[\frac{a_H}{32\pi^2 G\gamma} - \frac{\theta}{16\pi^2}\right]\int_H \delta A \wedge A$$
$$= \int_\Sigma \mathrm{Tr}\delta A \wedge (\epsilon \cdot {}^{\gamma\theta}P) + \frac{k(\theta)}{8\pi}\int_H \delta A \wedge A,$$

where $W[A] = \theta S_{CS}(A)$ and we used that

$$\delta S_{CS}[A] = \frac{1}{8\pi^2}\int_\Sigma \mathrm{Tr}[F(A) \wedge \delta A] - \frac{1}{16\pi^2}\int_H A \wedge \delta A + \text{term vanishing at } \infty.$$

So in addition to the transformation ${}^{\gamma}P \to {}^{\gamma\theta}P$, θ shifts the CS level:

$$k(\theta) = \frac{a_H}{4\pi G\gamma} - \frac{\theta}{2\pi}.$$

The simplectic form takes the form

$$\Omega(\delta_1, \delta_2) = \frac{1}{8\pi G}\int_\sigma \mathrm{Tr}[\delta_1 A \wedge \delta_2(\epsilon \cdot {}^{\gamma\theta}P) - \delta_2 A \wedge \delta_1(\epsilon \cdot {}^{\gamma\theta}P)] - \frac{k(\theta)}{4\pi}\int_H \delta_1 A \wedge \delta_2 A,$$

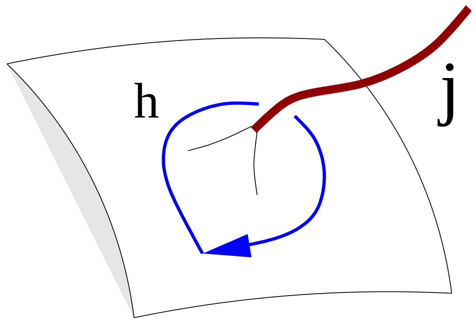
The quantum boundary conditions [Ashtekar-Corichi-Krasnov-Baez]

$$F_{ab}(A) = -\frac{2\pi}{a_H}\epsilon_{abc}E^c{}_i r^i \Rightarrow$$

$$\frac{a_H}{2\pi}F_{ab}(A) = -(8\pi G\gamma)\epsilon_{abc}({}^{\gamma\theta}P^c{}_i r^i - \frac{\theta}{8\pi}B^c_i r^i) \Rightarrow$$

$$\boxed{\frac{1}{4\pi}\left[\frac{a_H}{(4\pi G\gamma)} - \frac{\theta}{2\pi}\right]F_{ab} = -\epsilon_{abc}{}^{\gamma\theta}P^c{}_i r^i}$$

As the boundary condition and the spectrum of $\widehat{F}_{ab}$ depend on the θ only through the CS level the quantum boundary condition imposes the θ-independent matching

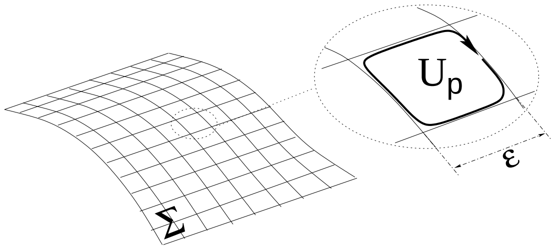


$$h(A) \triangleright \psi_n = e^{iF_n}\psi_n$$
$$\text{with} \quad F_n = \frac{2\pi n}{k}$$
$$\text{Quantum boundary condition } n = -2m$$

One can implement the constraints at the horizon as for $\theta = 0$.

The black hole horizon area spectrum. Using the quantum boundary condition

$$B^a n_a = -\frac{4\pi}{k(\theta)} P^a_i n_a r^i$$



$$A_H \left|n; \{j_i, m_i\}_{i=1}^n\right\rangle =$$
$$\lim_{\epsilon\to 0} \sum_{n,m} \sqrt{E(S^{nm}, \tau^i)E(S^{nm}, \tau_i)} \left|n; \{j_i, m_i\}_{i=1}^n\right\rangle =$$
$$= 8\pi\gamma\ell_p^2 \sum_{i=1}^n \sqrt{C(\theta)m_i^2 + j_i(j_i+1)} \left|n, \{j_i, m_i\}_{i=1}^n\right\rangle$$

$$C(\theta) = \frac{\theta}{k(\theta)\pi}\left(\frac{\theta}{k(\theta)\pi} + 1\right)$$

Therefore, here the quantum isolated horizon constraint implies that the quantum operator associated to the (Dirac) physical observable A_H is well defined. The counting techniques of [Meissner, Domagala-Lewandowski] one finds that the θ-dependence does not change the leading term in the entropy: explicitly $S_H := \log[\mathcal{N}(a_H)] \approx (4\ell_p\gamma)^{-1}\gamma_M a_H$, where $\mathcal{N}(a_H)$ is the number of horizon states compatible with a macroscopic horizon area a_H and $\gamma_M = 0.23...$

Conclusions:

- As in QCD the effects of large $SU(2)$ gauge transformations are encoded in a real parameter $\theta \in [0, 2\pi]$. Effects are expected in parity violating systems, e.g. Black Holes.

- From dimensional reasons we expect the former effects to be important in the deep Planckian regime. However, we discover drastic implications for certain kinematical geometric operators (Area and volume are ill defined).

- But what about quantum horizon area? Quantum horizon area remains well defined thanks to the IH boundary condition BH entropy remains finite and agrees with standard results in the semiclassical regime (polynomial corrections in $\epsilon = \theta \ell_p^2 / a_H$).

- Some aspects of the result are reminiscent of the BH entropy calculation in the presence of nonminimaly coupled scalar fields [Ashtekar-Corichi-Sudarsky]

Additional questions:

- Dirac vs. Kinematical observables [Thiemann-Dittrich]
- Can one study analytically the BH entropy behaviour for small black holes for which the θ effects will be important?
- It seems that for *physical area and volume* to be well defined for arbitrary θ we need the curvature to be distributional. Link with simplicial like geometry? Strings and branes of the kind studied in [Baez-AP, Montesinos-AP, Fairbairn-AP]