K4-FREE GRAPHS AS A FREE ALGEBRA

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E. Cosme, D. Pous

Laboratoire de l'Informatique du Parallélisme
École Normale Supérieure de Lyon

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Algebras of relations appear naturally in many contexts in computer science as they constitute a framework well suited to the semantics of imperative programs.

Many objects of interest either are relations or can be seen as relations. A major benefit of a relational approach in computer science is the surprisingly small number of relations needed to express complex notions.

SYNTAX

We consider algebras of the following type

$$u, v ::= u \cdot v \mid u \parallel v \mid u^{\circ} \mid 1 \mid \top \mid a \qquad\qquad (a \in \Sigma).$$

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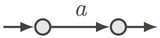
One model for this algebra is the set of relations on a given set with the usual interpretation of the operators.

TERMS AS GRAPHS

$\mathsf{g}(a)$



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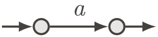
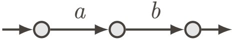


$\mathsf{g}(a \cdot b)$



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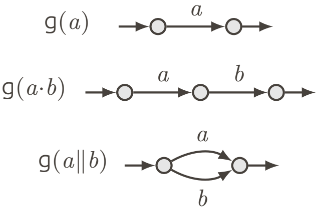

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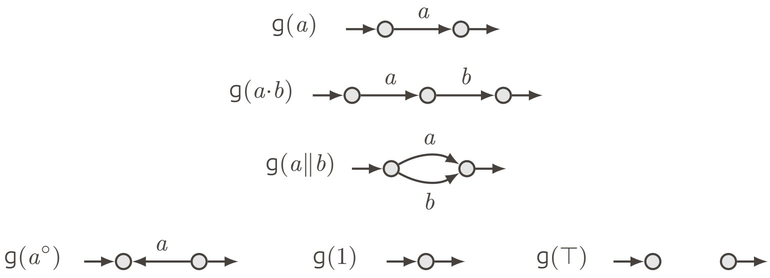

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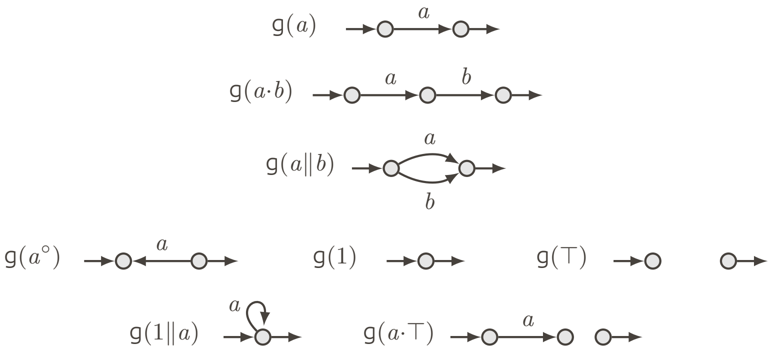

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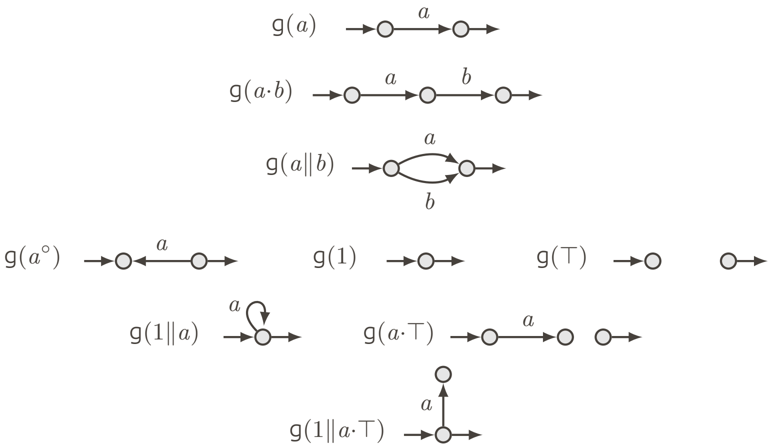
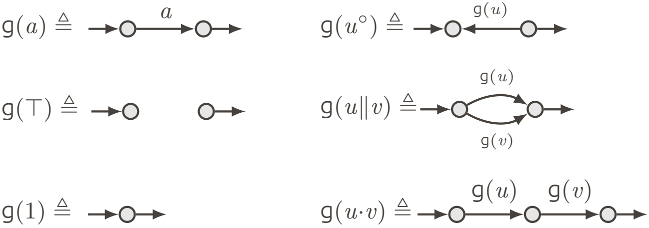


ASSOCIATED GRAPH



$$\text{Terms} \xrightarrow{\mathsf{g}} \text{Graphs}$$

Theorem [FS90]

For any terms u, v, we have

$$\mathsf{Rel} \models u \subseteq v \qquad \Leftrightarrow \qquad \mathsf{g}(u) \blacktriangleleft \mathsf{g}(v).$$

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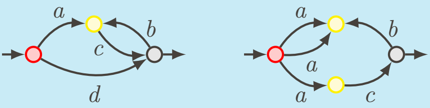
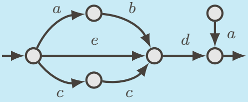
Example

$$\mathsf{Rel} \models a{\cdot}(b^{\circ}{\cap}c){\cap}d \subseteq (a{\cap}a){\cdot}b^{\circ}{\cap}a{\cdot}c$$

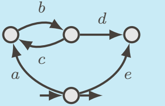
$\mathsf{g}(a{\cdot}(b^{\circ} \parallel c) \parallel d)$ $\blacktriangleleft$ $\mathsf{g}((a \parallel a){\cdot}b^{\circ} \parallel a{\cdot}c)$



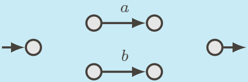
Example



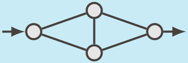
$\mathsf{g}((a \cdot b \| c^2 \| e) \cdot d \cdot (1 \| a^\circ \cdot \top))$



$\mathsf{g}(1 \| a \cdot (b \| c^\circ) \cdot d \cdot e^\circ)$



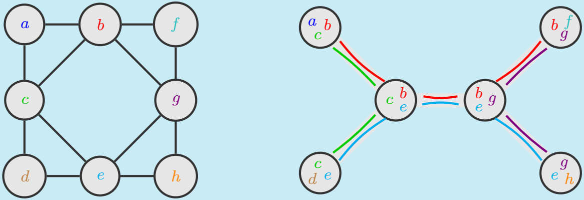
$\mathsf{g}(\top \cdot a \cdot \top \cdot b \cdot \top) \cong \mathsf{g}(\top \cdot b \cdot \top \cdot a \cdot \top)$



It is not a term graph [FS90]

TREE DECOMPOSITION

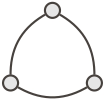
Example



The width of a tree decomposition is the size of the largest node minus one. The treewidth of a graph is the minimal width of a tree decomposition for this graph.

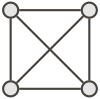
Bounded treewidth can be described by minor exclusion.

TW_1



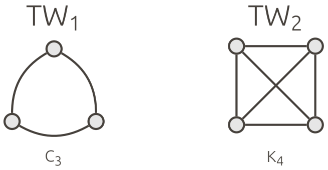
C_3

TW_2



K_4

Bounded treewidth can be described by minor exclusion.

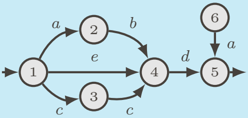


Proposition

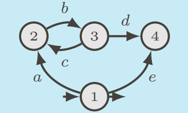
Every term graph has treewidth bounded by 2 with one node containing input and output.

$$\text{Term} \xrightarrow{g} \mathrm{TW}_2$$

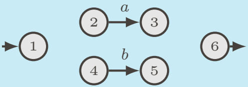
Example



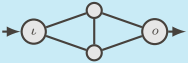
$\mathsf{g}((a{\cdot}b \| c^2 \| e){\cdot}d{\cdot}(1 \| a^{\circ}{\cdot}\top))$



$\mathsf{g}(1 \| a{\cdot}(b \| c^{\circ}){\cdot}d{\cdot}e^{\circ})$



$\mathsf{g}(\top{\cdot}a{\cdot}\top{\cdot}b{\cdot}\top) \cong \mathsf{g}(\top{\cdot}b{\cdot}\top{\cdot}a{\cdot}\top)$



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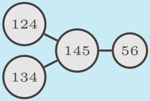
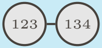
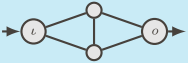
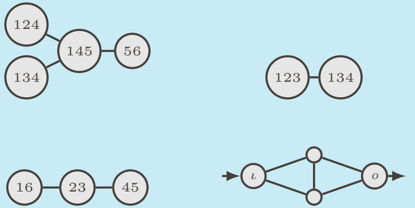






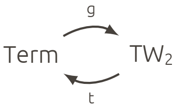
No such tree decomposition

Example



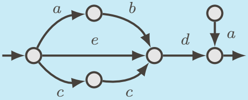
No such tree decomposition

Extract a term from a graph with compatible input and output.



CASE 1: CONNECTED WITH INPUT DIFFERENT FROM OUTPUT

Example

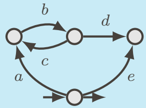


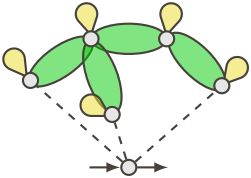
Case 1.1

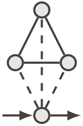
Case 1.2

CASE 2: CONNECTED WITH INPUT EQUALS OUTPUT

Example



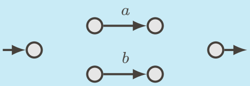




$1 \| u \cdot \top$

CASE 3: DISCONNECTED

Example



$\top \cdot u$	Disconnects the input.
$u \cdot \top$	Disconnects the output.

Theorem

For any 2-pointed graph G with compatible input and output,

$$g(t(G)) \cong G.$$

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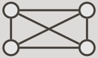
Corollary

Let G be a graph. The following statements are equivalent.

1. G is a term graph.
2. G has treewidth bounded by 2.
3. G is K_4 minor free.



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1. $G \cong H$ implies $\mathsf{t}(G) \equiv \mathsf{t}(H)$
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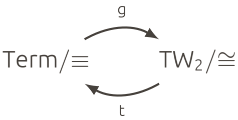

The reduct $(\|, \top)$ is a commutative monoid, the reduct $(\cdot, 1)$ is a monoid. The converse $^\circ$ is an involution.

$$
\begin{aligned}
1\|1 &\equiv 1 \\
u\cdot(1\|v) &\equiv u\|\top\cdot(1\|v) \\
1\|u\cdot v &\equiv 1\|(u\|v^\circ)\cdot\top \\
u\cdot\top\|v &\equiv (1\|u\cdot\top)\cdot v
\end{aligned}
$$

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$$u{\cdot}\top\|v \equiv (1\|u{\cdot}\top){\cdot}v$$

Theorem

The axioms listed above give a complete axiomatisation of isomorphism of graphs of treewidth bounded by 2.

$\mathrm{TW}_2/\cong$ is a free algebra.

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