VARIETIES AND COVARIETIES OF LANGUAGES

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CONTENTS

PRELIMINARIES

ALGEBRA-COALGEBRA

Given a category $\mathbf{X}$ and an endofunctor $F : \mathbf{X} \to \mathbf{X}$.

Definition

A F-algebra consists of a pair (X, α), where X is an object of $\mathbf{X}$ and $\alpha : FX \to X$ an arrow in $\mathbf{X}$.

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We call X the base and α the structure map of the (co)algebra.

AUTOMATA

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$$\alpha : X \to X^A$$

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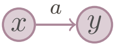

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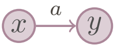

$$\Leftrightarrow \quad \alpha(x)(a) = y$$

We will also write $x_a = \alpha(x)(a)$ and, more generally,

$$x_\varepsilon = x \qquad x_{wa} = \alpha(x_w)(a)$$

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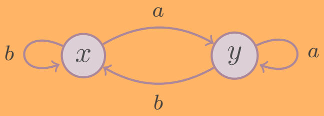

AUTOMATA

Because of the isomorphism

$$(X \times A) \to X \cong X \to X^A$$

the transition structure of an automaton X with inputs from an alphabet A can be viewed both as an G-algebra and as a F-coalgebra for the endofunctors on the category $\mathbf{Set}$ given by:

$$\begin{array}{rcl} G(X) & = & X \times A \\ F(X) & = & X^A \end{array}$$

POINTED AUTOMATA

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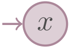

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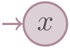

Note that pointed automata are $(1 + G)$-algebras.

COLOURED AUTOMATA

Definition

An automaton can be decorated by means of a colouring function

$$c : X \to 2$$

We call a state x accepting if $c(x) = 1$ otherwise it is called non-accepting. We call the triple (X, α, c) a coloured automaton

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AUTOMATA HOMOMORPHISMS

Definition

A function $h : X \to Y$ is a homomorphism between automata (X, α) and (Y, β) if it makes the following diagram commute

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\
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A homomorphism of pointed automata and of coloured automata must preserve initial values and colours, respectively.

If $X \subseteq Y$ and $h : X \hookrightarrow Y$ is the inclusion function, we will say that X is a subautomaton of Y. It will be denoted by $X \leq Y$.

BISIMULATION

Definition

We call a relation $R \subseteq X \times Y$ a bisimulation of automata if for all $(x, y) \in X \times Y$,

$$(x, y) \in R \;\Rightarrow\; \forall a \in A,\ (x_a, y_a) \in R$$

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For pointed automata (X, α, x) and (Y, β, y), R is a pointed bisimulation if, moreover, $(x, y) \in R$.

For coloured automata (X, α, c) and (Y, β, d), R is a coloured bisimulation if, moreover,

$$(x, y) \in R \;\Rightarrow\; c(x) = d(y)$$

BISIMULATION EQUIVALENCE

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The quotient map of a bisimulation equivalence on X is a homomorphism automata:

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "\alpha"'] & X/E \arrow[d, "{[\alpha]}"] \\ X^A \arrow[r, "\pi^A"'] & (X/E)^A \end{tikzcd}$$

SETTING THE SCENE

INITIAL ALGEBRA

The set A forms a pointed automaton $(A\ ,\sigma,\varepsilon)$ with initial state ε and transition function defined by

$$\sigma : A \ \to (A\ )^A \quad \sigma(w)(a) = wa$$

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For any given automaton (X,α) and every choice of initial state $x : 1 \to X$, it induces a unique function $r_x : A\ \to X$, given by

$$r_x(w) = x_w$$

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This is equivalent to say that the following diagram commutes:

$$\begin{tikzcd}
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The function r_x maps a word w to the state x_w reached from the initial state x on input w and is therefore called the reachability map for (X, α, x).

FINAL COALGEBRA

The set 2^{A^*} of languages forms a coloured automaton $(2^{A^*}, \tau, \varepsilon?)$ with colour function $\varepsilon?$ defined by

$$\varepsilon? : 2^{A^*} \to 2 \quad \varepsilon?(L) = \begin{cases} 1 & \text{if } \varepsilon \in L \\ 0 & \text{otherwise} \end{cases}$$

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For any automaton (X, α) and every choice of colouring function $c : X \to 2$, it induces a unique function $o_c : X \to 2^{A}$, given by

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The function o_c maps a state x to the language $o_c(x)$ accepted by x and is therefore called the observability map for (X, α, c).

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If the reachability map r_x is surjective then we call (X, α, x) reachable. Note that $(A \ \ , \sigma, \varepsilon)$ is reachable.

If the observability map o_c is injective then we call (X, α, c) observable.

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EQUATIONS AND COEQUATIONS

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$$(X, \alpha) \models E \quad \Leftrightarrow \quad \forall x : 1 \to X, \;\; (X, \alpha, x) \models E$$

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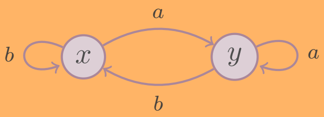

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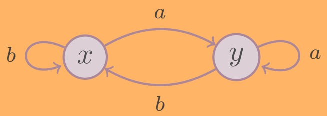

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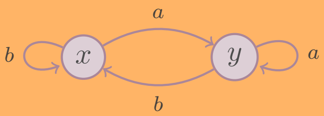

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Proposition

$(X, \alpha, x) \models E \quad \Leftrightarrow \quad E \subseteq \mathsf{ker}(r_x)$

We have, equivalently, that $(X, \alpha, x) \models E$ iff the reachability map r_x factors through $A\ /E$.

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Definition

We define $\mathsf{Eq}(X, \alpha)$ to be the largest set of equations satisfied by the automaton (X, α).

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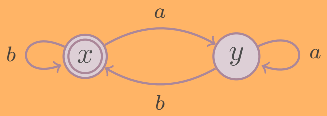

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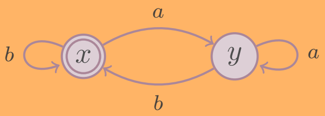

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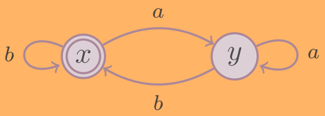

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therefore,

$$(X, \alpha, c) \models \{(a \ \ b)\ \ , (a \ \ b)^+\}$$

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$$(X, \alpha, c) \models D \quad \Leftrightarrow \quad \mathrm{im}(o_c) \leq D$$

We have, equivalently, that $(X, \alpha, c) \models D$ iff the observability map o_c factors through D.

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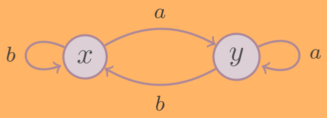
Definition

We define coEq(X, α) to be the smallest set of coequations satisfied by the automaton (X, α).

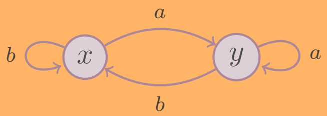
ALL TOGETHER NOW

Example



ALL TOGETHER NOW

Example



$$\mathsf{Eq}(X, \alpha) = \{aa = a, bb = b, ab = b, ba = a\}$$

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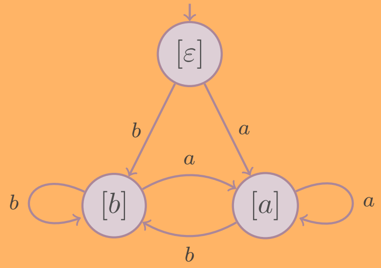

$$A\ \ /\mathsf{Eq}(X,\alpha)$$

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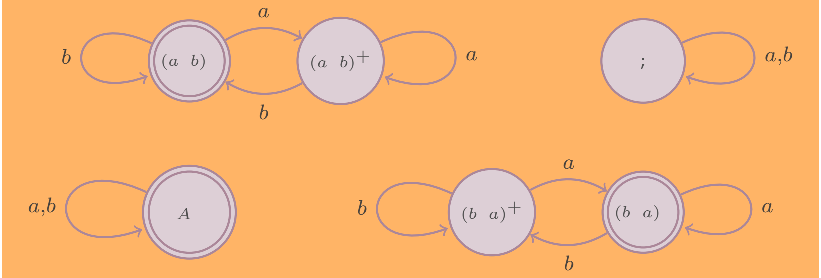

$\mathsf{coEq}(X, \alpha)$

VARIETIES AND COVARIETIES

VARIETIES

Definition

For every set E of equations we define the variety V_E by

$$V_E = \{(X, \alpha) \mid (X, \alpha) \models E\}$$

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COVARIETIES

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For every set D of coequations we define the covariety C_D by

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LANGUAGES

Definition

Let V_E be a variety. We define the variety of languages $L(V_E)$ by

$$L(V_E) = \{L \in 2^A \ \mid < L > \in V_E\}$$

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ON EQUATIONS AND VARIETIES

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Theorem

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The following statements are equivalent:

i. E is a congruence.
ii. $E = \mathsf{Eq}(X, \alpha)$ for some automaton (X, α).
iii. $(A \ /E, [\sigma]) \models E$.
iv. $\mathsf{Eq}(A \ /E, [\sigma]) = E$.

ON EQUATIONS AND VARIETIES

Under any of the statements above, we have:

Corollary

$$L(V_E) = \{L \in 2^A \;\mid \forall (v,w) \in E,\;\; L_v = L_w\}$$

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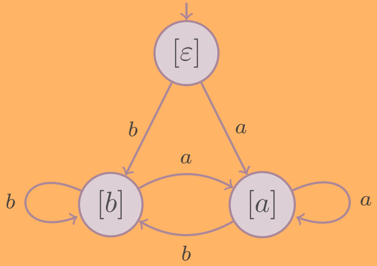
Note that if E is a bisimulation on A , $A\ /E$ has structure of automaton.

When E is a congruence on A , $A\ /E$ can be both seen as an automaton and as a monoid.

ON EQUATIONS AND VARIETIES

Example



Multiplication law is given by: $[w][v] = [w]_v = [wv]$

ON EQUATIONS AND VARIETIES

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For every automaton (X, α), A /Eq(X, α) is a monoid.

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Theorem

$$A\ /\mathrm{Eq}(X, \alpha) \cong \mathrm{trans}(X, \alpha)$$

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iii. $\text{coEq}(D, \tau) = D$.

iv. $L(C_D) = D$.

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