Toy black holes, generating functions, and entropy quantization

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0. Generating functions

Generating functions

Combinatorial problem $N(n)$ hard? Try and compute

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In principle can get $N(n)$ from that, but even more:

$$\sum_n N(n) = G(1),$$

$$G_{\leq}(z) \doteq \sum_n \left[\sum_{n' \leq n} N(n') \right] z^n = \frac{1}{1-z} G(z),$$

etc., and asymptotic behavior. Heuristically:

$$R = |\text{Pole of } G(z) \text{ closest to } 0| \qquad \Rightarrow \qquad N(n) \propto R^{-n}.$$

Sometimes generating functions are easy to calculate due to identities such as

$$N(n) = \sum_{n'=0}^{n} A(n')B(n-n') \qquad \Rightarrow \qquad G_N(z) = G_A(z)G_B(z).$$

1. The Toy Black Hole

Black holes in LQG

Long, beautiful story (Rovelli, Ashtekar, Baez, Corichi, Krasnov, …).
BH horizon punctured by spin-network edges.

- ✘ Surface states: $|(b_1, b_2, \ldots)\rangle$. $b_i \in \mathbb{Z}$ mod k.
- ✘ Bulk states: $|(j_1, m_1; j_2, m_2; \ldots)(\text{ more })\rangle$. $j_i \in \mathbb{N}_*/2$, $m_i \in \{-j, -j+1, \ldots, j\}$.
- ✘ j-labels $\longleftrightarrow$ area: $A_j = 8\pi\gamma l_P^2 \sqrt{j(j+1)}$

Details in previous talks.

We can expand

$$\sqrt{j(j+1)} = j + \frac{1}{2} - \frac{1}{4(2j+1)} - \frac{1}{16(2j+1)^3} + \dots$$

so let us do fantasy LQG:

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This simplifies things tremendously.

Counting

When counting BH states, two ways to count:

- ✗ b-labels only
- ✗ b-, m-, and j-labels

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- ✘ $\mathfrak{b}$-labels only
- ✘ $\mathfrak{b}$-, $\mathfrak{m}$-, and $\mathfrak{j}$-labels

This talk: $\mathfrak{b}$-labels only.

A side remark: when counting in this way for the toy black hole

- ✘ number of surface states for area in a small shell around A
- ✘ number of surface states for area smaller or equal A

are exactly equal.

Following Lewandowski and Domagala, but with new spectrum, number of states is

$$N(a) \doteq \left| \left\{ (m_1, m_2, \ldots),\ m_i \in \mathbb{Z}_* \ : \ \sum_i m_i = 0, \sum_i (|m_i| + 1) = a \right\} \right| .$$

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Will actually look at slightly more general problem:

$$N(a, j) \doteq \left| \left\{ (m_1, m_2, \ldots),\ m_i \in \mathbb{Z}_* \,:\, \sum_i m_i = j, \sum_i (|m_i| + 1) = a \right\} \right|.$$

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$$G(g, z) \doteq \sum_{a=0}^{\infty} \sum_{j=-a}^{a} N(a, j) g^a z^j.$$

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First determine the generating function for sequences of length 1:

$$G_1(g,z) = g \sum_{m=1}^{\infty} (gz)^m + \left(\frac{g}{z}\right)^m = g^2 \left(\frac{1}{z-g} + \frac{z}{1-gz}\right).$$

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GF for sequences of length 2 is $(G_1)^2$ etc. So altogether

$$G(g,z) = \sum_{m=1}^{\infty} (G_1(g,z))^m = \frac{g^2 \left(z^2 - 2gz + 1\right)}{(g+1)\left(2zg^2 - \left(z^2 + z + 1\right) g + z\right)}.$$

Derived generating functions

$$G^{(j=0)}(g) \doteq \sum_a N(a, 0) g^a = \frac{1}{2\pi i} \oint_C \frac{1}{z} G(g, z)\, dz$$

$$= \frac{(1-g)g}{(g+1)\sqrt{(g-1)(2g-1)(2g^2+g+1)}} - \frac{g}{g+1}$$

$$= 2g^4 + 2g^6 + 6g^7 + 8g^8 + 12g^9 + 34g^{10} + 58g^{11} + \dots$$

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$$T(g) \doteq \sum_{j,a} N(a,j)g^{a} = G(g,1) = -\frac{2g^2}{2g^2+g-1} = \frac{1}{3}\sum_{a=1}(2(-1)^{a} + 2^{a})g^{a}$$

Asymptotics

Heuristics: If

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad \text{and} \qquad R = |\text{Pole of } f(x) \text{ closest to } 0|$$

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expect $c_n \propto R^{-n}$. For example:

$$G^{(j=0)}(g) = \frac{(1-g)g}{(g+1)\sqrt{(g-1)(2g-1)(2g^2+g+1)}} - \frac{g}{g+1}$$

Expect $N(a,0) \propto 2^a$.

Theorems show

$$N(a,0) \sim \frac{1}{6\sqrt{\pi}} \frac{2^a}{\sqrt{a}}, \qquad \sum_{b}^{a} N(b,0) \sim \frac{1}{3\sqrt{\pi}} \frac{2^a}{\sqrt{a}}$$

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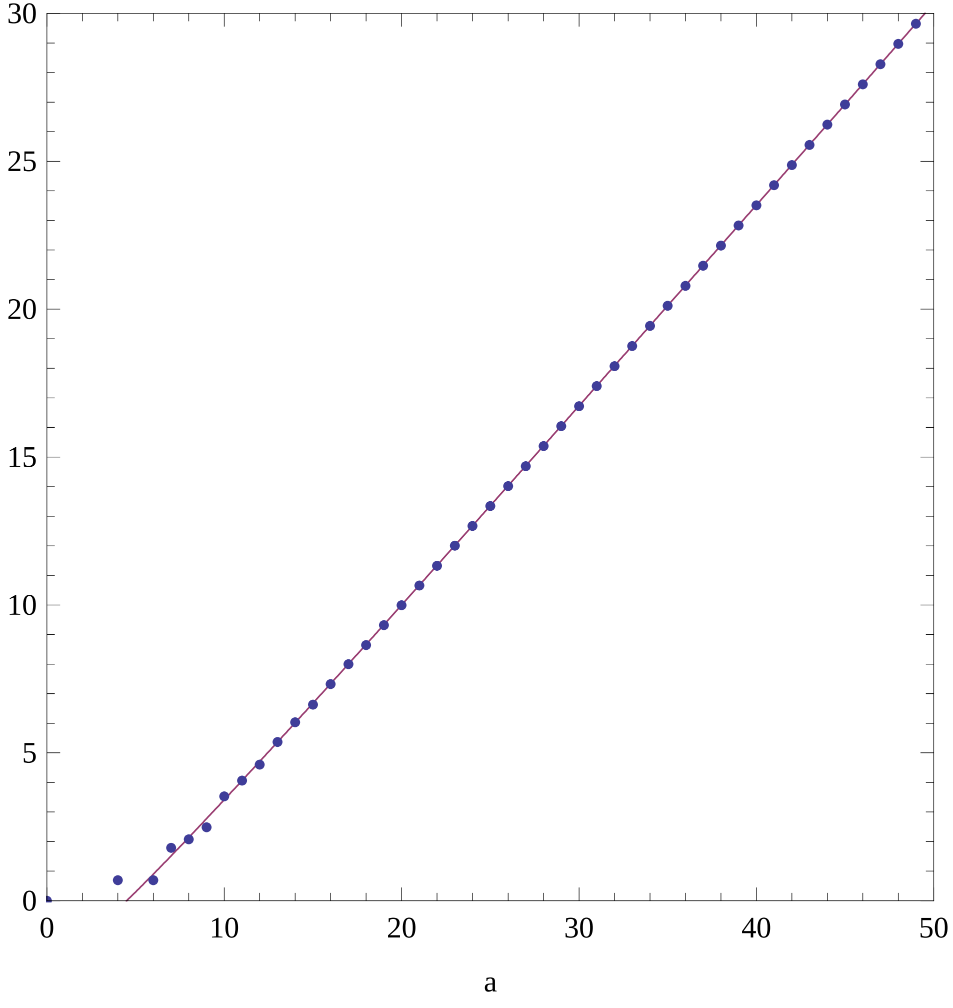
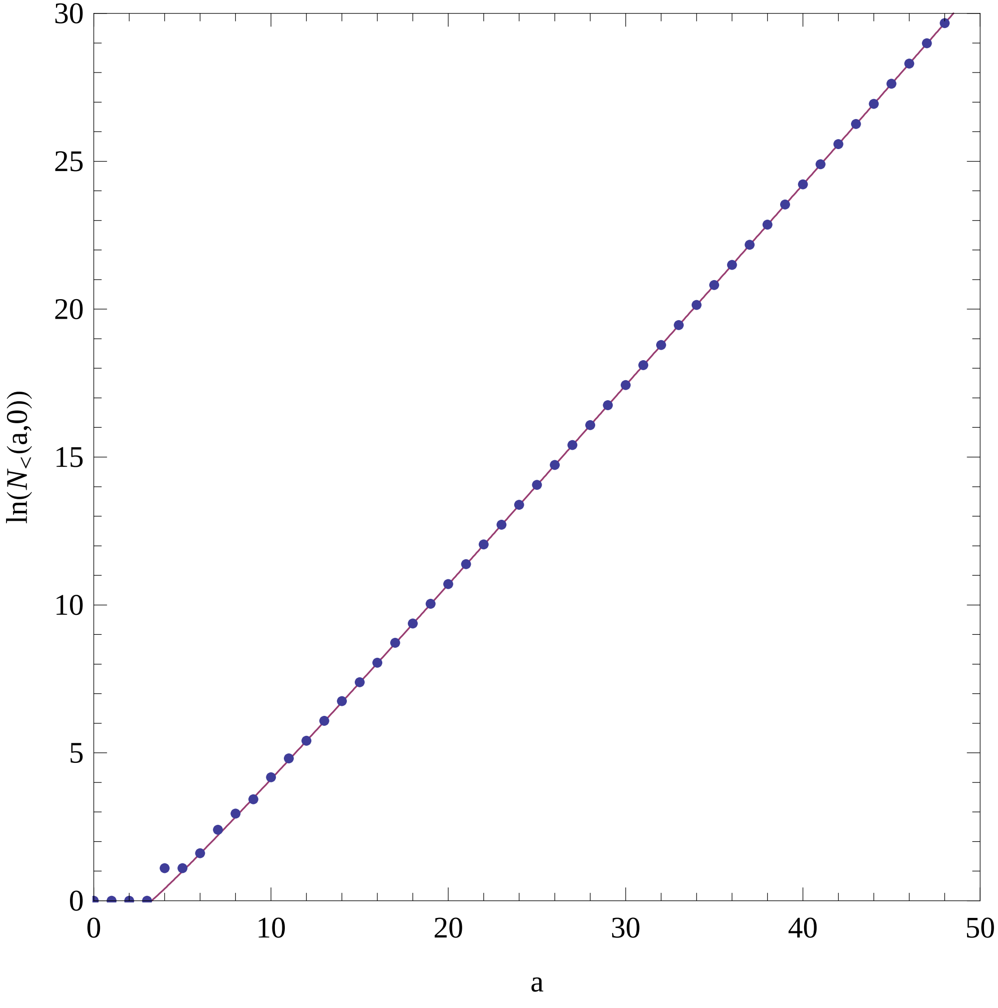
and we already saw that

$$\sum_{j} N(a,j) \sim 2^a.$$



a



ln($N_<$(a,0))
a

2. Entropy Quantization

A. Corichi, J. Diaz-Polo and E. Fernandez-Borja, "Black hole entropy quantization," Phys. Rev. Lett. **98** (2007) 181301

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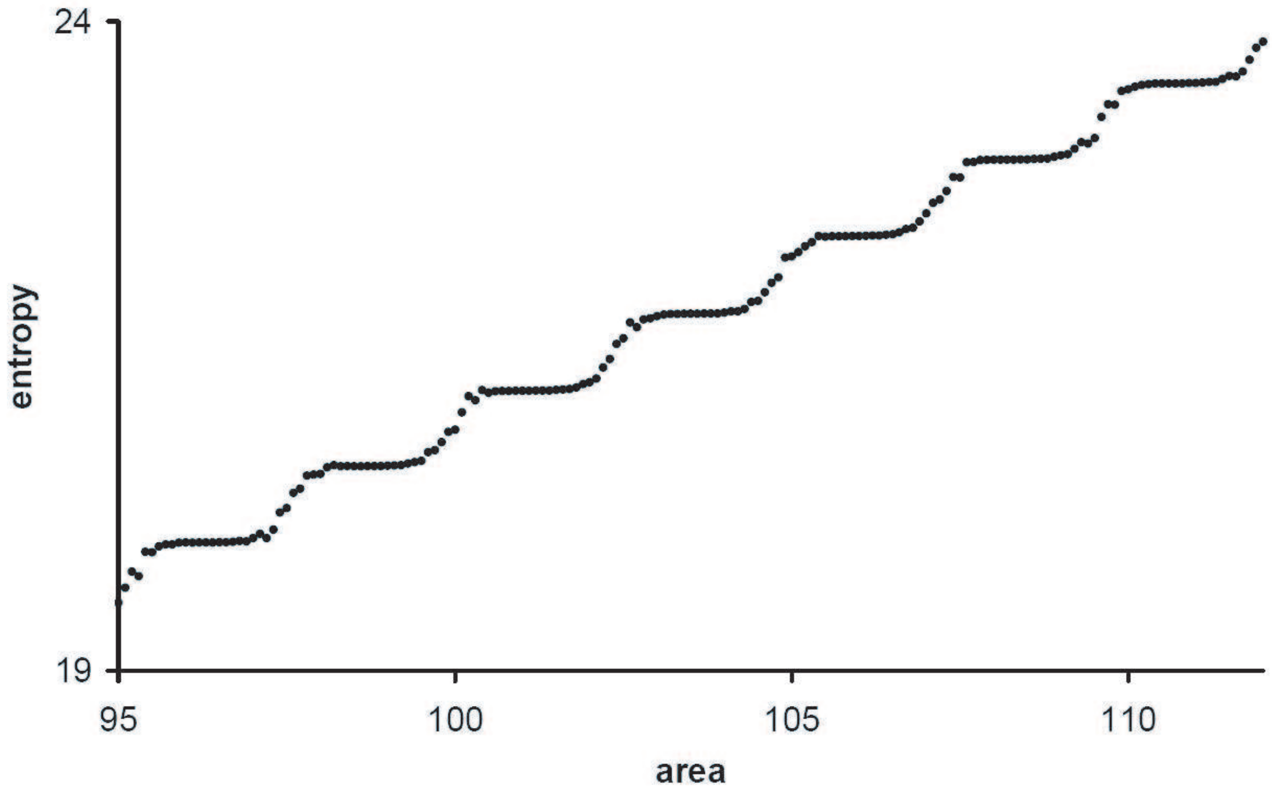

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$$\Delta A = \gamma \chi l_P^2, \qquad \text{with } \chi \approx \chi_{\text{CDF}} = 8.80$$

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$$\chi \approx 8 \ln(3)$$

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Furthermore: Phenomenon contingent on implementing quantum boundary conditions

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Toy black hole shows staircase to perfection. (No surprise.)

Why is the real black hole so similar?

GF not directly applicable (see however Fernando's talk!)

But can take some lessons over:

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Why is the real black hole so similar?

GF not directly applicable (see however Fernando's talk!)

But can take some lessons over:

1. Look at the problem in terms of $N(I, j)$:

$$N(I, j) = \left| \left\{ (m_1, m_2, \ldots),\ m_i \in \mathbb{Z}_*/2 : \sum_i m_i = j, \sum_i \sqrt{|m_i|(|m_i|+1)} \in I \right\} \right|$$

2. View state labels as paths in a certain space

Then use statistics of steps in these paths to explain pattern.

States as paths

Space (Area $\times \sum_i m_i$): $\mathbb{R}_+ \times \mathbb{Z}/2$

State label $(m_1, m_2, \ldots)$ gives a **path** through this space, starting at the point $(0, 0)$.

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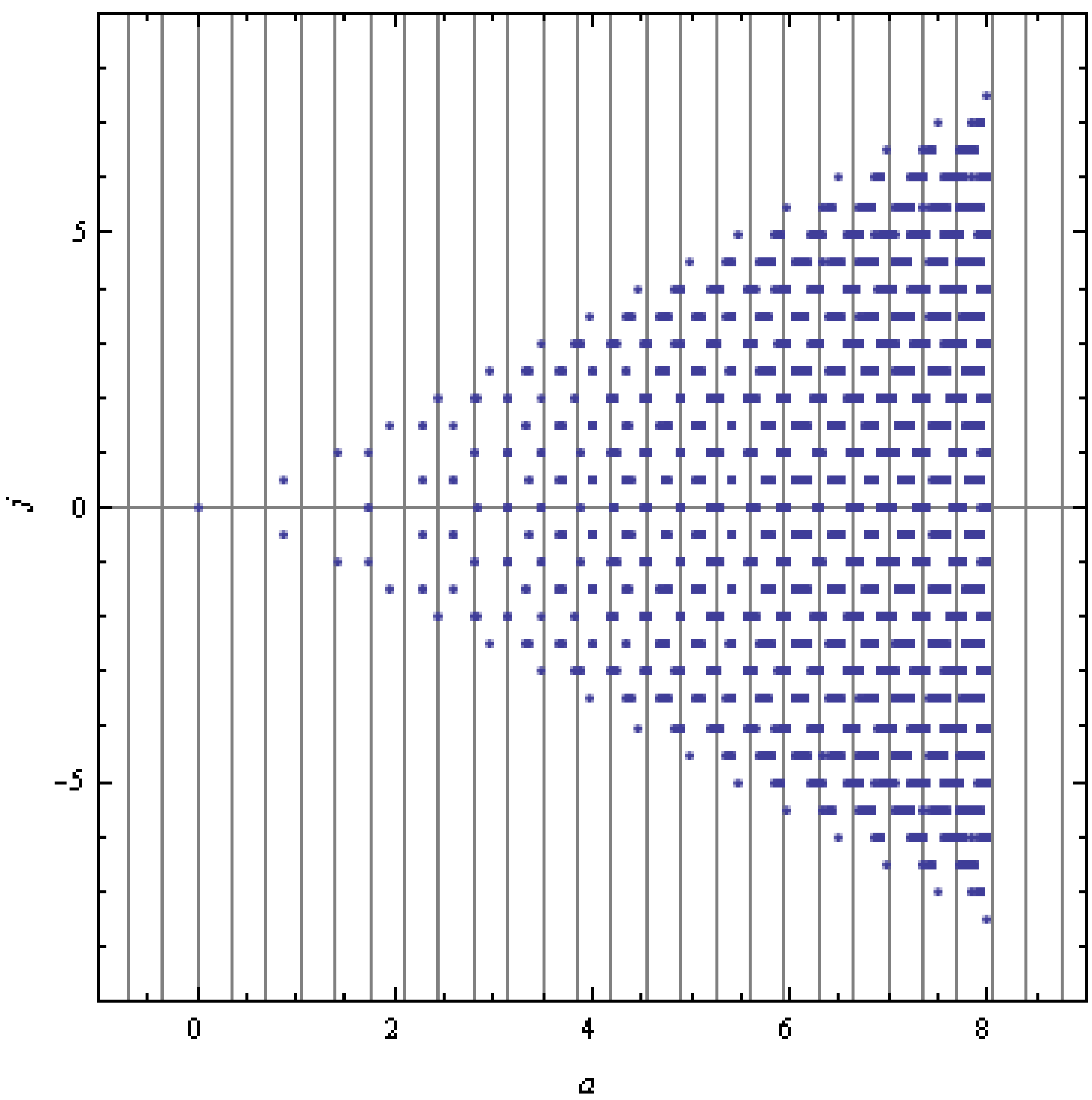
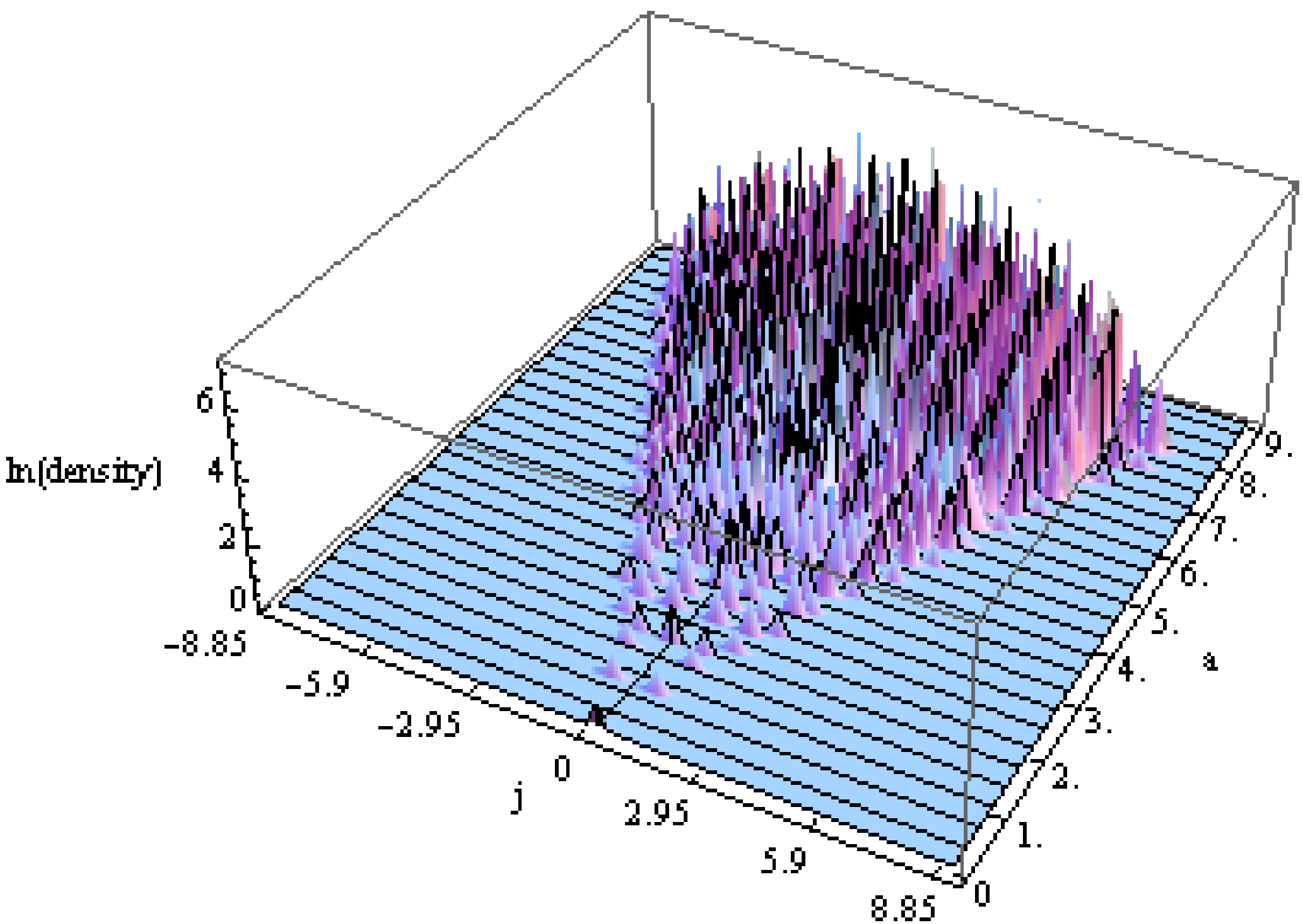
Physical states $\longleftrightarrow$ paths that end on $\mathbb{R}_+ \times \{0\}$

Plot endpoints of all paths




ln(density)
0
2
4
6
-8.85
-5.9
-2.95
0
2.95
5.9
8.85
j
0
1.
2.
3.
4.
5.
6.
7.
8.
9.
a

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- ✘ we don't have that
- ✘ we have zillions of paths (with may steps each) whose endpoints cluster in a pattern
- ✘ use statistics!
- ✘ maybe we have $s(m) = I(m)s_0$, with $I(m) \in \mathbb{N}$ *on average*?

Statistics of steps

Write

$$a(m) = I(m)\Delta a + \epsilon(m)$$

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For path $\mathcal{P} = (m_1, m_2, \ldots, m_n)$ let

$$\delta(n, \mathcal{P}) = \sum_{i=1}^{n} \epsilon(m_i).$$

Then the cental limit theorem says

$$\langle \delta(n) \rangle \approx n \langle \epsilon(m) \rangle, \qquad \langle \delta(n)^2 - \langle \delta(n) \rangle^2 \rangle \approx n \langle \epsilon(m)^2 - \langle \epsilon(m) \rangle^2 \rangle,$$

This furnishes explanation of the clustering if

- ✘ $\langle \epsilon(m) \rangle = 0$
- ✘ small variance:

$$\sqrt{n\langle \epsilon(m)^2 \rangle} << \Delta a, \qquad \text{or} \qquad n << \frac{(\Delta a)^2}{\langle \epsilon(m)^2 \rangle}.$$

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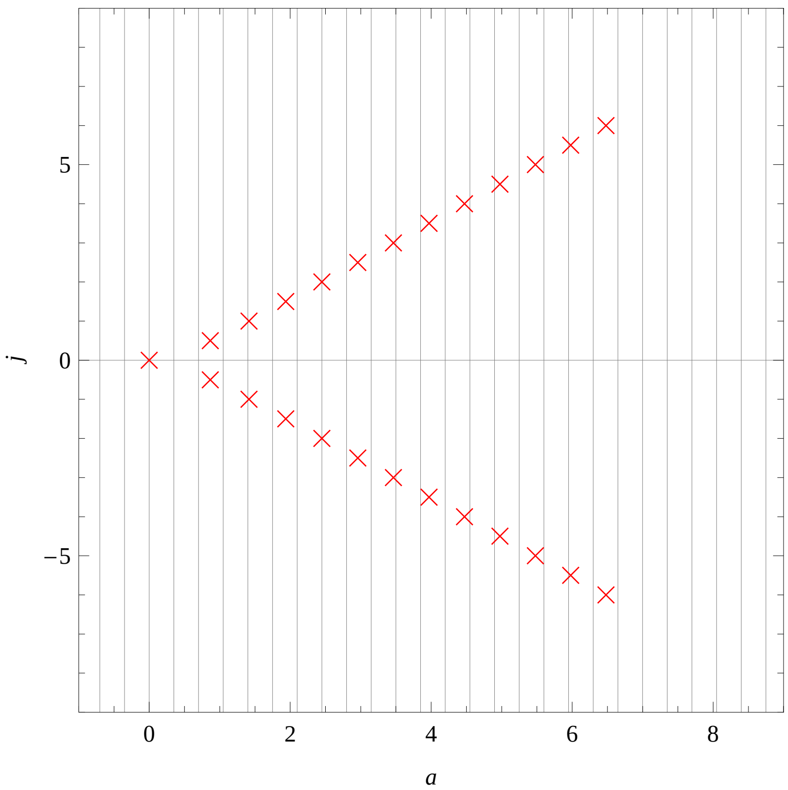
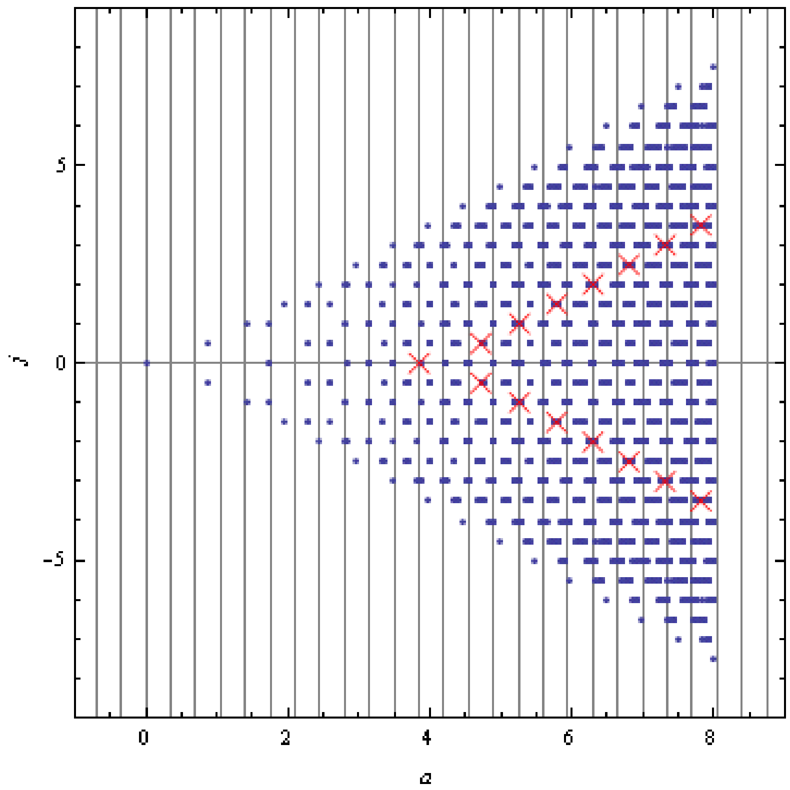
How does this work out in practice? Will need

- information about probability distributions
- educated guess for $I(m)$

For probability distribution: Approximate by Lewandowski-Domagala.

Determining $I(m)$





$$a(m) = \left(\frac{3}{2} \cdot 2m + 1\right) \Delta a + \epsilon(m).$$

Caveat

$$a(m) = \left(\frac{3}{2} \cdot 2m + 1\right) \Delta a + \epsilon(m).$$

is the right thing to use to determine Δa.

But it only explains a periodicity $3\Delta a$ for physical states.

The rest is in the initial conditions .

Better explanation by Agulló, Borja, Díaz-Polo: Uses

- Maximal Degeneracy Distribution
- Area degeneracy relation $4\sqrt{1/2(1/2+1)} = \sqrt{3(3+1)}$

to arrive at the same result.

Results

The requirement $\langle \epsilon(m) \rangle = 0$ implies

$$\Delta a = \frac{\langle a(m) \rangle}{3\langle m \rangle + 1} \quad \text{and} \quad \epsilon(m) = a(m) - (3m + 1)\frac{\langle a(m) \rangle}{3\langle m \rangle + 1}.$$

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This can be evaluated numerically. We find

$$\Delta a \approx 0.34952, \qquad \langle \epsilon(m)^2 \rangle \approx 0.00019156$$

What does that mean?

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$$\frac{\Delta a}{\sqrt{\langle \epsilon(m)^2 \rangle}} \approx 25$$

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- Result for Δa compares nicely with CDF:

$$\chi \approx 8.7843, \qquad \chi_{\mathrm{CDF}} \approx 8.80 \qquad \frac{\chi_{\mathrm{CDF}} - \chi}{\chi_{\mathrm{CDF}}} \approx 0.00129$$

- We seem to be even closer to the conjectured value:

$$8\ln(3) \approx 8.7889, \qquad \frac{8\ln(3)-\chi}{\chi} \approx 0.00053.$$

Conclusions

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Entropy quantization due to “resonance” in area spectrum.

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Entropy quantization due to "resonance" in area spectrum.

How good is this explanation?

- ✗ can explain why pattern independent of counting
- ✗ can explain why implementing boundary condition matters
- ✗ does not say wether $\chi = 8\ln(3)$
- ✗ does not say why not $3\Delta a$
- ✗ does not say wether pattern persists for large black holes

Better: Analytic approach. See following talks.

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Space: $\mathbb{R}_+ \times \mathbb{Z}/2$

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$$p'(m) \approx \exp\left(-2\pi\gamma_M \sqrt{|m|(|m|+1)}\right)$$

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