THE DUAL EQUIVALENCE OF EQUATIONS AND COEQUATIONS FOR AUTOMATA

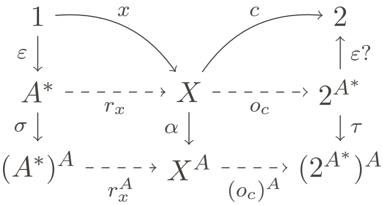
COIN: COalgebra In the Netherlands

Nijmegen, 2014

Enric Cosme Llópez [1]
Jan Rutten[2]

[1] **Universitat de València**
[2] **CWI and Radboud Universiteit Nijmegen**

THE SCENE



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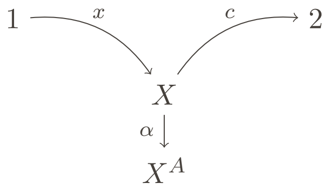

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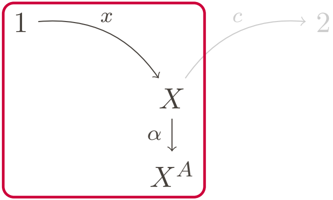

Because of the isomorphism $X \to X^A \cong X \times A \to X$, the above frame can be seen as an algebra.

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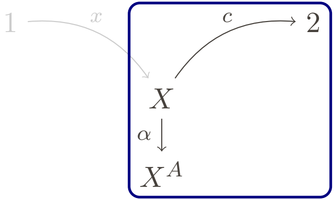

THE SCENE

$$
\begin{array}{ccccc}
1 & \xrightarrow{\quad x \quad} & & & \\
\varepsilon \big\downarrow & & \searrow & & \\
A^* & & X & \xrightarrow{\quad c \quad} & 2 \\
\sigma \big\downarrow & & \alpha \big\downarrow & & \\
(A^*)^A & & X^A & &
\end{array}
$$

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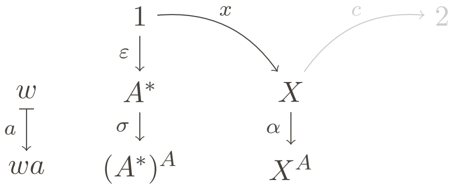

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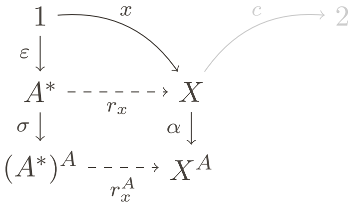

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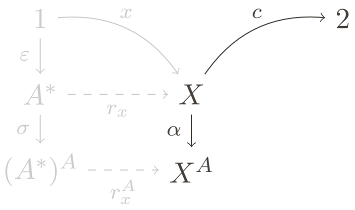

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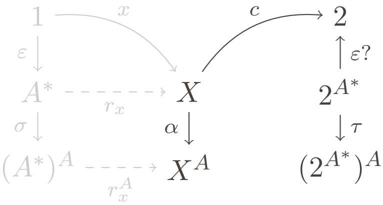

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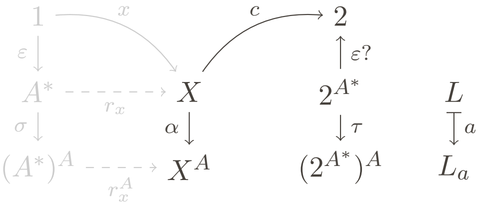

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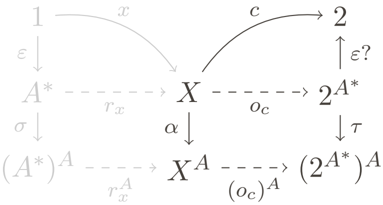

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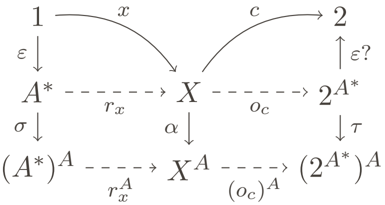
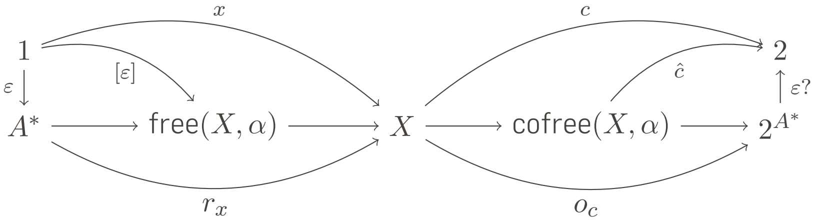


THE EXTENDED SCENE



EQUATIONS

Definition

A set of equations is a bisimulation equivalence $E \subseteq A^* \times A^*$ on the initial automaton (A^*, σ).

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We say that the pointed automaton (X, α, x) satisfies E

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We define:

$$(X, \alpha) \models E \quad \Leftrightarrow \quad \forall x : 1 \to X,\ \ (X, \alpha, x) \models E$$

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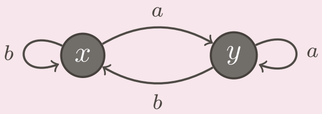

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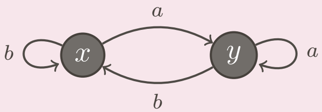

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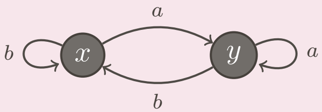

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$$(X, \alpha, c) \models D \quad \Leftrightarrow \quad \forall x \in X, \;\; o_c(x) \in D$$

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$$(X, \alpha) \models D \quad \Leftrightarrow \quad \forall c : X \to 2,\; (X, \alpha, c) \models D$$

COEQUATIONS

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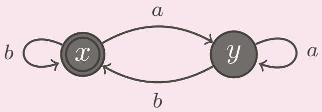

COEQUATIONS

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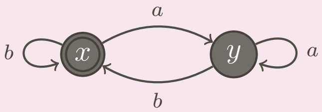

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$$o_c(x) = (a^*b)^* \qquad o_c(y) = (a^*b)^+$$

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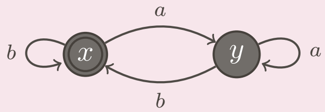

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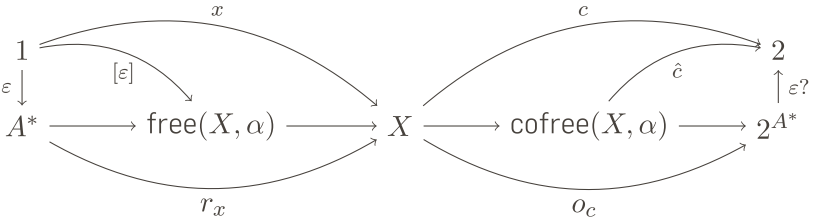
therefore,

$$(X, \alpha, c) \models \{(a^*b)^*, (a^*b)^+\}$$

THE EXTENDED SCENE



FREE

Let (X, α) be an arbitrary automaton. We show how to construct an automaton that corresponds to the largest set of equations satisfied by (X, α). And, dually, we construct an automaton that corresponds to the smallest set of coequations satisfied by (X, α).

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For notational convenience we assume X to be finite but nothing will depend on that assumption.

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(i) First, we take the product of the n pointed automata (X, x_i, α) that we obtain by letting the initial element x_i range over X. This yields a pointed automaton $(\Pi X, \bar{x}, \bar{\alpha})$ with

$$\Pi X = \prod_{x:1 \to X} X_x \quad \cong \quad X^n$$

(where $X_x = X$), with $\bar{x} = (x_1, \ldots, x_n)$, and with $\bar{\alpha} : \Pi X \to (\Pi X)^A$ defined by

$$\bar{\alpha}(y_1, \ldots, y_n)(a) = ((y_1)_a, \ldots, (y_n)_a)$$

FREE

(ii) Next we consider the reachability map $r_{\bar{x}} : A^* \to \Pi X$ and define:

$$\mathsf{Eq}(X, \alpha) = \ker(r_{\bar{x}}) \qquad \mathsf{free}(X, \alpha) = A^*/\mathsf{Eq}(X, \alpha)$$

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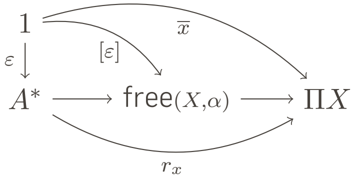
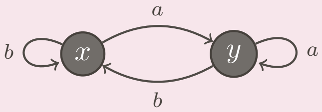
This yields the pointed automaton $(\mathsf{free}(X, \alpha), [\varepsilon], [\sigma])$:



ALL TOGETHER NOW

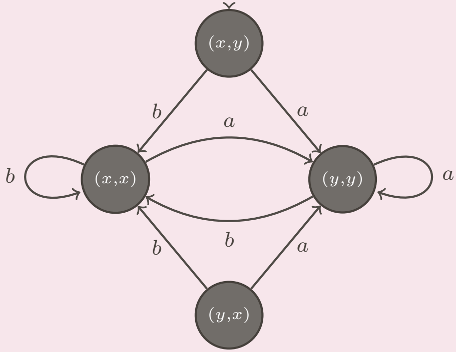
Example



ALL TOGETHER NOW

1st Step. Construct the product automaton.

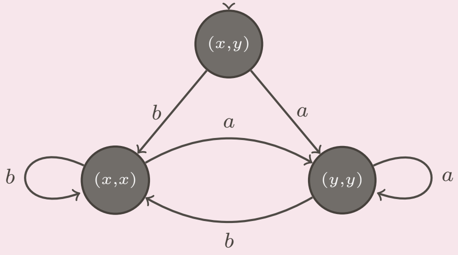
Example



ALL TOGETHER NOW

2nd Step. Take the image under the reachability map $r_{\overline{x}}$

Example



ALL TOGETHER NOW

We define $\mathsf{Eq}(X, \alpha)$ as $\ker(r_{\overline{x}})$.

Example

$$\mathsf{Eq}(X, \alpha) = \{aa = a, bb = b, ab = b, ba = a\}$$

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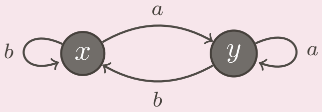
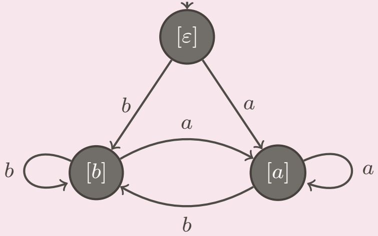


ALL TOGETHER NOW

free(X, α) is the quotient automaton of A^* over Eq(X, α).

Example



$$\text{free}(X, \alpha) = A^* / \text{Eq}(X, \alpha)$$

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Dually, let $X = \{x_1, \ldots, x_n\}$ be the set of states of a finite automaton (X, α). We define a coloured automaton cofree(X, α) in two steps, as follows:

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(i) First, we take the coproduct of the 2^n coloured automata (X, c, α) that we obtain by letting c range over the set $X \to 2$ of all colouring functions. This yields a coloured automaton $(\Sigma X, \hat{c}, \hat{\alpha})$ with

$$\Sigma X = \sum_{c:X\to 2} X_c$$

(where $X_c = X$), and with $\hat{c}$ and $\hat{\alpha}$ defined component-wise.

COFREE

(ii) Next we consider the observability map $o_{\hat{c}} : \Sigma X \to 2^{A^*}$ and define:

$$\mathsf{coEq}(X, \alpha) = \mathsf{im}(o_{\hat{c}}) \qquad \mathsf{cofree}(X, \alpha) = \mathsf{coEq}(X, \alpha)$$

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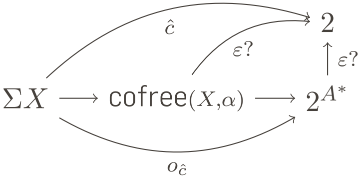
This yields the coloured automaton (cofree$(X, \alpha), \varepsilon?, \tau$):



ALL TOGETHER NOW

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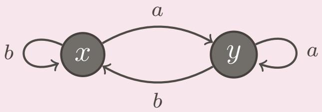
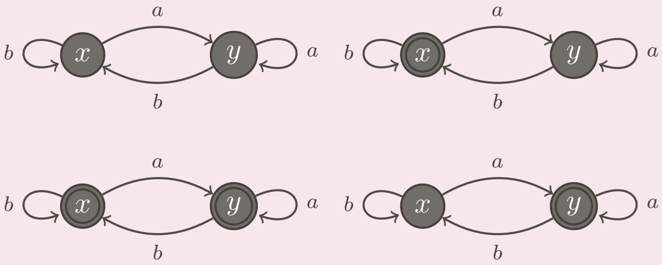


ALL TOGETHER NOW

1st Step. Construct the coproduct automaton.

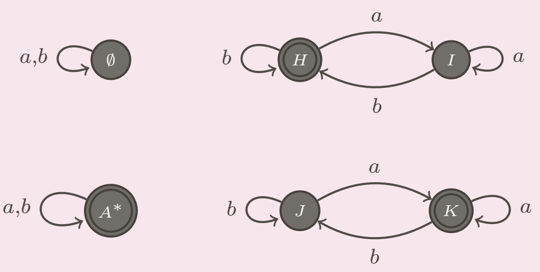
Example



ALL TOGETHER NOW

2nd Step. Take the image under the observability map $o_{\hat{c}}$.

Example



$H = (a^* b)^*$

$I = (a^* b)^+$

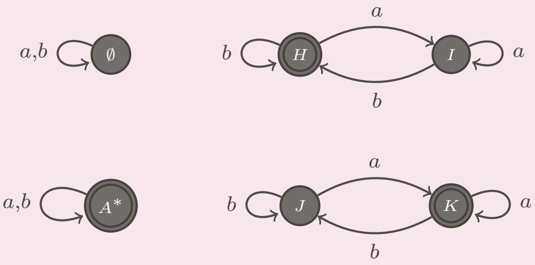
$J = (b^* a)^+$

$K = (b^* a)^*$

ALL TOGETHER NOW

We define $\text{coEq}(X, \alpha)$ as $\text{im}(o_{\hat{c}})$ and $\text{cofree}(X, \alpha) = \text{coEq}(X, \alpha)$.

Example



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$$\text{cofree}(X, \alpha) = \text{coEq}(X, \alpha)$$

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In this section, we shall first show that -- when suitably restricted -- the constructions of free and cofree are in fact functorial, that is, they act not only on automata but also on homomorphisms.

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We will be using the following categories:

- $\mathcal{A}$: the category of automata (X, α) and automata homomorphisms
- $\mathcal{A}_m$: the category of automata (X, α) and automata monomorphisms
- $\mathcal{A}_e$: the category of automata (X, α) and automata epimorphisms

FUNCTORIAL FREE

As it turns out, we can extend the definition of free to monomorphisms, such that we obtain functors of the following type:

$$\text{free} : \mathcal{A}_m \to (\mathcal{A}_e)^{\text{op}}$$

Here the superscript op indicates a reversal of arrows.

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For monomorphisms,

$$\begin{array}{ccc} (X, \alpha) & & \text{free}(X, \alpha) \\ m \big\downarrow & \overset{\text{free}}{\longmapsto} & \big\uparrow \text{free}(m) \\ (Y, \beta) & & \text{free}(Y, \beta) \end{array}$$

where free(m) is defined simply by quotienting. Recall that the existence of the monomorphism m implies Eq$(Y, \beta) \subseteq$ Eq(X, α).

FUNCTORIAL COFREE

Dually, we can extend the definition of cofree to epimorphisms, such that we obtain functors of the following type:

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For epimorphisms,

$$\begin{array}{ccc} (X, \alpha) & & \text{cofree}(X, \alpha) \\ e \Big\downarrow & \overset{\text{cofree}}{\longmapsto} & \Big\uparrow \text{cofree}(e) \\ (Y, \beta) & & \text{cofree}(Y, \beta) \end{array}$$

where cofree(e) is just set inclusion. Recall that the existence of the epimorphism e implies coEq$(Y, \beta) \subseteq$ coEq(X, α).

CONGRUENCE QUOTIENTS

Definition

We introduce the category $\mathcal{C}$ of congruence quotients, which is defined as follows:

$$\begin{aligned} \text{objects}(\mathcal{C}) &= \{ (A^*/C, [\sigma]) \mid C \text{ is a congruence relation } \} \\ \text{arrows}(\mathcal{C}) &= \{ e : A^*/C \to A^*/D \mid e \text{ is an epimorphism} \} \end{aligned}$$

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Theorem

$\text{free}(\mathcal{A}_m) = \mathcal{C}^{\text{op}}$

VARIETIES OF LANGUAGES

Definition

A variety of languages, is a set $V \subseteq 2^{A^*}$ such that:

(i) V is a complete atomic Boolean subalgebra of 2^{A^*}.

(ii) if $L \in V$ then for all $a \in A$, both L_a and ${}_aL \in V$

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Theorem

$\text{cofree}(\mathcal{C}) = \mathcal{V}^{\text{op}}$

MAIN THEOREM

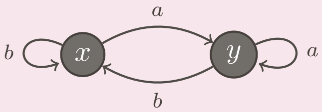
Our main result is a dual equivalence.

Theorem

cofree: $\mathcal{C} \cong \mathcal{V}^{\mathrm{op}}$: free

ILLUSTRATING THE DUALITY

Example



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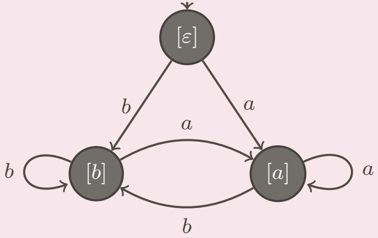

$\mathsf{free}(X, \alpha)$

ILLUSTRATING THE DUALITY

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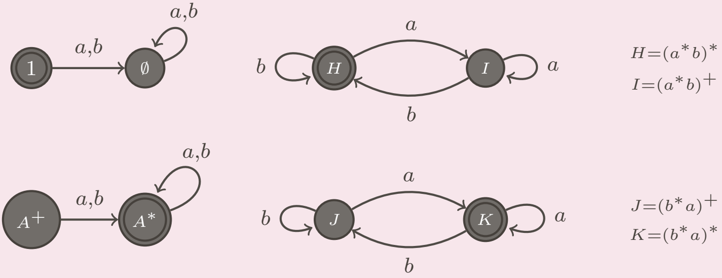

cofree ∘ free(X, α)

ILLUSTRATING THE DUALITY

To better understand the duality, consider a quotient automaton $(A^*/C, [\sigma])$. For a word $w \in A^*$, consider the following colouring:

$$\begin{array}{rccl} \delta_{[w]} : & A^*/C & \longrightarrow & 2 \\ & [v] & \longmapsto & \begin{cases} 1 & \text{if } [v] = [w] \\ 0 & \text{otherwise} \end{cases} \end{array}$$

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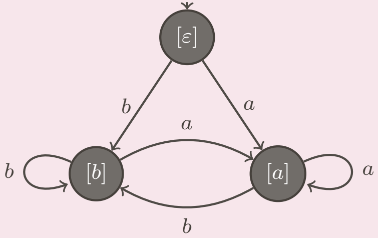
Under this colouring, it holds:

$$o_{\delta_{[w]}}([\varepsilon]) = [w]$$

Hence, every state $[w] \in A^*/C$ also belongs to cofree$(A^*/C, [\sigma])$.

ILLUSTRATING THE DUALITY

Example



$\text{free}(X, \alpha)$

ILLUSTRATING THE DUALITY

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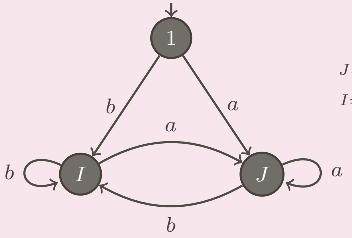

$J = (b^* a)^+$

$I = (a^* b)^+$

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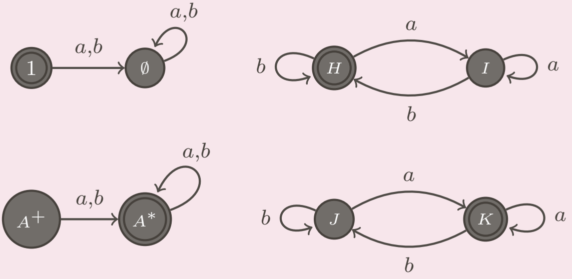

$H = (a^* b)^*$
$I = (a^* b)^+$

$J = (b^* a)^+$
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$$\text{cofree} \circ \text{free}(X, \alpha)$$

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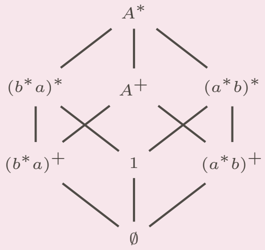

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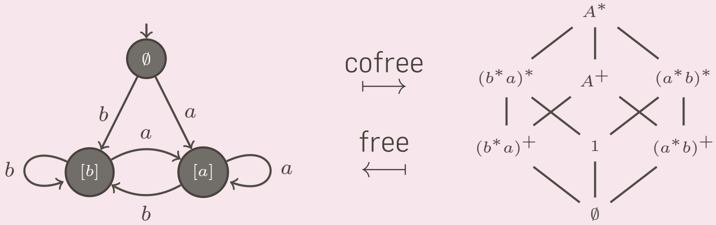
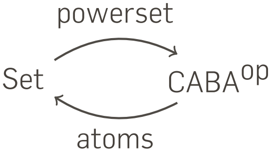


ILLUSTRATING THE DUALITY

That is, forgetting all the automata structure we recover the classical duality:



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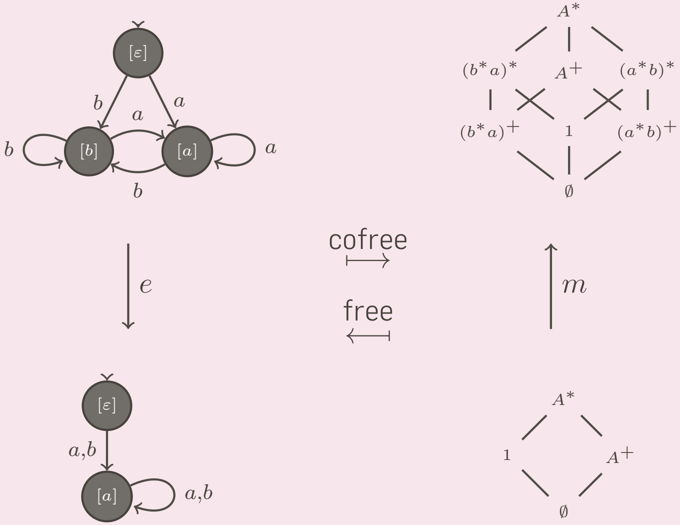
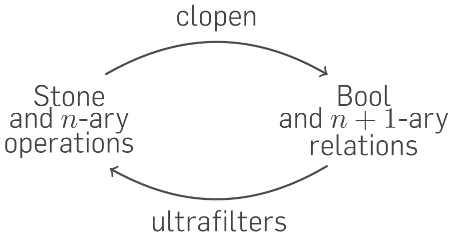


RELATED WORK

In the literature we have found examples of such duality [GGP08 Geh11, Rou11]. They start directly from the extended Stone duality:



RELATED WORK

For a regular language L. They state the duality:

$$(\text{trans}(\langle L \rangle), \cdot) \underset{\text{ultrafilters}}{\overset{\text{clopen}}{\rightleftarrows}} (\mathcal{B}(L), /, \backslash)$$

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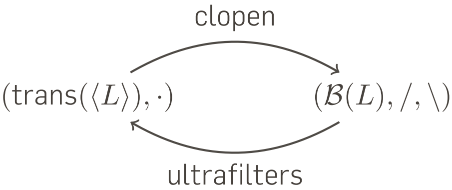

Where $(\text{trans}(\langle L \rangle), \cdot)$ is the transition monoid of $\langle L \rangle$ and $(\mathcal{B}(L), /, \backslash)$ is a boolean algebra with some residuation properties.

RELATED WORK

Theorem

$\text{free}(X, \alpha) \cong \text{trans}(X, \alpha)$

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For finite automata, our duality coincides with that of Gehrke et al.

COMMUTATIVE LANGUAGES

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Let $A = \{a, b\}$ and let ab=ba denote the smallest congruence on A^* containing the equation (ab, ba).

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Languages $[w]$ in the congruence quotient $A^*/\text{ab=ba}$ satisfy

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By the duality Theorem, we have that $V = \text{cofree}(A^*/\text{ab=ba})$ is a variety of languages.

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Corollary

$L \in \text{coEq}(A^*/C, [\sigma]) \;\Leftrightarrow\; C \subseteq \text{Eq}(\langle L \rangle, \tau)$

EQUATIONAL BISIMULATIONS

Definition

Let $C \subseteq A^* \times A^*$ be a congruence. We call a relation $R \subseteq 2^{A^*} \times 2^{A^*}$ a C-bisimulation if for all $(K, L) \in R$,

1. $\varepsilon \in K \Leftrightarrow \varepsilon \in L$
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Proposition

If R is a C-bisimulation and $(K, L) \in R$ then,

i. $K = L$
ii. $\langle K \rangle \models C$

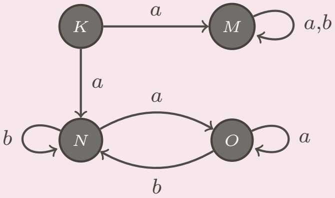
COMMUTATIVE LANGUAGES

Example

Let $K = aA^* + b(a^*b)^* + b(b^*a)^+$. We shall use last proposition to show that K is commutative. Referring to the example of commutative languages, we need to prove that $\langle K \rangle \models$ ab=ba.

COMMUTATIVE LANGUAGES

Example



$$K = aA^* + b(a^*b)^* + b(b^*a)^+$$
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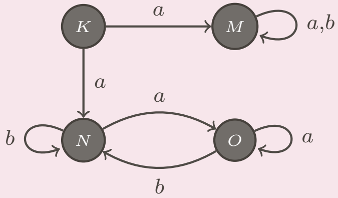

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Let

$$R = \{(K,K)\} \cup \{M,N,O\}^2$$

Then R is an (ab=ba)-bisimulation. Thus $\langle K \rangle \models$ ab=ba.

BIBLIOGRAPHY

M. Gehrke,
Duality and recognition,
MFCS, volume 6907 of LNCS, pages 3-18, 2011.

M. Gehrke, S. Grigorieff, J.-E. Pin,
Duality and equational theory of regular languages,
In Proceedings of ICALP, pages 246-257, 2008.

J.J.M.M. Rutten, A. Ballester-Bolinches, E. Cosme-Llópez,
Varieties and covarieties of languages (preliminary version),
In Proceedings of MFPS XXIX, pages 7-28, 2013.

F. Roumen,
Canonical automata via duality,
Master's thesis, Radboud Universiteit Nijmegen, 2011.

HSRM Beamer Theme, by Benjamin Weiss.